Positive values of inhomogeneous indefinite ternary quadratic forms of type (2,1)

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Abstract. Let $\Gamma_{2,1}^{(k)}$ denote the kth successive inhomogeneous minima for positive values of real indefinite ternary quadratic forms of type (2,1). Earlier the first four minima for the class of zero forms were obtained. Here it is proved that for all the forms, whether zero or non zero, $\Gamma_{2,1}^{(2)} = 8/3$. All the critical forms have also been obtained.

Key words: inhomogeneous minimum, quadratic forms, lattices, admissible, continued fractions.

1. Introduction

Let $Q(x_1, x_2, \ldots, x_n)$ be a real indefinite quadratic form in n variables of determinant $D \neq 0$ and of type $(r, n-r)$. Let $\Gamma_{r,n-r}$ denote the infimum of all numbers $\Gamma > 0$ such that for any real numbers $c_1, c_2, \ldots, c_n$ there exist integers $x_1, x_2, \ldots, x_n$ satisfying

$$0 < Q(x_1 + c_1, x_2 + c_2, \ldots, x_n + c_n) \leq (\Gamma|D|)^{1/n}. \tag{1.1}$$

The values of $\Gamma_{r,n-r}$ are known for various n. See for reference Aggarwal and Gupta [1]. Let $\Gamma_{r,n-r}^{(k)}$ denote the kth successive inhomogeneous minimum for positive values of indefinite quadratic forms of type $(r, n-r)$. Bambah et al [2] proved that $\Gamma_{2,3}^{(2)} = 16$. Dumir and Sehmi [5, 6] obtained $\Gamma_{r+1,r}^{(2)}$ for all $r \geq 2$. For incommensurable forms (forms that are not multiple of rational forms) (1.1) is true with arbitrary small constant by a result of Watson [13] and Oppenheim's conjecture proved by Margulis [10]. Rational forms in $n \geq 5$ variables are zero forms by Meyer's Theorem. Ternary and quaternary forms are not necessarily zero forms. So Dumir and Sehmi [5, 6] just needed to consider zero forms. Raka [9] obtained the first four minima for ternary forms of the type $(2, 1)$ for the class of zero forms. For zero forms there is a standard method using a result of Macbeath [8].

In this paper we prove $\Gamma_{2,1}^{(2)} = 8/3$ for all forms. We apply a different method using the work of Barnes and Swinnerton Dyer, which contained a mistake. For a complete and elaborate proof of their work see Grover and Raka [7]. For ternary and quaternary forms our method is more powerful than the ones used earlier. In another paper [12] we will prove that $\Gamma_{3,1}^{(2)} = 4$, giving a correct proof of a result of R. Rieger.

Definition   We say that $(x, y, z) \equiv (x_0, y_0, z_0) \pmod 1$ if and only if $x - x_0,\ y - y_0,\ z - z_0$ are integers.

Thus the statement that given any real numbers x_0, y_0, z_0 there exist integers x, y, z satisfying

$$\alpha < Q(x + x_0, y + y_0, z + z_0) < \beta$$

is equivalent to saying that there exist $(x, y, z) \equiv (x_0, y_0, z_0) \pmod 1$ satisfying

$$\alpha < Q(x, y, z) < \beta.$$

Here we prove:

Theorem 1   *Let $Q(x, y, z)$ be a real indefinite quadratic form of type $(2, 1)$ and determinant $D < 0$. Then for any real numbers x_0, y_0, z_0, there exist $(x, y, z) \equiv (x_0, y_0, z_0) \pmod 1$ such that*

$$0 < Q(x, y, z) < (8|D|/3)^{1/3} \tag{1.2}$$

except for the forms $Q \sim \rho Q_i$, $i = 1, 2, 3$, $\rho > 0$; further for Q_i, (1.2) is solvable unless $(x_0, y_0, z_0) \equiv (x_0^{(i)}, y_0^{(i)}, z_0^{(i)}) \pmod 1$ where Q_i and $(x_0^{(i)}, y_0^{(i)}, z_0^{(i)})$ are given in the following table:

i	Q_i	$(x_0^{(i)}, y_0^{(i)}, z_0^{(i)})$	$\Gamma(Q_i)$
1	$xy + z^2$	$(0, 0, 0)$	4
2	$\left(x + \dfrac{1}{2}y\right)y + z^2$	$\left(\dfrac{1}{2}, 0, 0\right)$	4
3	$2xy + y^2 + yz + 3z^2$	$\left(\dfrac{1}{2}, 0, 0\right)$	8/3

where $\Gamma(Q_i)$ is the positive inhomogeneous minimum of Q_i.

Note that Q_i, $i = 1, 2, 3$ are all inequivalent and are zero forms.

2. Some Lemmas and general Reduction

Lemma 1 Let $Q(x, y, z)$ be as in Theorem 1. Then there exist integers x, y, z such that

$$0 < Q(x, y, z) \leq (9|D|/4)^{1/3} \tag{2.1}$$

except for the form $Q \sim \rho Q_1 = \rho(xy + z^2)$.

 This is a result of Oppenheim [11].

Lemma 2 Let α, β, γ be real numbers with $\gamma > 1$. Let p be an integer such that $p < \gamma \leq p + 1$. Then given any real number x_0, there exist $x \equiv x_0$ (mod 1) such that

$$0 < (x + \alpha)^2 + \beta < \gamma \tag{2.2}$$

provided that

$$-p^2/4 < \beta < \gamma - 1/4.$$

This follows from Lemma 6 of Dumir [4].

Lemma 3 Let $\varphi(y, z)$ be an indefinite binary quadratic form of discriminant Δ. Let ν be any positive real number. Then for any real number y_0, z_0, there exist $(y, z) \equiv (y_0, z_0)$ (mod 1) satisfying

$$-\Delta/4\nu < \varphi(y, z) < \nu\Delta/4.$$

This is Theorem 1 of Blaney [3].

 For a matrix V, we use the same symbol V to denote the transformation defined by the matrix V.

Lemma 4 Let U be a 2×2 unimodular matrix of infinite order and $\mathcal{R}$ be a bounded set in $\mathbb{R}^2$. Let $\mathcal{R}$ have the property

$$U(\mathcal{R}) \cap (\mathcal{R} + A) \neq \emptyset \qquad \text{for some} \quad A \in \mathbb{Z}^2$$

but

$$U(\mathcal{R}) \cap (\mathcal{R} + \beta) = \emptyset \qquad \forall B \in \mathbb{Z}^2, \quad B \neq A.$$

If P is a point such that for each integer n positive or negative, $U^n(P)$ is congruent (mod 1) to a point of $\mathcal{R}$, then P is the unique fixed point of $\mathcal{R}$ given by $U(P) - A = P$.

This is a result of Cassels stated as Lemma 18 *in Raka* [9].

If Q is an incommensurable form in $n \geq 3$ variables, it takes arbitrary small values by a result of Margulis [10]. For such a form the inequality (1.1) is true for arbitrary small Γ by Watson [13]. So we can assume that Q is a multiple of a rational form and hence a multiple of an integral form. Dividing (1.2) throughout by that multiple, if necessary, we can suppose that Q is an integral form.

Let

$$M = M(Q) = \inf Q(x, y, z) \tag{2.3}$$
$$x, y, z \in \mathbb{Z}$$
$$Q(x, y, z) > 0.$$

By Lemma 1,

$$0 < M \leq (9|D|/4)^{1/3}$$

except for the form $Q \sim \rho Q_1$, $\rho > 0$.

Lemma 5   *If* $Q \sim \rho Q_1 = \rho(x^2 + yz)$, $\rho > 0$ *then* (1.2) *is solvable in* $(x, y, z) \equiv (x_0, y_0, z_0)$ (mod 1) *except when* $(x_0, y_0, z_0) \equiv (0, 0, 0)$ (mod 1).

Proof.   Because of homogeneity we can suppose that $\rho = 1$. Let $d = (8|D|/3)^{1/3} = (2/3)^{1/3}$.

If y_0 or z_0 is not congruent to 0 (mod 1), say without loss of generality $y_0 \not\equiv 0$ (mod 1), choose $y \equiv y_0$ (mod 1) such that $0 < |y| \leq 1/2$, $x \equiv x_0$ (mod 1) arbitrarily and $z \equiv z_0$ (mod 1) such that:

$$0 < x^2 + yz \leq |y| \leq 1/2 < d.$$

Therefore let $(y_0, z_0) \equiv (0, 0)$ (mod 1). Take $y = z = 0$, and choose $x \equiv x_0$ (mod 1) such that $0 \leq |x| \leq 1/2$. Then

$$0 \leq x^2 + yz \leq 1/4 < d.$$

Thus (1.2) is solvable unless $x_0 \equiv 0$ (mod 1).   □

Let now $Q \not\sim \rho Q_1$. Since the set $\{Q(x, y, z) : x, y, z \in \mathbb{Z}, Q(x, y, z) > 0\}$ consists of positive integers, the infimum M is attained at some point (x_1, y_1, z_1) with $\gcd(x_1, y_1, z_1) = 1$. i.e.

$$Q(x_1, y_1, z_1) = M \leq (9|D|/4)^{1/3}$$

Applying a suitable unimodular transformation we can suppose that

$$Q(1,0,0) = M$$

and write

$$Q(x,y,z) = M\{x + hy + gz)^2 + \varphi(y,z)\} \qquad (2.4)$$

where $|h| \leq 1/2$, $|g| \leq 1/2$ and $\varphi(y,z)$ is a rational indefinite binary quadratic form of discriminant

$$\Delta^2 = 4|D|M^{-3} \geq 16/9. \qquad \text{(using (2.4))}$$

Also by definition of M, we have for all integers x, y, z either $Q(x,y,z) \leq 0$ or $Q(x,y,z) \geq M$.

Because of homogeneity, it suffices to prove.

Theorem A *Let $Q(x,y,z) = (x + hy + gz)^2 + \varphi(y,z)$, where $\varphi(y,z)$ is an indefinite binary quadratic form of discriminant*

$$\Delta^2 = 4|D| \geq 16/9 \qquad (2.5)$$

and

$$-1/2 < h \leq 1/2, \quad -1/2 < g \leq 1/2. \qquad (2.6)$$

Suppose for integers x, y, z we have

$$either \quad Q(x,y,z) \leq 0 \quad or \quad Q(x,y,z) \geq 1. \qquad (2.7)$$

Let

$$d = (8|D|/3)^{1/3}. \qquad (2.8)$$

Then either there exist $(x,y,z) \equiv (x_0,y_0,z_0) \pmod 1$ satisfying

$$0 < Q(x,y,z) < d \qquad (2.9)$$

or $Q \sim \rho Q_i$. Further for Q_i, (2.9) is solvable unless $(x_0,y_0,z_0) \equiv (x_0^{(i)}, y_0^{(i)}, z_0^{(i)}) \pmod 1$, $i = 1,2,3$, where Q_i and $(x_0^{(i)}, y_0^{(i)}, z_0^{(i)})$ are as listed in Theorem 1.

Lemma 6 *Let $Q(x,y,z)$ be as defined in Theorem A. Then for integers y, z we have either*

$$\varphi(y,z) = 0 \quad or \quad \varphi(y,z) \leq -1/4 \quad or \quad \varphi(y,z) \geq 3/4. \qquad (2.10)$$

The proof is similar to that of Lemma 8 of Dumir [4].

From (2.5) and (2.8) we get $d \geq (32/27)^{1/3} > 1$. Let n be an integer (≥ 1) such that $n < d \leq n + 1$. If there exist $(y, z) \equiv (y_0, z_0)$ (mod 1), such that

$$-n^2/4 < \varphi(y, z) < d - 1/4 \qquad (2.11)$$

then by Lemma 2, there exists $x \equiv x_0$ (mod 1) satisfying (2.9).

Lemma 7   *If $n \geq 2$, then (2.11) and hence (2.9) is solvable in $(x, y, z) \equiv (x_0, y_0, z_0)$ (mod 1).*

Proof.   Apply Lemma 3, with $\nu = \Delta/n^2$ to get

$$-n^2/4 = -\Delta/4\nu < \varphi(y, z) < \nu\Delta/4 = \Delta^2/4n^2.$$

Then (2.11) will be satisfied if

$$\Delta^2/(4d - 1) = 3d^3/2(4d - 1) < n^2.$$

This is easily seen to be true for $d \leq n + 1$ and $n \geq 2$.   □

Lemma 8   *Let $n = 1$, so that $(32/27)^{1/3} \leq d \leq 2$. Suppose (2.11) i.e.*

$$-1/4 < \varphi(y, z) < d - 1/4$$

has no solution in $(y, z) \equiv (y_0, z_0)$ (mod 1), then we have

$$\varphi \sim \rho\varphi_4 = \rho(y^2 - 2z^2), \quad or \quad \varphi \sim \rho\varphi_5 = \rho(3y^2 + 11z^2 + 18yz),$$

$\rho > 0$, *and $(y_0, z_0) \equiv (1/2, 1/2)$ (mod 1).*

3.   Proof of Lemma 8

Let $\mathcal{L}$ be the inhomogeneous lattice associated with $4\varphi(y, z)$ with determinant $\Delta(\mathcal{L}) = 4\Delta$. i.e. $\mathcal{L}$ is given by the set of points

$$\xi = \alpha y + \beta z, \quad \eta = \gamma y + \delta z$$

where (y, z) run through all numbers congruent to (y_0, z_0) (mod 1), and $4\varphi(y, z) = (\alpha y + \beta z)(\gamma y + \delta z)$. We say that $\mathcal{L}$ is admissible for the region $R_m : -1 \leq \xi\eta \leq m$, if it has no point in the interior of R_m. To prove Lemma 8, it is enough to prove that if $\mathcal{L}$ is admissible for the region R_m with $m = 4d - 1$, then $\mathcal{L}$ must correspond to the special forms φ_4 and φ_5.

Barnes and Swinnerton Dyer have developed a general theory to obtain the critical determinant of R_m i.e. the lower bound of $\Delta(\mathcal{L})$ over all R_m-admissible lattices $\mathcal{L}$. For this see Grover and Raka [7]. For any inhomogeneous lattice $\mathcal{L}$ of determinant $\Delta(\mathcal{L})$ with no points on the co-ordinate axis, there corresponds a chain of divided cells and a sequence of non-zero integral pairs (h_n, k_n) for $-\infty < n < \infty$; h_n and k_n having the same sign. The condition that the chain does not break off is simply that $\mathcal{L}$ has no lattice vector parallel to a co-ordinate axis. Set $a_{n+1} = h_n + k_n$ for all n, so that $|a_{n+1}| \geq 2$. If $h_n = k_n > 0$ for each n, the lattice $\mathcal{L}$ is called a symmetrical lattice, otherwise nonsymmetrical. For a symmetrical lattice, it follows that $a_n \geq 4$ for arbitrarily large values of $|n|$ for n of each sign.

Let $[b_1, b_2, b_3, \ldots]$ denote the continued fraction

$$b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - \cdots}}$$

where b_i's are integral and $|b_i| \geq 2$.

We need the following Lemmas 9–12 due to Barnes and Swinnerton Dyer stated as Lemmas 1, 2, 3 & 5 in Grover and Raka [7].

Lemma 9 *Let $b_i > 0$ for all i and $b_i \geq 4$ for some arbitrary large i, then*

$$[b_1, b_2, \ldots, b_n, b_{n+1}, \cdots] < [b_1, b_2, \ldots, b_n, b'_{n+1}, \ldots] \tag{3.1}$$

provided that $b_{n+1} < b'_{n+1}$. In particular

$$[b_1, b_2, \ldots, b_n - 1] < [b_1, b_2, \ldots, b_n, \ldots] < [b_1, b_2, \ldots, b_n] \tag{3.2}$$

Lemma 10 *Let $\{a_n\}_{-\infty}^{\infty}$ be a sequence associated to a symmetrical lattice $\mathcal{L}$. Let*

$$\theta_n = [a_n, a_{n-1}, a_{n-2}, \ldots], \quad \phi_n = [a_{n+1}, a_{n+2}, \ldots] \tag{3.3}$$

so that $\theta_n > 1$, $\phi_n > 1$ by Lemma 9 above. Then the lattice $\mathcal{L}$ is given by the set of points (ξ, η)

$$\begin{aligned} \xi &= \alpha_n(y - 1/2) + \beta_n(z - 1/2), \\ \eta &= \gamma_n(y - 1/2) + \delta_n(z - 1/2) \end{aligned} \tag{3.4}$$

where $\delta_n/\gamma_n = \phi_n$ and $\alpha_n/\beta_n = \theta_n$ and y, z are integers. The quadratic

form associated with $\mathcal{L}$ is given by

$$\frac{\Delta(\mathcal{L})}{\theta_n \phi_n - 1}[(\theta_n y + z)(y + \phi_n z)],$$

$$(y_0, z_0) \equiv (1/2, 1/2) \pmod{1}. \tag{3.5}$$

Lemma 11   *A symmetrical lattice $\mathcal{L}$ is admissible for R_m if and only if the inequalities*

$$\frac{\Delta}{m} \geq \frac{4(\theta_n \phi_n - 1)}{(\theta_n + 1)(\phi_n + 1)} = \Delta_n^+ \tag{3.6}$$

and

$$\Delta \geq \frac{4(\theta_n \phi_n - 1)}{(\theta_n - 1)(\phi_n - 1)} = \Delta_n^-, \quad \text{hold for all } n. \tag{3.7}$$

Lemma 12   *If $0 < L < 2(k+1)$ and for any n*

$$\Delta_n^+ \leq L/k, \quad \Delta_n^- \leq L$$

then

$$\frac{L(\theta_n - 1) - 4}{L(\theta_n - 1) - 4\theta_n} \leq \phi_n \leq \frac{4 + (L/k)(\theta_n + 1)}{4\theta_n - (L/k)(\theta_n + 1)} \tag{3.8}$$

and

$$\left| \theta_n - \frac{2(k-1)}{2(k+1) - L} \right| \leq \frac{\sqrt{L^2 - 16k}}{2(k+1) - L}. \tag{3.9}$$

These inequalities also hold if θ_n and ϕ_n are interchanged.

Lemma 13   *Let $\mathcal{L}$ be a non-symmetrical lattice of $\det \Delta(\mathcal{L})$, which is admissible for R_m, $3 < m \leq 7$. Then*

$$\Delta(\mathcal{L}) \geq (1.8251)(m+1). \tag{3.10}$$

This follows from Lemma 8 of Grover and Raka [7].

Lemma 14   *Let $\{a_n\}_{-\infty}^{\infty}$ be a sequence associated with a symmetrical lattice, where a_n's take value among 2, 4, 6, 8 or 10 only. Suppose*

$$\Delta_n^- \leq 8\sqrt{3} = L \quad (say) \tag{3.11}$$

$$\Delta_n^+ \leq 8\sqrt{3}/7 = L/k \quad (say) \text{ for all } n. \tag{3.12}$$

Then the sequence satisfies the following:

$$\text{if } a_r \geq 4 \text{ for some } r, \text{ then } a_{r+1} = a_{r-1} = 2. \tag{3.13}$$

$$\text{if } a_r = a_{r-1} = 2 \text{ for some } r, \text{ then } a_{r+1} \geq 6 \text{ and } a_{r-2} \geq 6. \tag{3.14}$$

Proof. Let $a_r \geq 4$. Suppose if possible $a_{r+1} \geq 4$, then $\theta_r > 3$, $\phi_r > 3$, Δ_r^+ being an increasing function of θ_r and ϕ_r gives $\Delta_r^+ \geq 2 > L/k$; a contradiction to (3.12). Hence $a_{r+1} = 2$. Similarly by symmetry $a_{r-1} = 2$.

If $a_r = a_{r-1} = 2$ for some r, then by (3.1) and (3.13),

$$\theta_r = [a_r, a_{r-1}, \ldots] \leq [2, \overset{\times}{2}, \overset{\times}{10}] = \sqrt{20} - 3.$$

(The crosses denote the infinite repetition)
(3.8) gives

$$\phi_r \geq \frac{L(\theta_r - 1) - 4}{L(\theta_r - 1) - 4\theta_r} > 3.88$$

which implies that $a_{r+1} \geq 4$. But if $a_{r+1} = 4$, then using (3.1) and (3.13)

$$\phi_r = [a_{r+1}, a_{r+2}, \ldots] \leq [4, \overset{\times}{2}, \overset{\times}{10}] = \sqrt{20} - 1.$$

$$\theta_r = [a_r, a_{r-1}, \ldots] \leq [2, \overset{\times}{2}, \overset{\times}{10}] = \sqrt{20} - 3.$$

then Δ_r^- being a decreasing function of θ_r and ϕ_r we have

$$\Delta_r^- \geq \frac{4((\sqrt{20} - 1)(\sqrt{20} - 3) - 1)}{(\sqrt{20} - 2)(\sqrt{20} - 4)} > 14.09 > L$$

a contradiction to (3.11). Therefore we must have $a_{r+1} \geq 6$. Similarly by symmetry $a_{r-2} \geq 6$. $\qquad\square$

Lemma 15 *Let $\mathcal{L}$ be the inhomogeneous lattice of determinant $\Delta(\mathcal{L}) = 4\Delta$, associated with $4\varphi(y, z)$ where (y, z) run over all numbers congruent to (y_0, z_0) (mod 1) and $\varphi(y, z)$ is as given in Theorem A. Then either $\mathcal{L}$ is not admissible for the region R_m, $m = 4d - 1$ or $\mathcal{L}$ corresponds to quadratic forms φ_4 and φ_5.*

Proof. Since $1 < d \leq 2$, $3 < m = 4d - 1 \leq 7$.

Case I: $\mathcal{L}$ is non symmetrical. One can easily check here that for $d \leq 2$,

$$\Delta(\mathcal{L}) = 4\Delta = 4(3d^3/2)^{1/2} < (1.8251)\,(4d) = (1.8251)(m + 1).$$

Therefore by Lemma 13, $\mathcal{L}$ is not admissible for R_m.

Case II: $\mathcal{L}$ is a symmetrical lattice. Let $\mathcal{L}$ be admissible for R_m, then by Lemma 11,

$$\max(m\Delta_n^+, \Delta_n^-) \le \Delta(\mathcal{L}) = 4\Delta \quad \text{for all} \quad n.$$

Let

$$\Delta_n^+ \le \frac{4\Delta}{m} = \frac{4(3d^3/2)^{1/2}}{4d-1} \le 8\sqrt{3}/7 = L/k$$

and

$$\Delta_n^- \le 4\Delta = 4(3d^3/2)^{1/2} \le 8\sqrt{3} = L.$$

Then hypothesis of Lemma 12 is satisfied with $k = 7$. Working up to 4 places of decimals we get from (3.9)

$$|\theta_n - 5.598| < 4.1726.$$

This gives $1.4 < \theta_n < 9.78$.

Since $\theta_n < a_n < \theta_n + 1$, we must have $a_n = 2, 4, 6, 8$ or 10. Now by Lemma 14, the sequence $\{a_n\}$ satisfies (3.13) or (3.14). The quadratic form $\varphi(y, z)$ associated with the symmetric lattice $\mathcal{L}$ is given by (from Lemma 10)

$$\varphi(y, z) = \frac{\Delta}{\theta_n \phi_n - 1} \left[\theta_n y^2 + \phi_n z^2 + (\theta_n \phi_n + 1)yz \right].$$

Subcase (I):   If in the sequence $\{a_n\}$, no two 2's are consecutive, then by (3.13) it must be of the form

$$\ldots 2, a_{-2}, 2, a_0, 2, a_2 \ldots \text{ where } a_{2r} \ge 4 \text{ for all } r.$$

If $a_{2r} \ge 6$ for some r, then by (3.1)

$$\theta_{2r} \ge [6, \overset{\times}{2}, \overset{\times}{4}] = 4 + \sqrt{2}$$

$$\phi_{2r} \ge [\overset{\times}{2}, \overset{\times}{4}] = (2 + \sqrt{2})/2$$

then

$$0 < \varphi(0, 1) = \frac{\Delta \cdot \phi_{2r}}{\theta_{2r}\phi_{2r} - 1} \le \frac{\sqrt{12} \cdot (2 + \sqrt{2})/2}{(4 + \sqrt{2})(2 + \sqrt{2})/2 - 1}$$

$$= \frac{\sqrt{12}(2 + \sqrt{2})}{8 + 6\sqrt{2}} < \frac{3}{4}$$

as

$$\Delta = \left(\frac{3}{2}d^3\right)^{1/2} \leq \sqrt{12} \quad \text{for} \quad d \leq 2.$$

This contradicts (2.10).

Therefore $a_{2r} = 4$ for all r. Then the sequence $\{a_n\}$ is $\{\overset{\times}{2}, \overset{\times}{4}\}$ and the quadratic form associated to it is

$$\rho(y^2 + 2z^2 + 4yz) \sim \rho(y^2 - 2z^2) = \rho\varphi_4;$$

$$(y_0, z_0) \equiv \left(\frac{1}{2}, \frac{1}{2}\right) \quad (\text{mod } 1).$$

Subcase (II):   Let two 2's be consecutive in the sequence say $a_{r-1} = a_r = 2$ for some r. If $a_{r+1} \geq 8$, then $\phi_r > 7$ and $\theta_r > 1$ already, so

$$0 < \varphi(1, 0) = \frac{\Delta\theta_r}{\theta_r\phi_r - 1} \leq \frac{\sqrt{12}}{6} < \frac{3}{4},$$

a contradiction to (2.10). Therefore we must have $a_{r+1} = 6$, by (3.14). Similarly by symmetry $a_{r-2} = 6$. Now $a_{r+2} = 2$ by (3.13); but if $a_{r+3} \neq 2$, we will have

$$\phi_r = [a_{r+1}, a_{r+2}, a_{r+3}, \ldots] \geq [6, 2, 4, \overset{\times}{2}, 2, \overset{\times}{6}]$$

$$= \frac{221 + 38\sqrt{48}}{41 + 7\sqrt{48}} > 5.4,$$

$$\theta_r = [a_r, a_{r-1}, a_{r-2}, \ldots] \geq [\overset{\times}{2}, 2, \overset{\times}{6}] = \frac{9 + \sqrt{48}}{11} > 1.4,$$

and then

$$0 < \varphi(1, 0) = \frac{\Delta\theta_r}{\theta_r\phi_r - 1} \leq \frac{\sqrt{12} \cdot (1.4)}{(5.4)(1.4) - 1} < 3/4$$

a contradiction to (2.10). Therefore we must have $a_{r+3} = 2$. But then two consecutive 2's must be followed by a 6, and repeating the argument we must have $\{a_n\} = (\overset{\times}{2}, \overset{\times}{2}, 6)$. The quadratic form associated to it is

$$\varphi(y, z) = \rho(3y^2 + 11z^2 + 18yz) = \rho\varphi_5;$$

$$(y_0, z_0) \equiv (1/2, 1/2) \pmod 1;$$

where $\rho^2 = |D|/48 = d^3/128$. If $d < 2$, that is if $\rho < 1/4$, we have

$$0 < \varphi(1, 0) = 3\rho < 3/4.$$

This gives a contradiction to (2.10). Therefore for $\rho\varphi_5$ we must have $d = 2$. This proves Lemma 15 and hence Lemma 8. □

4. The Critical Forms

Lemma 16 *If $\varphi = \rho\varphi_5 = \rho(3y^2 + 11z^2 + 18yz)$, $(y_0, z_0) \equiv \left(\frac{1}{2}, \frac{1}{2}\right) \pmod 1$, $d = 2$, then (2.9) is solvable unless $Q \sim Q_3$ and $(x_0, y_0, z_0) \sim \left(\frac{1}{2}, 0, 0\right) \pmod 1$.*

Proof. Here $\varphi \sim \rho(3y^2 - 16z^2)$, $(y_0, z_0) \sim \left(0, \frac{1}{2}\right) \pmod 1$. Since $\Delta = \sqrt{12}$, we get $\rho = 1/4$. Let without loss of generality

$$Q(x, y, z) = (x + hy + gz)^2 + \frac{1}{4}(3y^2 - 16z^2).$$

Take $y = 0$, $z = 1/2$ and choose $x \equiv x_0 \pmod 1$ such that $1 \le |x + g/2| \le 3/2$, so that

$$0 = 1 - 1 \le Q(x, y, z) \le 9/4 - 1 < 2.$$

Therefore (2.9) is solvable unless

$$x_0 + g/2 \equiv 0 \pmod 1. \tag{4.1}$$

Similarly taking $y = 0$, $z = -1/2$, (2.9) is solvable unless

$$x_0 - g/2 \equiv 0 \pmod 1. \tag{4.2}$$

From (4.1), (4.2) and (2.6) we get

$$g = 0 \quad \text{and} \quad x_0 \equiv 0 \pmod 1. \tag{4.3}$$

Therefore, if (2.9) is not solvable, we have

$$Q(x, y, z) = (x + hy)^2 + \frac{1}{4}(3y^2 - 16z^2).$$

Take $x = 1$, $y = 1$, $z = 1/2$ and using (2.6) we get

$$0 = \frac{1}{4} - \frac{1}{4} < Q(x, y, z) = (1 + h)^2 - \frac{1}{4} \le \frac{9}{4} - \frac{1}{4} = 2$$

So (2.9) is solvable unless $h = 1/2$. Therefore

$$Q(x, y, z) = \left(x + \frac{1}{2}y\right)^2 + \frac{1}{4}(3y^2 - 16z^2)$$
$$= x^2 + xy + y^2 - 4z^2 \underset{\sim}{} 2xy + y^2 + yz + 3z^2$$

by means of the unimodular transformation

$$x \longrightarrow -2x + 3z, \quad y \longrightarrow 2x + y - z, \quad z \longrightarrow -x + z.$$

Also then $(x_0, y_0, z_0) \sim (1/2, 0, 0)$ (mod 1). □

Lemma 17 *If* $\varphi = \rho\varphi_4 = \rho(y^2 - 2z^2)$, $(y_0, z_0) \equiv \left(\frac{1}{2}, \frac{1}{2}\right)$ (mod 1), $d \leq 2$ *then* (2.9) *is solvable unless*

$$Q \sim 2Q_2, (x_0, y_0, z_0) \sim \left(\frac{1}{2}, 0, 0\right) \quad \text{(mod 1)}.$$

Proof. If $\rho < 1$, take $y = 1/2$, $z = 1/2$, so that

$$-\frac{1}{4} < \varphi(y, z) = \rho\left(-\frac{1}{4}\right) < 0 < d - \frac{1}{4}.$$

Therefore (2.11) and hence (2.9) has a solution. Let now $\rho = \Delta/\sqrt{8} \geq 1$. This gives $d^3 \geq 16/3$. If (2.9) has no solution, we must have for all integers p, q, r

$$\left.\begin{array}{ll} \text{either} & Q(p + x_0, q + 1/2, r + 1/2) \geq d \\ \text{or} & Q(p + x_0, q + 1/2, r + 1/2) \leq 0. \end{array}\right\} \tag{4.4}$$

Take $q = r = 0$ and choose an integer p such that

$$1/2 \leq \alpha = |p + x_0 + h/2 + g/2| \leq 1. \tag{4.5}$$

Then from (4.4) we must have

$$\text{either} \quad \alpha^2 \geq d + \rho/4 \geq (16/3)^{1/3} + 1/4 > 1.9971$$
$$\text{or} \quad \alpha^2 \leq \rho/4 = \Delta/4\sqrt{8} \leq \sqrt{12}/4\sqrt{8} < 0.3062$$

i.e. either $\alpha > 1.4131$ or $\alpha < 0.5534$. From (4.5) we must have

$$0.446 < p + x_0 + h/2 + g/2 < 0.5534 \quad \text{(mod 1)}$$

i.e.

$$1/2 - 0.0534 < x_0 + h/2 + g/2 < 1/2 + 0.534 \quad (\text{mod } 1). \qquad (4.6)$$

Similarly taking $(q, r) = (-1, 0)$ and $(0, -1)$, we must have

$$1/2 - 0.0534 < x_0 - h/2 + g/2 < 1/2 + 0.0534 \quad (\text{mod } 1). \qquad (4.7)$$

$$1/2 - 0.0534 < x_0 + h/2 - g/2 < 1/2 + 0.0534 \quad (\text{mod } 1). \qquad (4.8)$$

Subtracting (4.7) and (4.8) respectively from (4.6) we get

$$-0.1068 < h, \quad g < 0.1068 \quad (\text{mod } 1). \qquad (4.9)$$

Since from (2.6), $|h| \leq 1/2$, $|g| \leq 1/2$, we have

$$P = (h, g) \in \mathcal{R}, \text{ where } \mathcal{R} \text{ is the region given by}$$

$$\mathcal{R} = \left\{ (x, y) \in \mathbb{R}^2 : -0.1068 < x, y < 0.1068 \right\}.$$

Let $A = (0, 0)$, $U = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$ be an automorph of φ_4. Then

$$U(\mathcal{R}) \subseteq \left\{ \begin{array}{ll} (x, y) \in \mathbb{R}^2 : & -0.75 < x < 0.75 \\ & -0.56 < y < 0.56 \end{array} \right\}.$$

Clearly $U(\mathcal{R}) \cap \mathcal{R} + B = \emptyset$ for all $B \in \mathbb{Z}^2$, $B \neq A$.

Now for all integers n positive or negative, the unimodular transformation $\begin{pmatrix} 1 & 0 \\ 0 & U^n \end{pmatrix}$ transforms Q into

$$Q(x, y, z) = (x + h_n y + g_n z)^2 + \rho(y^2 - 2z^2).$$

The above argument shows that if (2.9) has no solution then $U^n(P) = (h_n, g_n)$ must also satisfy (4.9) and hence must be congruent to a point of $\mathcal{R}$ (mod 1). Therefore by Lemma 4, we must have $U(P) - A = P$, which gives $h = 0$, $g = 0$, since $U(P) = (3h + 4g, 2h + 3g)$. Thus $Q(x, y, z) = x^2 + \rho(y^2 - 2z^2)$ and

$$\frac{1}{2} - 0.0534 < x_0 < \frac{1}{2} + 0.0534 \quad (\text{mod } 1). \qquad (4.10)$$

If $1 < \rho < 9/8$, then

$$0 < Q(3, 0, 2) = 9 - 8\rho < 1.$$

This contradicts (2.7).

If $9/8 \le \rho = \Delta\sqrt{8} \le \sqrt{3/2}$, take $y = 1/2$, $z = 3/2$ and choose $x \equiv x_0$ (mod 1) such that $5/2 - 0.0534 < x < 5/2 + 0.0534$ so that

$$0 < (2.4466)^2 - \frac{17}{4}\sqrt{\frac{3}{2}} < Q$$

$$= x^2 - \frac{17}{4}\rho \le (2.5534)^2 - \frac{17}{4} \cdot \frac{9}{8} < d.$$

Thus if (2.9) has no solution, we must have $\rho = 1$.

Now if $x_0 \not\equiv 1/2$ (mod 1), choose x such that $\frac{1}{2} < |x| \le 1$, take $y = 1/2$, $z = 1/2$, so that $0 < Q = x^2 - 1/4 \le 1 - 1/4 < d$. Thus for (2.9) to have no solution we must have

$$Q = x^2 + y^2 - 2z^2, \quad (x_0, y_0, z_0) \equiv \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \quad (\text{mod } 1)$$

$$\sim 2Q_2 = 2\left[\left(x + \frac{1}{2}y\right)y + z^2\right]$$

by means of transformation

$$x \longrightarrow x + 2z, \quad y \longrightarrow x + y, \quad z \longrightarrow x + z.$$

Also then $(x_0, y_0, z_0) \sim \left(\frac{1}{2}, 0, 0\right)$ (mod 1). This completes the proof of the theorem. $\square$

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