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Cramér-von Mises-Watson type statistics for testing normality with censored data

Toshio HASHIMOTO

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Abstract. The modified form of the Cramér-von Mises-Watson type statistics is employed for the problem of testing normality with type I censored samples in the presence of an unknown location parameter θ. Asymptotic distribution theory is developed for such statistics when θ is estimated by its maximum likelihood estimator $\hat{\theta}$. Small sample properties of the statistics are also considered.

Key words: Cramér-von Mises statistic. Watson statistic. Goodness of fit test. Type I censored sample.

1. Introduction

Let $x_1, x_2, \ldots, x_n$ be independent observations on random variable X with a continuous distribution function $F(x)$. Suppose that we have an ordered set of observations $x_1 \leqq x_2 \leqq \ldots \leqq x_n$ and that we wish to test the simple null hypothesis

$$H_0 : F(x) = F_0(x) ,$$

where $F_0(x)$ is a completely specified distribution function. Then a test can be based on the Cramér-von Mises goodness of fit test statistic

$$W_n^2 = n \int_{-\infty}^{\infty} \{F_n(x) - F_0(x)\}^2 dF_0(x)$$

where $F_n(x)$ is the empirical distribution function of the sample, that is, $F_n(x)$ is the proportion of $x_1, x_2, \ldots, x_n$ not greater than x, and $F_0(x)$ is the cumulative distribution function assumed under H_0.

We also consider the Watson statistic U_n^2, introduced by Watson [29], as a goodness of fit statistic on the circle but which can be used also on the line,

$$U_n^2 = n \int_{-\infty}^{\infty} [F_n(x) - F_0(x) - \int_{-\infty}^{\infty} \{F_n(y) - F_0(y)\} dF_0(y)]^2 dF_0(x) .$$

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The distribution of W_n^2 is distribution-free, because it can be seen by making the probability transformation $t_i = F_0(x_i)$ $(i = 1, 2, ..., n)$ and defining $F_n(t)$ to be the proportion of observations t_i's not greater than t. Under the transformation above, the null hypothesis H_0 reduces to the hypothesis that $t_1 \leqq t_2 \leqq ... \leqq t_n$ is an ordered sample from a uniform population on [0,1]. Then the test statistics take the forms

$$W_n^2 = n \int_0^1 \{F_n(t) - t\}^2 dt,$$
$$U_n^2 = n \int_0^1 [F_n(t) - t - \int_0^1 \{F_n(s) - s\} ds]^2 dt \ .$$

Pettitt and Stephens [21] modified the statistics W_n^2, U_n^2 so that they could be used to the goodness of fit of a censored sample of N observations $0 < t_1 < ... < t_N \leqq p < 1$, when the distribution function $F_0(x)$ is known completely. They considered the asymptotic distribution of the modified Cramér-von Mises statistic

$${}_pW_n^2 = n \int_0^p \{F_n(t) - t\}^2 dt \quad (0 < p < 1)$$

and the modified Watson statistic

$${}_pU_n^2 = n \int_0^p \{F_n(t) - t - \frac{1}{p} \int_0^p [F_n(s) - s] ds\}^2 dt \ ,$$

and obtained the percentiles of the asymptotic distributions of ${}_pW_n^2$ and ${}_pU_n^2$ for various values of p.

In this paper we consider the corresponding composite null hypothesis $H_0 : F(x) = F_0(x;\theta)$, where $F_0(x;\theta)$ is a known distribution function but θ is an unknown location parameter which must be estimated from a censored sample.

If we denote by $\hat{\theta}$ the estimator for θ, we may consider $\hat{t}_i = F_0(x_i;\hat{\theta})$ and $\hat{F}_n(t)$, the sample disribution of the $\hat{t}_i$'s. We then modify the statistics to give

$${}_p\hat{W}_n^2 = n \int_0^p \{\hat{F}_n(t) - t\}^2 dt \tag{1.1}$$

and

$${}_p\hat{U}_n^2 = n \int_0^p \{\hat{F}_n(t) - t - \frac{1}{p} \int_0^p [\hat{F}_n(s) - s] ds\}^2 dt \tag{1.2}$$

as a measure of the discrepancy or 'distance' between $F_n(t)$ and the uniform [0,1] distribution. In what follows we employ ${}_p\hat{W}_n^2$ and ${}_p\hat{U}_n^2$ for testing normality when the location parameter θ in

$$H_0 : F_0(x) = \int_0^x \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(x-\theta)^2}{2}\right\} dx$$

is unspecified. We assume the sample is time-truncated, where the truncation point is $p = F_0(T;\theta)$.

In the next section we consider the maximum likelihood estimator (abbreviated to henceforth as MLE) $\hat{\theta}$ and investigate the asymptotic properties in the censored normal sample. To find the asymptotic distribution of the statistics, we need to consider the limiting distribution of the empirical process $\hat{y}_n(t) = \sqrt{n}\{\hat{F}_n(t) - t\}$, and we consider this situation in section 3. In section 4, asymptotic distributions of ${}_p\hat{W}_n^2$ and ${}_p\hat{U}_n^2$ for testing normality are studied using the techniques similar to those employed by Durbin, Knott and Taylor [10], Pettitt and Stephens [21]. Small sample applications are discussed in final section.

2. MLE and its asymptotic properties in the censored normal sample

One of the standard sampling procedures used in life testing experiment is the time-truncated sampling scheme, i.e. type I censoring, where the failure times $0 \leqq x_1 \leqq x_2 \leqq \ldots \leqq x_n \leqq T$ of the items which fail prior to a prefixed time T are recorded.

Let $[0,T]$ be duration of experiment and n be the number of items being put on test. It is usually assumed that each items on test have an exponential distribution with probability density function

$$f(x;\theta) = \frac{1}{\theta}\exp\left(-\frac{x}{\theta}\right) , \quad (x \geq 0,\ \theta > 0).$$

The problem of estimating the scale parameter θ in such a time truncated sampling has been investigated by many authors such as Epstein - Sobel [11], Bartholomew [2], [3] and Yang -Sirvanci [30]. Especially based on the null hypothesis

$$H_0 : F(x;\theta) = 1 - \exp\left(-\frac{x}{\theta}\right) \quad (x \geq 0,\ \theta > 0) ,$$

Pettitt [20] and Sirvanci-Levent [24] have obtained the percentiles of the asymptotic distribution of the Cramér -von Mises statistic ${}_pW_n^2$ in connection with finding the covariance function of the limiting Gaussian process $\hat{y}(t)$ in the censored sample.

Using the failure times and the number of failure N, observed in $[0,T]$, the maximum likelihood estimator $\hat{\theta}$ of θ under the hypothesis of exponentiality can be formed as explicitely

$$\hat{\theta} = \frac{1}{N}\left\{\sum_{i=1}^{N} x_i + (n-N)T\right\} .$$

The object of this section is to give a simple derivation of several asymptotic properties of $\hat{\theta}$, as $n \to \infty$, under the assumption that each item on test has a normal distribution $N(\theta,1)$, therefore the null hypothesis which is to be considered here can be stated as

$$H_0 : F_0(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(x-\theta)^2}{2}\right\} dx .$$

Such a sampling procedures as defined above is of marked interest since it is frequently assumed in life testing that the logarithum of time to death is normally distributed. Note that it is not essential to suppose the known varianceis 1, because if it were σ^2, values of x/σ would be tested to come from a normal disribution with variance 1.

We will consider situations in which $n-N$ greatest observations are censored (i.e. not recorded) leaving only $x_1, x_2, ..., x_N$. Best unbiased estimators, based on these order statistic, are particular useful in these circumstances, as the MLE of θ is much more difficult to calculate than they are for complete samples. We will first, however, discuss maximum likelihood estimator, and possible approximations thereto.

The joint probability density functions of $x_1, x_2, ..., x_N$ is

$$p(x_1, x_2, ..., x_N) = \frac{n!}{(n-N)!}\{1-\Phi(t-\theta)\}^{n-N} \prod_{i=1}^{N} \phi(x_i - \theta)$$

where

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

and

$$\Phi(x) = \int_{-\infty}^{x} \phi(y)dy \ .$$

The maximum likelihood estimator $\hat{\theta}$ satisfies the following equation

$$\frac{\partial \log L(\theta)}{\partial \theta} = (n-N)\frac{-\frac{\partial}{\partial \theta}\Phi(T-\theta)}{1-\Phi(T-\theta)} + \sum_{i=1}^{N}(x_i - \theta) = 0 \ .$$

Using the notation $T-\theta = \xi$ and noting the

$$\frac{\partial \Phi(T-\theta)}{\partial \theta} = -\phi(\xi) \ ,$$

we have the equation

$$(n-N)\frac{\phi(\xi)}{1-\Phi(\xi)} + \sum_{i=1}^{N}(x_i - \theta) = 0 \ .$$

Bearing in mind the fact in the equation above the censored observation are replaced by the expected value of the appropriate tail of the normal distribution truncated at T, we may obtain approximate solution of the likelihood equation by replacing $T-\hat{\theta} = \hat{\xi}$ by U_α with $\alpha = N/(n+1)$. We have therefore the approximate equation

$$\hat{\theta} = \frac{1}{N}\left\{\sum_{i=1}^{N} x_i + (n-N)\mu(\alpha)\right\} = \bar{x} + \frac{n-N}{N}\mu(\alpha) \tag{2.1}$$

where

$$\mu(\alpha) = \frac{\phi(U_\alpha)}{1-\Phi(U_\alpha)}$$

are introduced as the notation for the moment of the tails.

According to Johnson [16], we used here the system defined by $\Phi(U_\alpha) = \alpha$ with $\alpha = N/(n+1)$ so that U_α is the lower $100\alpha\%$ point of the standard normal distribution and the values of $\mu(\alpha)$ can be obtained from the table of Owen [18] noting the formula

$$\frac{1-\Phi(U_\alpha)}{\phi(U_\alpha)} = \frac{1}{\mu(\alpha)} \ .$$

In the following, we discuss the asymptotic unbiasedness and consistency of MLE $\hat{\theta}$ of the mean θ of a normal distribution assuming that at least one failure was observed in the interval $(-\infty, T]$.

First we remark that the total number of failure N is a random variable and its probability law is a binomial $B(n,p)$, where $p = \Phi(T-\theta)$.

The conditional joint density of $x_1, x_2, ..., x_N$ given $N = m$ is

$$g(x_1, x_2, ..., x_m) = \frac{m!}{\sqrt{2\pi}^m} \exp\left\{-\frac{1}{2}\sum_{i=1}^{m}(x_i-\theta)^2\right\} \frac{1}{\{\Phi(T-\theta)\}^m} \quad (2.2)$$

where $x_1 < x_2 < ... < x_m \leqq T$ and $m = 1, 2, ..., n$.

Equation (2.2) shows that $x_1, x_2, ..., x_m$ have same joint probability density as the ordered statistics of a random sample of m observations $u_1, u_2, ..., u_m$ from the density

$$f(t) = \frac{1}{\sqrt{2\pi}\Phi(T-\theta)} \exp\left\{-\frac{(t-\theta)^2}{2}\right\}, \; t \leq T \; . \quad (2.3)$$

We have directly

$$\begin{aligned} E(u_1) &= \int_{-\infty}^{T} t f(t) dt = \theta - \frac{\phi(T-\theta)}{\Phi(T-\theta)} \\ &= \theta - \frac{\phi(T-\theta)}{p} \; . \end{aligned}$$

In the following, the notation E_c will be used to denote conditional expectation conditioned on $[N > 0]$. The condional mean for the maximum likelihood estimator $\hat{\theta}$ in (2.1) is

$$\begin{aligned} E_c(\hat{\theta}) &= E\left\{E(\hat{\theta}/N = m)/N > 0\right\} \\ &= E\left\{E(u_1) + \frac{n-m}{m}\mu(\alpha)/N > 0\right\} \\ &= \theta - \frac{\phi(T-\theta)}{p} - n\mu(\alpha)E_c\left(\frac{1}{N}\right) - \mu(\alpha). \end{aligned} \quad (2.4)$$

In order to evaluate the con0ditional mean, it is necessary to calculate the conditional expectation $E_c(1/N)$, but it is already obtained by Yang-Sirvanci [30] in their appendix as follows

$$E_c\left(\frac{1}{N}\right) = \frac{1}{1-q^n} \sum_{k=1}^{n} C_k^n \frac{p^k q^{n-k}}{k}$$

$$= \frac{1}{1-q^n}\sum_{k=1}^{n}\frac{q^{n-k}}{k} - \frac{q^n}{1-q^n}\sum_{k=1}^{n}\frac{1}{k} \tag{2.5}$$

where $q = 1 - p$, and $p = \Phi(x - \theta)$. The full expression for $E_c(\hat{\theta})$ can be obtained by substiting (2.5) into (2.4), and it follows from (2.4) that

$$\lim_{n\to\infty} E_c(\hat{\theta}) = \theta - \frac{\phi(T-\theta)}{p} - \frac{\phi(T-\theta)}{1-p} - \frac{\phi(T-\theta)}{1-p}\lim_{n\to\infty} E_c\left(\frac{n}{N}\right) .$$

According to the results

$$\lim_{n\to\infty} E_c\left(\frac{n}{N}\right) = \frac{1}{p}$$

proved by Yang-Sirvantci [30],

$$\lim_{n\to\infty} E_c(\hat{\theta}) = \theta - \frac{\phi(T-\theta)}{p} - \frac{\phi(T-\theta)}{1-p} + \frac{\phi(T-\theta)}{1-p}\frac{1}{p} = \theta .$$

This shows that MLE $\hat{\theta}$ is an asymptotically unbiased estimator for θ. Next we shall compute the second moment of u_1.

$$\begin{aligned} E(u_1^2) &= \int_{-\infty}^{r} \frac{t^2}{\sqrt{2\pi}\Phi(T-\theta)} \exp\left\{-\frac{(t-\theta)^2}{2}\right\}dt \\ &= \frac{1}{\Phi(T-\theta)}\left[\int_{-\infty}^{T} \frac{(t-\theta)^2}{\sqrt{2\pi}} \exp\left\{-\frac{(t-\theta)^2}{2}\right\}dt \right. \\ &\quad + 2\theta \int_{-\infty}^{T} \frac{t}{\sqrt{2\pi}} \exp\left\{-\frac{(t-\theta)^2}{2}\right\}dt \\ &\quad \left. - \theta^2 \int_{-\infty}^{T} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(t-\theta)^2}{2}\right\}dt\right] \\ &= \frac{1}{\Phi(T-\theta)}[I_1 + I_2 + I_3] , \end{aligned}$$

say.

Then,

$$I_1 = -(T-\theta)\phi(T-\theta) + \Phi(T-\theta),$$
$$I_2 = 2\theta\{-\phi(T-\theta) + \theta\Phi(T-\theta)\} ,$$

and

$$I_3 = \theta^2\Phi(T-\theta) ,$$

and the second moment is thus given by

$$E(u_1^2) = \frac{1}{\Phi(T-\theta)}\{-T\phi(T-\theta) - \theta\phi(T-\theta) + \theta^2\Phi(T-\theta) + \Phi(T-\theta)\} . \tag{2.6}$$

To evaluate $Var_c(\hat{\theta})$, we make use of the standard decomposition

$$Var_c(\hat{\theta}) = E_c[Var(\hat{\theta}/N = m)] + Var[E(\hat{\theta}/N = m)] . \tag{2.7}$$

We also need to compute

$$\begin{aligned} E(\hat{\theta}^2/N = m) &= \frac{1}{m^2}E\Big\{\sum_{i=1}^{m} x_i^2 + 2\sum_{1\le i<j\le m} x_i x_j \\ &\quad + 2(n-m)\mu(\alpha)\sum_{i=1}^{m} x_i + (n-m)^2\mu^2(\alpha)\Big\} \\ &= \frac{1}{m}E(u_1^2) + \frac{m-1}{m}E^2(u_1) + \frac{(n-m)^2}{m^2}\mu^2(\alpha) \\ &\quad + \frac{2(n-m)}{m}\mu(\alpha)E(u_1) . \end{aligned} \tag{2.8}$$

Substituting $E(u_1) = \theta - \phi(T-\theta)/p$ and (2.6) into (2.8), we then have

$$\begin{aligned} Var_c(\hat{\theta}/N = m) &= \frac{1}{m}\{E(u_1^2) - E^2(u_1)\} \\ &= \frac{1}{m}\Big\{1 - \frac{T}{p}\phi(T-\theta) + \frac{\theta}{p}\phi(T-\theta) - \frac{\phi^2(T-\theta)}{p^2}\Big\} , \end{aligned}$$

and

$$\begin{aligned} &E_c\{Var(\hat{\theta}/N = m)\} \\ &\quad = E_c\left(\frac{1}{N}\right)\Big\{1 - \frac{T}{p}\phi(T-\theta) + \frac{\theta}{p}\phi(T-\theta) - \frac{\phi^2(T-\theta)}{p^2}\Big\} . \end{aligned} \tag{2.9}$$

Now noting the result of $E(\hat{\theta}|N = m)$, it follows that

$$\begin{aligned} Var_c\{E(\hat{\theta}/N = m)\} &= Var_c\Big\{\frac{\phi(T-\theta)}{1-p}\frac{n}{m} + \theta - \frac{\phi(T-\theta)}{p} - \frac{\phi(T-\theta)}{1-p}\Big\} \\ &= \frac{n^2\phi^2(T-\theta)}{(1-p)^2}Var_c\left(\frac{1}{N}\right) . \end{aligned} \tag{2.10}$$

The variance is thus obtained by substituting (2.9) and (2.10) into (2.7)

$$\begin{aligned} Var_c(\hat{\theta}) &= E_c\left(\frac{1}{N}\right)\left\{1 - \frac{T\phi(T-\theta)}{p} + \frac{\theta\phi(T-\theta)}{p} - \frac{\phi^2(T-\theta)}{p^2}\right\} \\ &\quad + \frac{n^2\phi^2(T-\theta)}{(1-p)^2} Var_c\left(\frac{1}{N}\right) . \end{aligned} \tag{2.11}$$

Note that the result above shows the necessity of evaluation of conditional variance $Var_c(1/N)$ adding to the conditional mean $E_c(1/N)$ given by (2.5). As to the derivation of finding the second moment $E_c(1/N^2)$, we may mention the following formula as in a similar manner employed by Yang-Sirvanti [30]. The conditional second moment $E_c(1/N^2)$ is given by

$$E_c\left(\frac{1}{N^2}\right) = \frac{1}{1-q^m}\sum_{k=1}^{m}\frac{1}{k}\left(\sum_{l=1}^{k}\frac{q^{m-1}}{l}\right) - \frac{q^m}{1-q^m}\sum_{k=1}^{m}\frac{1}{k}\left(\sum_{l=1}^{k}\frac{1}{l}\right),$$

where we used the elementary relations

$$\sum_{k=1}^{m} C_k^m \frac{\rho^k}{k} = \sum_{k=1}^{m}\frac{(1+\rho)^k - 1}{k}$$

and

$$\sum_{k=1}^{m} C_k^m \frac{\rho^k}{k^2} = \sum_{k=1}^{m}\frac{1}{k}\left\{\sum_{l=1}^{k}\frac{(1+\rho)^l - 1}{l}\right\} .$$

The consistency of $\hat{\theta}$ will follow from verifying the sufficient condition, $Var_c(\hat{\theta}) \to 0$, as $n \to \infty$. It suffies to show for this purpose that $E_c(1/N) \to 0$, and $n^2\,Var_c(1/N^2) \to 0$ according to (2.11) but it is clear from the theorem of Yang-Sirvanci [30], observing that

$$n^2\,Var_c\left(\frac{1}{N}\right) = E_c\left(\frac{n^2}{N^2}\right) - E_c^2\left(\frac{n}{N}\right) \to \frac{1}{p^2} - \frac{1}{p^2} = 0 .$$

Hence we have an important result.

Theorem 1 *Let $\hat{\theta}$ be the MLE for the censored normal sample $N(\theta, 1)$. Then $\hat{\theta}$ is an asymptotically unbiased estimator for θ and is a consistent estimator for θ.*

3. The limiting distribution of the empirical process

Pettitt and Stephens [21] modified the W_n^2, U_n^2 so that they could be used to goodness of fit test of a censored sample of N observations $0 < t_1 < t_2 < \dots < t_N \leqq p < 1$, when the distribution $F_0(x)$ is known completely. They considered the asymptotic distribution of the modified Cramér-von Mises statistic

$$ {}_pW_n^2 = n\int_0^p \{F_n(t) - t\}^2 dt \quad (0 < p < 1) $$

and the modified Watson's statistic

$$ {}_pU_n^2 = n\int_0^p \left[F_n(t) - t - \frac{1}{p}\int_0^p \{F_n(s) - s\}ds\right]^2 dt\ , $$

and obtained percentiles of the asymptotic distributions of ${}_pW_n^2$ and ${}_pU_n^2$ for various values of p.

In this section we consider corresponding composite null hypothesis $H_0 : F(x) = F_0(x;\theta)$, where $F(x;\theta)$ is a known distribution but θ is a unknown location parameter which must be estimated from the censored sample. If we denote $\hat{\theta}$ the estimator for θ, we may consider $\hat{t}_i = F_0(x_i;\hat{\theta})$ and $\hat{F}_n(t)$, the empirical distribution of the $\hat{t}_i$'s. We then modify the statistics to give

$$ {}_p\hat{W}_n^2 = n\int_0^p \{\hat{F}_n(t) - t\}^2 dt \quad (0 < p < 1), \tag{3.1} $$

$$ {}_p\hat{U}_n^2 = n\int_0^p \left\{\hat{F}_n(t) - t - \frac{1}{p}\int_0^p [\hat{F}_n(s) - s]ds\right\}^2 dt\ , \tag{3.2} $$

as a measure of the discrepancy or "distance" between $\hat{F}_n(t)$ and the uniform (0,1) distribution $U(0,1)$.

In what follows we employ ${}_p\hat{W}_n^2$ and ${}_p\hat{U}_n^2$ for testing normality when the location parameter θ in

$$ H_0 : F_0(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(x-\theta)^2}{2}\right\} dx $$

is unspecified. We assume that the sample is time-truncated, where the truncation point is $p = F(T;\theta)$.

To find the asymptotic distributions of the statistics ${}_p\hat{W}_n^2$ and ${}_p\hat{U}_n^2$, we need to consider the limiting distribution of the empirical process $\hat{y}_n(t) = \sqrt{n}\{\hat{F}_n(t) - t\}$. We first investigate the convergence of $\hat{y}_n(t)$.

In his excellent paper, Durbin [8] proved that the empirical process $\hat{y}_n(t)$

converge weakly to a Gaussian process $\hat{y}(t)$ under fairly general conditions. We now consider the possibility of the applications of his results to our case when θ is estimated from censored data. Durbin assumes that the estimator $\hat{\theta}$, used in the transformation, which are function of the vector-valued function $l(\ ,\ )$, so that

$$\sqrt{n}(\hat{\theta}-\theta)=\frac{1}{n}\sum_{i=1}^{n} l(x_i;\theta)+\varepsilon_n\ , \tag{3.3}$$

where $x_1, x_2, ..., x_n$ is a random samples from a continuous distribution $F(x;\theta)$ and the function $l(\ ,\ )$ is such that, for a random observation x,
(i) $E\{l(x;\theta)\}=0$
(ii) $E\{l^2(x;\theta)\}=V_\theta$ is finite,
and
(iii) $\varepsilon_n \to 0$ in probability as $n\to\infty$.

Durbin's corollary 1 of Theorem 1 gives that $\hat{y}_n(t)$ converges weakly to the normal process $\hat{y}(t)$, $0 \leqq t \leqq 1$ under the asuumption described above (Assumption A.1.) with mean function $E\{\hat{y}(t)\}=0$ and covariance function

$$\begin{aligned} &Cov\{\hat{y}(x),\hat{y}(t)\}\\ &\quad=\min(s,t)-st-h'(s)g(t)-h'(t)g(s)+g'(s)V_\theta g(t)\ , \end{aligned} \tag{3.4}$$

where

$$h(t)=\int_{-\infty}^{x(t;\theta)} l(x;\theta)dF(x;\theta),\quad g(t)=\left[\frac{\partial F(x;\theta)}{\partial\theta}\right]_{x=x(t;\theta)}$$

with $t=F(x;\theta)$ and $x=x(t;\theta)$. The matrix V_θ is given by $V_\theta = E\{l(x;\theta)\ l(x;\theta)'\}$.

The maximum likelihood estimator $\hat{\theta}$ given in (2.1) satisfies asymptotically the conditions above for $F(x;\theta)=\Phi(x-\theta)$. To show this, we may first write the $\sqrt{n}(\hat{\theta}-\theta)$ in the form (3.3) in terms of the indicator function

$$\begin{aligned} I_T(x_i) &= 1 \qquad \text{if } x_i \le T\\ &= 0 \qquad \text{if } x_i > T \end{aligned}$$

with

$$l(x_i)=\frac{1}{p}\{[x_i-\mu(\alpha)-\theta]I_T(x_i)+\mu(\alpha)\}\ ,$$

$$\varepsilon_n = \sqrt{n}(\hat{\theta} - \theta)\Big(1 - \frac{N}{np}\Big) , \tag{3.5}$$

where

$$p = \Phi(T-\theta) = \int_{-\infty}^{T-\theta} \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{t^2}{2}\Big) dt ,$$

$x_1, x_2, ..., x_n$ is a random sample from the normal population $N(\theta, 1)$. We shall show the condition (i) are verified asymptotically by taking the first moment of $l(x;\theta)$ in (3.3) as follows.

$$E\{l(x_i, \theta)\} = -\frac{1}{p}\phi(T-\theta) + \frac{1}{p}\mu(\alpha)\{1 - \Phi(T-\theta)\} .$$

Thus if we may regard $\mu(\alpha)$ as

$$\frac{\phi(T-\theta)}{1-\Phi(T-\theta)} = \frac{\phi(U_\alpha)}{1-\Phi(U_\alpha)}$$

with $\Phi(U_\alpha) = \alpha$, we then have

$$\lim_{n\to\infty} E\{l(x_i;\theta)\} = 0 .$$

It is seen after some algebra that the second moment of $l(x_i;\theta)$ yields

$$\begin{aligned} E\{l^2(x_i;\theta)\} &= \frac{1}{p^2}\int_{-\infty}^{T} (x-\theta)^2 \frac{1}{\sqrt{2\pi}}\Big\{-\frac{(x-\theta)^2}{2}\Big\}dx \\ &\quad + \frac{\mu^2(\alpha)}{p^2}\int_{T}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\Big\{-\frac{(x-\theta)^2}{2}\Big\}dx \\ &= -\frac{1}{p^2}(T-\theta)\phi(T-\theta) + \frac{1}{p^2}\Phi(T-\theta) \\ &\quad + \frac{\mu^2(\alpha)}{p^2}\{1 - \Phi(T-\theta)\} . \end{aligned}$$

Hence the condition (ii) can be written as

$$\begin{aligned} \lim_{n\to\infty} E\{l^2(x_i;\theta)\} &= \frac{1}{p^2}\Big\{-(T-\theta)\phi(T-\theta) + p + \frac{\phi^2(T-\theta)}{1-p}\Big\} \\ &= V_\theta . \end{aligned} \tag{3.6}$$

To verify the condition (iii), we use a result of Halperin [12] which states that the first factor $\sqrt{n}(\hat{\theta}-\theta)$ in the expression ε_n, converges in distribution.

Since the second factor $1 - N/np \to 0$ in probability, then by a standard result, such as Rao [22], $\varepsilon_n \to 0$ in probability.

Since assumption A.1 of Durbin is satisfied by the maximum likelihood estimator $\hat{\theta}$ in the censored data as indicated above, convergence of $\hat{y}_n(t)$ to a Gaussian process $\hat{y}(t)$ follows immediately by Theorem 1 of Durbin.

Halperin considered conditions under which the maximum likelihood estimator for type II censored samples (i.e. the smallest r observations $x_1, x_2, ..., x_r$ are known) is asymptotically normally distributed with minimum variance. For this case the likelihood is given by

$$L = \frac{n!}{(n-r)!}\{1 - F(x_r;\theta)\}^{n-r} \prod_{i=1}^{r} f(x_i;\theta)$$

and the MLE, which satisfies $\partial \log L/\partial\theta = 0$, is asymptotically efficient if the condition of Halperin [12] are satisfied, and we have

$$\sqrt{n}(\hat{\theta} - \theta) = \frac{1}{\sqrt{n}} K^{-1} \frac{\partial \log L}{\partial \theta} + \varepsilon_n \ ,$$

where

$$K = n^{-1} E\left(\frac{\partial \log L}{\partial \theta} \frac{\partial \log L}{\partial \theta'}\right) ,$$

provided K^{-1} is a finite positive-definite matrix and $\varepsilon_n \to 0$ in probability.

We can apply his result for type I censoring, with observation less than some fixed point T, and if we write

$$\begin{aligned} l(x;\theta) \ &= K^{-1}\frac{\partial}{\partial\theta}\log f(x;\theta) && (x \le T) \\ &= -K^{-1}\frac{\partial}{\partial\theta}F(T;\theta)\{1 - F(T;\theta)\}^{-1} && (x > T) \ , \end{aligned}$$

then

$$\sqrt{n}(\hat{\theta} - \theta) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} l(x_i;\theta) + \varepsilon_n \ ,$$

and θ meets Durbin's assumptions, with $\hat{\theta}$ satisfying (3.3).

Now we see that, for asymptotically efficient estimator for censored samples satisfying Halperin's condition,

$$h(t;\theta) = \int_{-\infty}^{x(t;\theta)} l(x;\theta) dF(x;\theta) = K^{-1} g(t)$$

for $t \leqq p = F(x;\theta)$, and

$$\begin{aligned} h(t;\theta) &= \int_{-\infty}^{T} l(x;\theta)dF(x;\theta) + \int_{T}^{x(t;\theta)} l(x;\theta)dF(x;\theta) \\ &= K^{-1}\int_{-\infty}^{T} \frac{\partial f(x;\theta)}{\partial\theta}dx \\ &\quad - \frac{1}{1-p}K^{-1}\frac{\partial}{\partial\theta}\int_{T}^{x(t;\theta)} F(T;\theta)dF(x;\theta) \\ &= K^{-1}g(t) - \frac{t-p}{1-p}K^{-1}\frac{\partial}{\partial\theta}F(T;\theta) \end{aligned}$$

for $t > p$, and so the covariance function (3.4) reduces to

$$Cov\{\hat{y}(t),\hat{y}(s)\} = \min(s,t) - st - g'(t)K^{-1}g(s) \ (0 \leq s,\ t \leq p)\ .$$

For our normal population case $N(\theta,1)$, since the function $l(\ ,\ )$ has a single parameter,

$$K^{-1} = \frac{1}{p^2}\left\{-(T-\theta)\phi(T-\theta) + p - \frac{\phi^2(T-\theta)}{1-p}\right\}$$

as indicated in (3.6) and taking the substitution $x = J(t) = \Phi^{-1}(t)$ gives

$$\begin{aligned} g(t) &= -\phi\{J(t)\} = -\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{J^2(t)}{2}\right\} \\ &= -B(t), \qquad\qquad \text{say.} \end{aligned}$$

Hence we obtain the covariance function of the limiting Gaussian process $\hat{y}(t)$ for $0 \leqq s,\ t \leqq p$, as

$$\begin{aligned} \rho(s,t) &= Cov\{\hat{y}(s),\hat{y}(t)\} \\ &= \min(s,t) - st \\ &\quad - \frac{B(s)B(t)}{p^2}\left\{-(T-\theta)\phi(T-\theta) + p - \frac{\phi^2(T-\theta)}{1-p}\right\}. \quad (3.7) \end{aligned}$$

Next we shall discuss the covariance function of limiting Gaussian process $\hat{z}(t)$ defined for the statisic ${}_p\hat{U}_n^2$ in (3.2). We first note that the modification to U_n^2 is not straightforward, that is, the integral

$$\int_0^1 \{F_n(s) - s\}ds$$

can be considered as an estimate of the corresponding integral with $F(t)$, the distribution function of t_i, replacing $F_n(t)$. If only observations are available which are less than $p = F(T;\theta)$, then an estimate of this integral is

$$\frac{1}{p}\int_0^p \{F_n(s) - s\}ds \ .$$

For the statistic ${}_p\hat{U}_n^2$, modified empirical proces $\hat{z}_n(t)$ is considered as

$$\hat{z}_n(t) = \hat{y}_n(t) - \frac{1}{p}\int_0^p \hat{y}_n(s)ds$$

and then we have

$${}_p\hat{U}_n^2 = \int_0^p \hat{z}_n^2(t)dt \ .$$

The covariance function of the limiting Gaussian process $\hat{z}(t)$ may be obtained directly through the use of the kernel $k(s,t) = \min(s,t) - st$ by expanding $\hat{y}(t)\hat{y}(s)$.

Write

$$\begin{aligned}\rho_1(s,t) &= E\{\hat{z}(s)\hat{z}(t)\} \\ &= \rho(s,t) + \frac{1}{p^2}\iint_0^p \rho(s,t)dsdt \\ &\quad - \frac{1}{p}\int_0^p \rho(s,t)ds - \frac{1}{p}\int_0^p \rho(s,t)dt \ ,\end{aligned}$$

where $\rho(s,t)$ denote the covariance function of $\hat{y}(t)$ obtained as (3.7). Using this, we have

$$\begin{aligned}\rho_1(s,t) &= k(s,t) - B(s)B(t)V_\theta + \frac{1}{p^2}\iint_0^p \{k(s,t) - B(s)B(t)V_\theta\}dsdt \\ &\quad - \frac{1}{p}\int_0^p \{k(s,t) - B(s)B(t)V_\theta\}ds \\ &\quad - \frac{1}{p}\int_0^p \{k(s,t) - B(s)B(t)V_\theta\}dt \\ &= k(s,t) + \left[\frac{1}{p^2}\iint_0^p k(s,t)dsdt \right. \\ &\quad \left. - \frac{1}{p}\int_0^p k(s,t)ds - \frac{1}{p}\int_0^p k(s,t)dt\right] \\ &\quad - \left\{B(s) - \frac{1}{p}\int_0^p B(s)ds\right\}\left\{B(t) - \frac{1}{p}\int_0^p B(t)dt\right\}V_\theta \ .\end{aligned}$$

Now, for the kernel $k(s,t) = \min(s,t) - st$, we have well known results as

$$\frac{1}{p}\int_0^p k(s,t)ds = t - \frac{pt}{2} - \frac{t^2}{2p}, \quad \frac{1}{p^2}\iint_0^p k(s,t)dsdt = \frac{p}{3} - \frac{p^2}{4} .$$

Therefore, the integral in the bracket [ ] above is

$$\frac{p}{3} - \frac{p^2}{4} - (s+t) + \frac{p}{2}(s+t) + \frac{1}{2p}(s^2+t^2) = a(s,t), \quad \text{say} .$$

Note that $k(s,t) + a(s,t)$ is the covariance function of $z(t)$ obtained by Pettitt-Stephens [21] for the test statistic ${}_pU_n^2$ when $F(x;\theta)$ is completely specified. Hence, we have

$$\begin{aligned} \rho_1(s,t) &= k(s,t) + a(s,t) \\ &\quad - \left\{B(s) - \frac{1}{p}\int_0^p B(s)ds\right\}\left\{B(t) - \frac{1}{p}\int_0^1 B(t)dt\right\}V_\theta . \end{aligned}$$

The substitution $x = J(t)$, with $J(t) = \{x; t = \Phi(x)\}$, gives, after some algebra,

$$\int_0^p B(t)dt = \int_0^p \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{J^2(t)}{2}\right\}dt = \frac{1}{2\sqrt{\pi}}\Phi\{\sqrt{2}(T-\theta)\} ,$$

and finally we have

$$\begin{aligned} \rho_1(s,t) &= k(s,t) + a(s,t) - \left\{B(s) - \frac{1}{2p\sqrt{\pi}}\Phi[\sqrt{2}(T-\theta)]\right\} \\ &\qquad \left\{B(t) - \frac{1}{2p\sqrt{pi}}\Phi[\sqrt{2}(T-\theta)]\right\}V_\theta \\ &= \min(s,t) - st + \frac{p}{3} - \frac{p^2}{4} - (s+t) + \frac{p}{2}(s+t) + \frac{1}{2p}(s^2+t^2) \\ &\quad + \left\{B(s) - \frac{1}{2p\sqrt{\pi}}\Phi[\sqrt{2}(T-\theta)]\right\} \\ &\qquad \left\{B(t) - \frac{1}{2p\sqrt{\pi}}\Phi[\sqrt{2}(T-\theta)]\right\}V_\theta , \end{aligned} \tag{3.8}$$

where V_θ is given by

$$V_\theta = \frac{1}{p^2}\left\{-(T-\theta)\phi(T-\theta) + p - \frac{\phi^2(T-\theta)}{1-p}\right\} .$$

In his paper [19], Pettitt gave the value of $\rho_1(s,t)$ as

$$\min(s,t) - st - \{c - g(s)\}' K^{-1} \{c - g(t)\} \ , \tag{3.9}$$

where

$$c = \frac{1}{p} \int_0^p g(t) dt \ ,$$

for the distribution $F(x;\theta)$, but his result for ${}_p\hat{U}_n^2$ seems to be incorrect. $\min(s,t) - st$ in (3.9) should replace as

$$\min(s,t) - st + \frac{p}{3} - \frac{p^2}{4} - (s+t) + \frac{p}{2}(s+t) + \frac{1}{2p}(s^2 + t^2) \ .$$

Therefore we arrive at the following theorem.

Theorem 2 *The empirical process $\hat{y}_n(t)$ converges weakly to a zero mean normal process $\hat{y}(t)$, $0 \leqq t \leqq 1$ in the metric space (D,d) with covariance function*

$$\begin{aligned} \rho(s,t) &= \min(s,t) - st \\ &\quad - \frac{B(s)B(t)}{p^2} \left\{ -(T-\theta)\phi(T-\theta) + p - \frac{\phi^2(T-\theta)}{1-p} \right\} , \end{aligned}$$

and the modified empirical process $\hat{z}_n(t)$ converges weakly also to a normal process $\hat{z}(t)$ in the same metric space (D,d) with mean 0 and covariance

$$\begin{aligned} \rho_1(s,t) &= \min(s,t) - st + \frac{p}{3} - \frac{p^2}{4} - (s+t) \\ &\quad + \frac{p}{2}(s+t) + \frac{1}{2p}(s^2 + t^2) \\ &\quad + \left\{ B(s) - \frac{1}{2p\sqrt{\pi}} \Phi[\sqrt{2}(T-\theta)] \right\} \\ &\quad \left\{ B(t) - \frac{1}{2p\sqrt{\pi}} \Phi[\sqrt{2}(T-\theta)] \right\} V_\theta \ . \end{aligned}$$

4. Asymptotic distribution theory of the test statistics

4.1. Convergences of ${}_p\hat{W}_n^2$ and ${}_p\hat{U}_n^2$

Our basic space will be the space D of right-continuous functions with left-hand limits on [0,1]. On D we use the Skorohod metric defined on pages 111 of Billingsley [4]. It follows from the treatment of Billingsley that ${}_pU_n^2 \rightarrow$

${}_pU^2$, ${}_pW_n^2 \to {}_pW^2$ whenever ${}_pW_n^2$ is continuous in metric d. Because of weak convergence of $\hat{y}_n(t)$ to $\hat{y}(t)$, where $\hat{y}(t)$ is a normal process in the metric space (D,d) with mean function zero and covariance function $E\{\hat{y}(s)\hat{y}(t)\}$, we observe if $h(\hat{y}_n(t))$ is a functional of $\hat{y}_n(t)$ which is continuous in metric d, then $h(\hat{y}_n(t))$ converges weakly to $h(\hat{y}(t))$ by the arguments of Billingsley.

Let

$$ {}_p\hat{W}^2 = \int_0^p \hat{y}^2(t)dt $$

and

$$ {}_p\hat{U}^2 = \int_0^p \hat{z}^2(t)dt $$

where $\hat{y}(t)$ and $\hat{z}(t)$ are Gaussian processes given in section 3 with means 0 and covariance functions $\rho(s,t)$ and $\rho_1(s,t)$ respectively defined by (3.7) and (3.8). We shall show that these two statistics ${}_p\hat{W}_n^2$, ${}_p\hat{U}_n^2$ converge in distribution to ${}_p\hat{W}^2$ and ${}_p\hat{U}^2$ respectively. This means that continuous functional of the empirical process $\hat{y}_n(t)$ converge weakly to the same functionals of $\hat{y}(t)$.

To begin, we consider the asymptotic distributions of the statistics in (3.1) and (3.2), where $p = \Phi(T-\theta)$ is replaced by the estimate $\hat{p} = \Phi(T-\hat{\theta})$, that is

$$ {}_{\hat{p}}\hat{W}_n^2 = \int_0^{\hat{p}} \hat{y}_n^2(t)dt \tag{4.1} $$

and,

$$ {}_{\hat{p}}\hat{U}_n^2 = \int_0^{\hat{p}} \hat{z}_n^2(t)dt \tag{4.2} $$

since in any application, $\hat{p}$ is to be used for the unkown value p.

As to the weak convergence of ${}_{\hat{p}}\hat{W}_n^2$ to ${}_{\hat{p}}\hat{W}^2$, Sirvanti-Levent [24] considered a method by employing two steps, i.e. first step is to show that ${}_{\hat{p}}\hat{W}_n^2$ is asymptotically equivalent to ${}_p\hat{W}_n^2$ and then second is to show that the asymptotic distribution of ${}_p\hat{W}_n^2$ is same as the distribution of ${}_p\hat{W}^2$. To show the first step, they partitioned the ${}_{\hat{p}}\hat{W}_n^2$ into two parts, i.e.

$$ {}_{\hat{p}}\hat{W}_n^2 = \int_0^p \hat{y}_n^2(t)dt + \int_p^{\hat{p}} \hat{y}_n^2(t)dt \ , \tag{4.3} $$

and they showed the second term on the right hand side of (4.3) is asymptotically negligible by observing the evaluation of the bound of

$$\sqrt{n}\int_p^{\hat{p}}\{F_n(t)-t\}^2dt\ .$$

We now show here the direct proof of ${}_{\hat{p}}\hat{W}_n^2 \to {}_p\hat{W}^2$ (in D).

Together with what we have obtained in section 2 that $\hat{\theta}$ is a consistent estimator of θ, it is obvious that $\hat{p}$ converges to p in probability since $\hat{p}$ is a continuous function of θ. On the other hand, we have shown that the limiting distribution of $\hat{y}_n(t)$ is a continuous Gaussian process $\hat{y}(t)$ with covariance function given by $\rho(s,t)$. An immediate consequence of theorem 4.4 of Billingsley [4] gives us that $(\hat{p},\hat{y}_n(t)) \longrightarrow (p,\hat{y}(t))$ in distribution in $R\times D$, the product space of real numbers and the space of functions on $D[0,1]$ that are right continuous and have left limits.

We are going to discuss that if $(\hat{p},\hat{y}_n(t))$ is a sequence converging to $(p,\hat{y}(t))$ in $R\times D$, then

$$\int_0^{\hat{p}}\hat{y}_n^2(t)dt \longrightarrow \int_0^p \hat{y}^2(t)dt\ .$$

We see that, for $\hat{p}>p$,

$$\left|\int_0^{\hat{p}}\hat{y}_n^2(t)dt-\int_0^p\hat{y}^2(t)dt\right| \le \int_0^1\left|\hat{y}_n^2(t)-\hat{y}^2(t)\right|dt+\left|\int_p^{\hat{p}}\hat{y}^2(t)dt\right|$$

and we note that Skorohod convergence does imply $\hat{y}_n^2(t)\to\hat{y}^2(t)$ for continuity pointst of $\hat{y}^2(t)$, and if $\hat{y}_n^2(t)$ is continuous on all of [0,1], then Skorohod convergence implies uniform convergence, so that

$$\sup_t\left|\hat{y}_n^2(t)-\hat{y}^2(t)\right|\to 0$$

as $n\to\infty$, and so the first term on the right-hand side above inequality tends to zero. Since $\hat{y}^2(t)$ is continuous, the second term on the right converges to zero and so the convergence of

$$\int_0^{\hat{p}}\hat{y}_n^2(t)dt$$

to

$$\int_0^p\hat{y}^2(t)dt$$

is established.

It is quite natural to talk about that the same method works for ${}_{\hat{p}}\hat{U}_n^2$.

As we have seen the asymptotic distribution of ${}_p\hat{W}_n^2$ and ${}_p\hat{U}_n^2$ are the same as the distributions of ${}_p\hat{W}^2$ and ${}_p\hat{U}^2$ respectively, we can now concentrate on the processes $\hat{y}(t)$ and $\hat{z}(t)$, and then attempt to find the distributions of ${}_p\hat{W}^2$ and ${}_p\hat{U}^2$ with $0 < p < 1$.

4.2. Moments of ${}_p\hat{W}^2$ and ${}_p\hat{U}^2$

By approach of Kac-Siegert [17] and Anderson-Darling [1] which are used for finding the asymptotic theory of the simpler goodness of fit statistics when testing completely spedified hypothesis with uncensored samples, we can show that ${}_p\hat{W}^2$ and ${}_p\hat{U}^2$ are infinite sums of identically disributed chi-squared random variables

$$\sum_{j=1}^{\infty} \frac{c_j}{\lambda_j} \ . \tag{4.4}$$

In (4.4) $c_1, c_2, \ldots$ are independent χ^2 random variables with one degree of freedom and $0 < \lambda_1 < \lambda_2 < \ldots$ are eigenvalues of the integral equations

$$\lambda \int_0^p \rho(s,t) f(s) ds = f(t) \quad (0 \le t \le p) \tag{4.5}$$

for ${}_p\hat{W}^2$ and

$$\lambda \int_0^p \rho_1(s,t) f(s) ds = f(t) \quad (0 \le t \le p) \tag{4.6}$$

for ${}_p\hat{U}^2$, where $\rho(s,t)$ and $\rho_1(s,t)$ are covariance functions of the limiting processes $\hat{y}(t)$ and $\hat{z}(t)$ and $f(.)$ are the eigenfunctions of the corresponding integral equations above. Thus the eigenvalues $\{\lambda_j\}$ in (4.4) are to be determined from (4.5) or (4.6), where the expression (3.7) or (3.8) is used for the covariance function $\rho(s,t)$ or $\rho_1(s,t)$. Sukhatme [28] and Darling [5] give details of the integral equation of the form (4.5), for the general positive definite kernel $\rho(s,t)$, and the method may be applied to our case.

Now, it would be more convenient, for a better understanding, to begin with the treatment of the cumulants of ${}_p\hat{W}^2$. This may be obtained directly from (4.5) through the use of covariance function of the empirical process $\hat{y}(t)$ as

$$\kappa_j = 2^{j-1}(j-1)! \int_0^p \rho^{(j)}(s,s) ds \ , \tag{4.7}$$

where

$$\begin{aligned}\rho^{(j)}(s,t) &= \int_0^p \rho^{(j-1)}(s,u)\rho(u,t)du \quad (j \geq 2),\\ &= \rho(s,t) \quad (j=1)\ ,\end{aligned}$$

with $\rho(s,t)$ being the covariance function given in (3.7). This result follows from a straightforward extension of those of Anderson-Darling [1]. Hence the value of the cumulants can be computed exactly from (4.7), but in practice, after the first two, the integral calculation becomes extremely long and only the means and variances have been calculated, as shown below.

First of all we need to keep in mind that the expressions for $E({}_pW^2)$, $Var({}_pW^2)$ with censored data, are given by Pettitt-Stephens [21] as

$$\begin{aligned}E({}_pW^2) = \kappa_1({}_pW^2) &= \frac{p^2}{2} - \frac{p^3}{3},\\ Var({}_pW^2) = \kappa_2({}_pW^2) &= \frac{p^4}{3} - \frac{8p^5}{15} + \frac{2p^6}{9}\ .\end{aligned}$$

For ${}_p\hat{W}^2$,

$$\begin{aligned}\rho(s,t) &= \min(s,t) - st\\ &\quad - \frac{B(s)B(t)}{p^2}\left\{-(T-\theta)\phi(T-\theta) + p - \frac{\phi^2(T-\theta)}{1-p}\right\},\end{aligned}$$

and so

$$\begin{aligned}E({}_p\hat{W}^2) &= \int_0^p (t-t^2)dt\\ &\quad - \frac{1}{p^2}\left\{-(T-\theta)\phi(T-\theta) + p - \frac{\phi^2(T-\theta)}{1-p}\right\}\int_0^p B^2(t)dt\\ &= \frac{1}{2}p^2 - \frac{1}{3}p^3 - \frac{\Phi\{\sqrt{3}(T-\theta)\}}{2\pi\sqrt{3}}V_\theta\ . \qquad (4.8)\end{aligned}$$

The variance is

$$\begin{aligned}\kappa_2({}_p\hat{W}^2) &= Var({}_p\hat{W}^2) = 2\iint_0^p \{k(s,t) - B(s)B(t)V_\theta\}^2 dsdt\\ &= 2(K^2 - 2KB + B^2)\ ,\end{aligned}$$

where

$$\begin{aligned}2K^2 &= 2\iint_0^p \{\min(s,t) - st\}^2 dsdt = \frac{2}{9}p^6 - \frac{8}{15}p^5 + \frac{1}{3}p^4\\ &= Var({}_pW^2)\end{aligned}$$

and

$$\begin{aligned} 2B^2 &= 2\iint_0^p B^2(s)B^2(t)V_\theta^2 dsdt = 2\left\{\int_0^p B^2(t)dt\right\}^2 \\ &= 2\{E({}_pW^2) - E({}_p\hat{W}^2)\}^2 , \end{aligned}$$

thus $4KB$ must be found.

The substitution $x = J(s)$, i.e., $J(.)$ is the inverse of $\Phi(.)$, gives, after much algebra,

$$\begin{aligned} KB &= \iint_0^p k(s,t)B(s)B(t)V_\theta dsdt \\ &= 2V_\theta \int_0^p (1-t)B(t)\left\{\int_0^1 sB(s)\right\}dt \\ &= \frac{1}{\pi}V_\theta \int_0^p (1-t)B(t)\left\{\int_{-\infty}^{J(t)} \left[\frac{1}{2} + \frac{1}{2\pi}\int_{-\infty}^{\infty} \right.\right. \\ &\qquad \left.\left. \exp(-2u^2)\frac{\sin 2xu}{u}du\right] \exp(-x^2)dx\right\}dt \\ &= 2V_\theta \int_0^p (1-t)B(t)\left\{\int_{-\infty}^{J(t)} [1-\Phi(x)]\exp(-x^2)dx\right\}dt . \end{aligned}$$

The resulting integral does not seem to be tractable by analytic method, and so the KB has to be found numerically.

Finally, we have

$$\begin{aligned} Var({}_p\hat{W}^2) &= Var({}_pW^2) + \{E({}_pW^2) - E({}_p\hat{W}^2)\}^2 - 4KB \\ &= \frac{1}{3}p^4 - \frac{8}{15}p^5 + \frac{2}{9}p^6 + \frac{\Phi^2[\sqrt{3}(T-\theta)]}{6\pi^2}V_\theta^2 - 4KB . \end{aligned} \quad (4.9)$$

For $\hat{U}_p^2$,

$$\rho_1(s,t) = k_1(s,t) - C(s)C(t)V_\theta$$

where

$$\begin{aligned} k_1(s,t) = \min(s,t) - st + \frac{1}{3}p - \frac{1}{4}p^2 - (s+t) \\ + \frac{1}{2}p(s+t) + \frac{1}{2p}(s^2+t^2), \end{aligned}$$

$$C(s) = B(s) - \frac{1}{2p\sqrt{\pi}}\Phi\{\sqrt{2}(T-\theta)\} ,$$

and so,

$$\kappa_1({}_p\hat{U}^2) = E({}_p\hat{U}^2) = \int_0^p \left(s - s^2 + \frac{p}{3} - \frac{p^2}{4} - 2s + ps + \frac{s^2}{p}\right) ds$$
$$- \int_0^p \left\{ B^2(s) - \frac{\Phi[\sqrt{2}(T-\theta)]}{p\sqrt{\pi}} B(s) + \frac{\Phi^2[\sqrt{2}(T-\theta)]}{4p^2\pi} \right\} V_\theta ds \; .$$

The integral of the first term on the right-hand side is $p^2/6 - p^3/12$ and this value corresponds exactly to $E({}_pU^2)$ obtained by Pettitt-Stephens [21]. Concerning the second term, we are going to make use of the results

$$\int_0^p B(s)ds = \frac{\Phi\{\sqrt{2}(T-\theta)\}}{2\sqrt{\pi}}, \quad \int_0^p B^2(s)ds = \frac{\Phi^2\{\sqrt{3}(T-\theta)\}}{2\sqrt{3}\pi} \; ,$$

hence the mean is

$$E({}_p\hat{U}^2) = E({}_pU^2) - V_\theta \left\{ \frac{\Phi[\sqrt{3}(T-\theta)]}{2\pi\sqrt{3}} - \frac{\Phi^2[\sqrt{2}(T-\theta)]}{4p\pi} \right\} . \quad (4.10)$$

Furthermore, the variance is given in a similar manner by

$$\begin{aligned} \kappa_2({}_p\hat{U}^2) &= Var({}_p\hat{U}^2) = 2\iint_0^p \{k(s,t) - C(s)C(t)V_\theta\}^2 dsdt \\ &= 2(K_1^2 - 2K_1C + C^2), \qquad \text{say.} \end{aligned}$$

This again needs only straightforward but tedious algebra, and with the notation $k(s,t)$ and $a(s,t)$ introduced in section 8, the variance for ${}_p\hat{U}^2$ can be calculated as shown below. Write

$$\begin{aligned} K_1^2 &= \iint_0^p k^2(s,t)dsdt + 2\iint_0^p k(s,t)a(s,t)dsdt + \iint_0^p a^2(s,t)dsdt \\ &= E_1 + E_2 + E_3, \qquad \text{say.} \end{aligned}$$

We then have

$$2E_1 = \frac{1}{3}p^4 - \frac{8}{15}p^5 + \frac{2}{9}p^6 = Var({}_pW^2)$$

as before. Straightforward but cumbersome algebra give

$$4E_2 = -\frac{28}{45}p^4 + p^5 - \frac{5}{16}p^6$$

and

$$2E_3 = \frac{14}{15}p^4 - \frac{1}{2}p^5 + \frac{4}{24}p^6 \; .$$

Hence we get

$$2K_1^2 = \frac{1}{45}p^4 - \frac{1}{30}p^5 + \frac{1}{72}p^6 .$$

Use of (4.10) gives us for the third integral,

$$\begin{aligned} 2C^2 &= 2\{E({}_pU^2) - E({}_p\hat{U}^2)\}^2 \\ &= 2V_\theta^2 \left\{ \frac{\Phi[\sqrt{2}(T-\theta)]}{2\sqrt{3}\pi} - \frac{\Phi^2[\sqrt{2}(T-\theta)]}{4p\pi} \right\}^2 . \end{aligned}$$

Therefore K_1C must be found. Let

$$\alpha = \frac{1}{2p\sqrt{\pi}}\Phi\{\sqrt{2}(T-\theta)\} ,$$

then we may write

$$\begin{aligned} K_1C &= V_\theta\left\{ \iint_0^p k(s,t)B(s)B(t)dsdt - 2\alpha \iint_0^p k(s,t)B(s)dsdt \right. \\ &\quad + \alpha^2 \iint_0^p k(s,t)dsdt + \iint_0^p a(s,t)B(s)B(t)dsdt \\ &\quad \left. - 2\alpha \iint_0^p a(s,t)B(s)dsdt + \alpha^2 \iint_0^p a(s,t)dsdt \right\} \\ &= V_\theta\{E_1 - 2\alpha E_2 + \alpha^2 E_3 + E_4 - 2\alpha E_5 + \alpha^2 E_6\}, \quad \text{say.} \end{aligned}$$

Using the similar algebra as before, we then see that

$$E_3 = \frac{1}{3}p^3 - \frac{1}{4}p^4 ,$$

and

$$E_6 = \frac{1}{4}p^4 - \frac{1}{3}p^3 .$$

As for the rest, we meet again the difficulty of integral evaluation as was shown in the case of ${}_p\hat{W}^2$. However, it turns out to be possible to get a fairly explicit formula for the integrals, and we have

$$\begin{aligned} &E_1 = KB, \qquad E_5 = \left(\frac{1}{2}p^2 - p\right)\alpha\beta_1(p) + \frac{1}{2}\beta_2(p), \\ &E_4 = \left(\frac{1}{3}p^3 - \frac{1}{4}p^4\right)\alpha^2 + (p^2 - 2p)\alpha\beta_1(p) + \alpha\beta_2(p), \\ &E_2 = \frac{1}{2}\alpha p^3 - \frac{1}{2}p^2\beta_1(p) + \int_0^p \beta_1(t)dt - \int_0^p t\left\{\int_0^1 B(s)ds\right\}dt , \end{aligned}$$

where

$$\beta_1(t) = \int_0^t sB(s)ds, \quad \beta_2(t) = \int_0^t s^2 B(s)ds \ .$$

Thus we have

$$\begin{aligned} K_1 C \ = \ & V_\theta \Big\{ KB + \alpha p^2 \beta_1(p) - \Big(\frac{2}{3}p^3 + \frac{1}{4}p^3\Big)\alpha^2 \\ & - 2\alpha \int_0^p \beta_1(t)dt + 2\alpha \int_0^p t \Big[\int_0^t B(s)ds\Big] dt \Big\} \ . \end{aligned}$$

and

$$\begin{aligned} Var({}_p\hat{U}^2) \ = \ & \frac{1}{45}p^4 - \frac{1}{30}p^5 + \frac{1}{72}p^6 \\ & + 2V_\theta^2 \Big\{ \frac{\Phi[\sqrt{3}(T-\theta)]}{2\pi\sqrt{3}} - \frac{\Phi^2[\sqrt{2}(T-\theta)]}{4p\pi} \Big\}^2 - 4K_1 C \ . \quad (4.11) \end{aligned}$$

which will also be found numerically. Therefore to state the theorem formally,

Theorem 3 *Asymptotic distribution of* ${}_p\hat{U}_n^2$ *is the same as the distribution of* ${}_p\hat{U}^2$*, where p is estimate* $p = F(T-\theta)$*. As for the moments of* ${}_p\hat{U}^2$*, we have*

$$E({}_p\hat{U}^2) = E({}_pU^2) - V_\theta \Big\{ \frac{\Phi[\sqrt{3}(T-\theta)]}{2\pi\sqrt{3}} - \frac{\Phi^2[\sqrt{2}(T-\theta)]}{4p\pi} \Big\},$$

$$\begin{aligned} Var({}_p\hat{U}^2) = & \frac{1}{45}p^4 - \frac{1}{30}p^5 + \frac{1}{72}p^6 \\ & + 2V_\theta^2 \Big\{ \frac{\Phi[\sqrt{3}(T-\theta)]}{2\pi\sqrt{3}} - \frac{\Phi^2[\sqrt{2}(T-\theta)]}{4p\pi} \Big\}^2 - 4K_1 C \ . \end{aligned}$$

4.3. Solution of the integral equations

To find the percentiles of the distributions of ${}_p\hat{W}^2$ and ${}_p\hat{U}^2$, the integral equations (4.5) and (4.6) must be solved by using the procedure of Darling [5]. This method was suucessfully used by Stephens [25] in determining the distribution of $\hat{W}^2$ for the completely sample. Since it is not feasible to compute all the eigenvalues $\{\lambda_j\}$, the sum is usually truncated after a finite number of terms and the limiting random variable ${}_p\hat{W}^2$ or ${}_p\hat{U}^2$ is approximated by

$$S = \sum_{j=1}^{k} \frac{c_j}{\lambda_j} + \lambda \chi_\beta^2 , \tag{4.12}$$

where k is the number of known eigenvalues being suitably large integers, the c_j $(j = 1, 2, ..., k)$ are independent chi-squared random variables with one degree of freedom, and χ_β^2 is an also χ^2 random variable with β degree of freedom. The weights λ and the constant β are chosen so that S and the respective statistic ${}_p\hat{W}^2$ or ${}_p\hat{U}^2$ have the same mean and variance.

The cumlants of S are given by

$$\kappa_{*j} = 2^{j-1}(j-1)! \left\{ \sum_{i=1}^{k} \left(\frac{1}{\lambda_i} \right)^j + \lambda^j \beta \right\} , \tag{4.13}$$

the values of the constants λ and β are determined by comparing the true cumulants with those given by κ_{*j}. Hence to determine λ and β, the first two moment of the statistics which are already obtained are needed. From well-known relation for characteristic function of S, we have

$$\psi_s(t) = \prod_{j=1}^{k} \left(1 - \frac{2it}{\lambda_j} \right)^{-1/2} (1 - 2i\lambda t)^{-\frac{\beta}{2}} .$$

The disribution of the statistic S can be found accurately by inverting the characteristic function of S, using the technique of Imhof [15]. These method have been used successfully before (Durbin,Knott and Taylor [10], Sirvanci and Levent [24], Pettitt [20]) for approximating to the asymptotic distribution of a Cramér-von Mises type statsitic.

In computations for finding the percentiles of the statistic S, it is necessary firstly to compute κ_1 and κ_2 exactly from (4.8) and (4.9), then to compute the first k eigenvalues $\{\lambda_j\}$ for each values of censoring levels p $(0 < p < 1)$ utilizing the results of Darling [5], Stephens [27]. For this purpose, we now discuss the solutions of the integral equations (4.5) and (4.6).

For the statistic ${}_p\hat{W}^2$, the kernel is given by

$$\rho(s,t) = k(s,t) - B(s)B(t)V_\theta ,$$

where

$$k(s,t) = \min(s,t) - st, \quad (0 \le s,\ t \le p)$$

and

$$V_\theta = \frac{1}{p^2}\left\{-(T-\theta)\phi(T-\theta) + p - \frac{\phi^2(T-\theta)}{1-p}\right\} .$$

In order to solve (4.5), knowledges of the eigenfunction of the integral equation

$$\lambda \int_0^p k(s,t) f(s) ds = f(t), \quad (0 \le t < p) \tag{4.14}$$

are required. The details of the solution of this kernel $k(s,t)$ are given by Pettitt-Stephens [21]. The normalized eigenfunction associated with (4.14) is given by

$$f(t) = \frac{\sqrt{2}\sin(\sqrt{\lambda_j} p)}{\sqrt{p - \frac{1}{\sqrt{\lambda_j}}\sin(\sqrt{\lambda_j}p)\cos(\sqrt{\lambda_j}p)}}$$

and the corresponding eigenvalues $\sqrt{\lambda_j}$ must satisfy

$$\tan(p\sqrt{\lambda}) = -\sqrt{\lambda}(1-p) .$$

For ${}_p\hat{U}^2$, this also requires knowledge of the eigenfunctions of the integral equation

$$\lambda \int_0^p k_1(s,t) f(s) ds = f(t), \quad (0 \le t < p) , \tag{4.15}$$

where

$$\begin{aligned} k_1(s,t) &= \min(s,t) - st - (s+t) \\ &\quad + \frac{p}{2}(s+t) + \frac{1}{2p}(s^2+t^2) + \frac{p}{3} - \frac{p^2}{4} . \end{aligned}$$

The discussion is slightly more complicated. If we denote the eigenvalues for ${}_pW^2$ by $\{{}_w\lambda_j\}$, that is the solution of (4.14), and $\{\lambda_j\}$, $\{\lambda_j^*\}$ denote the eigenvalues for ${}_pU^2$, λ_j solution of $\sin(\sqrt{\lambda}/2)p = 0$ and λ_j^* solution of $\tan\left(\frac{\sqrt{\lambda}}{2}\right)p = -\frac{\sqrt{\lambda}}{2}(1-p)$, then

$$\lambda_j = \frac{4\pi^2 j^2}{p^2}, \quad \lambda_j^* = 4{}_w\lambda_j \quad (j = 1, 2, ...)$$

with corresponding eigenfunctions, suitably normalized,

$$f_j(t) = \cos(\sqrt{\lambda_j} t)$$

and

$$f_j^*(t) = \sin(\sqrt{\lambda_j^*} t) + \frac{1}{2}\sqrt{\lambda_j^*}(1-p)\cos(\sqrt{\lambda_j^*} t) .$$

Suppose that $k(s,t)$ has Fredholm determinant $D_0(\lambda)$, whose simple roots are $0 \leqq \lambda_1 \leqq \lambda_2 \leqq \ldots$, and let the corresponding eigenfunctions be $f_1(x), f_2(x), \ldots$.

Define

$$a_j = \int_0^p f_j(x)\psi(x)dx, \quad S(\lambda) = 1 + \lambda \sum_{i=1}^{\infty} \frac{a_i^2}{1-\lambda/\lambda_i} ,$$

where $\psi(x)$ is a function already found as $\psi(x) = B(x)V_\theta$.

Darling [5] showed that the Fredholm determinant for the kernel $\rho(s,t)$ is

$$D(\lambda) = D_0(\lambda)S(\lambda) .$$

The weight λ_j in the representation

$$S = \sum_{j=1}^{k} \frac{C_j}{\lambda_j} + \lambda \chi_\beta^2$$

have to be found, and these are the solutions of the relevant Fredholm determinant equation $D(\lambda) = 0$.

Let the weight λ_j obtained for $D_0(\lambda) = 0$ be called the standard weight. Then the weights consists of a subset of the standards, plus a new set λ_j' labelled with a dash. For ${}_p\hat{W}^2$, the zeros of $D_0(\lambda)$ are simple zeros, and will not be the zero of $D(\lambda)$ unless, in $S(\lambda)$, the corresponding Fourier coefficient a_j is zero. The other zeros λ_j' are solution of $S(\lambda) = 0$.

Similar method may be applied for ${}_p\hat{U}^2$, but the discussion is slightly more complicated.

Suppose a_j and a_j^* are the Fourier coefficients obtained using normalized eigenfunctions

$$f_j(x) = \sqrt{\frac{2}{p}} \cos\frac{2\pi jx}{p}$$

and

$$f_j^*(x) = d\{\sin(2\sqrt{{}_w\lambda_j x}) + \sqrt{{}_w\lambda_j}(1-p)\cos(2\sqrt{{}_w\lambda_j x})\} ,$$

where

$$d = \left\{ \frac{p+1+2p(1-p)^2{}_w\lambda_j}{4} - \frac{1-p}{4}\cos(4p\sqrt{{}_w\lambda_j}) + \frac{(1-p)^2{}_w\lambda_j^{-1}}{8\sqrt{{}_w\lambda_j}}\sin(4p\sqrt{{}_w\lambda_j}) \right\}^{-1/2}$$

respectively. Then $S(\lambda)$ becomes

$$S(\lambda) = 1 + \lambda \sum_{j=1}^{\infty} \frac{a_j^2}{1-\lambda/\lambda_j} + \lambda \sum_{j=1}^{\infty} \frac{a_j^{*2}}{1-\lambda/\lambda_j} .$$

The function $D(\lambda)$ is such that no a_j is now zero. This means that no standard weight is a zero of $D(\lambda)$, and hence the λ_j' are those of $S(\lambda)$.

Thus for each case, numerical values of the first k smallest eigenvalues of the integral equation are found for each censoring level p $(0 < p < 1)$.

Finally, we talk about a property of ${}_p\hat{W}_n^2$ or ${}_p\hat{U}_n^2$ test which will be a matter of central interest. As is shown implicitely in Sirvanci-Levent [24], the ${}_p\hat{W}_n^2$ test of exponentiality with type I censored samples in the presence of an unknown scale parameter has distribution-free property.

Further, this test is parameter-free since $\psi(u) = (1-u)\log(1-u)/\sqrt{p}$ does not depend on the parameter θ. On the other hand, Darling [5] pointed out that the $\hat{W}_n^2$ test of normality with no censoring is parameter-free when θ is a location or scale unknown parameter. Unlike these case, the situation considered in this paper is quite different. For our test, we see that

$$\psi(u) = B(u)\sqrt{V_\theta}$$

for ${}_p\hat{W}^2$, and

$$\psi(u) = \sqrt{V_\theta}\left\{ B(u) - \frac{\Phi[\sqrt{2}(T-\theta)]}{2p\sqrt{\pi}} \right\}$$

for ${}_p\hat{U}^2$, and in each case this clearly depends on the parameter θ in $F(x;\theta)$, hence our test is not even parameter-free. This result seems to reflect the complication of the situation for censored data.

5. Small-sample case

In this section we shall study the explicit expressions for the statistics involving the observations $\hat{t}_1, \hat{t}_2, ..., \hat{t}_n$. If

$$\hat{t}_1 < \hat{t}_2 < ... < \hat{t}_N \leq \hat{p} < \hat{t}_{N+1} \leq ... \leq \hat{t}_n \ ,$$

that is, N observation are less than $\hat{p}$ with the censored $n - N$ largest observations, then we can find

$$\begin{aligned} {}_{\hat{p}}\hat{W}_n^2 &= \sum_{i=1}^{N}\left(\hat{t}_i - \frac{2i-1}{2n}\right)^2 - \frac{N^3}{3n^2} + \frac{N}{12n^2} + \frac{\hat{p}N^2}{n} \\ &\quad - \hat{p}^2 N + \hat{p}^2 n - \hat{p}n + \frac{n}{3} \ . \end{aligned} \tag{5.1}$$

This shows that the exact value of ${}_{\hat{p}}\hat{W}_n^2$ may be simply calculated when the transformed individual sample values are known.

Similarly ${}_{\hat{p}}\hat{U}_n^2$ is given by

$${}_{\hat{p}}\hat{U}_n^2 = {}_{\hat{p}}\hat{W}_n^2 - n(2\hat{p}-1)\left\{\frac{1}{n\hat{p}}\sum_{i=1}^{N}\hat{t}_i + 1 - \frac{N}{n} - \frac{1}{2\hat{p}}\right\}^2 \ . \tag{5.2}$$

The statistics ${}_p\hat{W}_n^2$ and ${}_p\hat{U}_n^2$ are clearly defined for type I censoring when only the observations in the fixed interval $[0, p]$ are available. For type II censoring, when a fixed number of observations are censored, we may replace $\hat{p}$ by $\hat{t}_r$ in the integration, to obtain the statistics ${}_r\hat{W}_n^2$ and ${}_r\hat{U}_n^2$. Then the statistics

$$\begin{aligned} {}_r\hat{W}_n^2 &= n\int_0^{\hat{t}_r}\{\hat{F}_n(t) - t)\}^2 dt, \\ {}_r\hat{U}_n^2 &= n\int_0^{\hat{t}_r}\{\hat{F}_n(t) - t - \frac{1}{\hat{t}_r}\int_0^{\hat{t}_r}[\hat{F}_n(s) - s]ds\}^2 dt \end{aligned}$$

where $\hat{t}_i = F(x_i;\theta)$ and $\hat{F}_n(t)$ is the empirical distribution function of the $\hat{t}_i$'s for $t \leq \hat{t}_r$, can be used to test the null hypothesis.

The statistics ${}_r\hat{W}_n^2$ or ${}_r\hat{U}_n^2$ can then be calculated using the formula (5.1) or (5.2) by replacing $\hat{p}$ by $\hat{t}_r$ and N by r. Empirical percentage points of ${}_r\hat{W}_n^2$ and ${}_r\hat{U}_n^2$ may be found for $r = pn$ with each p $(0 < p < 1)$ and n.

On the other hand, the statistic ${}_r\hat{W}_n^2$ also converges to ${}_p\hat{W}^2$ provided $r/n \to p$ as $n \to \infty$. Similarly ${}_r\hat{U}_n^2$ converges to ${}_p\hat{U}^2$. As for the speed of covergence of ${}_r\hat{W}_n^2$ to ${}_p\hat{W}^2$, Monte Carlo experiment will be useful tool.

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Department of Mathematics
Faculty of Science
Hokkaido University
Sapporo 060, Japan