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Modular forms with coefficients involving class numbers and congruences of eigen values of Hecke operators

Masatoshi YAMAUCHI

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Abstract. Following F. Hirzebruch and D. Zagier's method, we construct a modular form $\Phi_D(z)$ of level D, weight 2 and of Nebentype, whose Fourier coefficients involve class numbers of orders of imaginary quadratic fields and the trace of elements of the real quadratic field $\mathbb{Q}(\sqrt{D})$. As one of apllications, we can prove two types of congruences, one is Shimura type and the other Doi-Brumer type of eigen values a_p of Hecke operators of cusp forms of Neben type by numerical data for primes $D \leq 97$.

Key words: modular form, class number, eigen values of Hecke operators.

Introduction

In their paper [4], F. Hirzebruch and D. Zagier constructed elliptic modular forms whose Fourier coefficients involve class numbers of orders of the imaginary quadratic number fields, counting intersection numbers of certain curves on Hilbert modular surfaces over the quadratic number fields. Following their method, we can give another elliptic modular form $\Phi_D(z)$ of level D, weight 2 and of Neben type. Further, expressing $\Phi_D(z)$ as a linear combination of the Eisenstein series and cusp forms in the the space of modular forms $M_2(\Gamma_0(D), \chi_D)$, We present two applications of our modular form. One is to show identities which involve eigen values of Hecke operators and solutions of the Pell equation corresponding to a quadratic field. For example, the simplest case is as follows.

Let $x = \frac{5}{4}(p+1) - \frac{1}{2}\sum_{\substack{t\in\mathbb{Z} \\ 4p-5t^2>0}} H(4p-5t^2)$, *where* $H(n)$ *denotes Hurwitz-Kronecker class number* (see §1), *then* x *together with an integer* y, *is an integral solution of the Pell equation* $x^2 - 5y^2 = 4p$ *for primes* $p \equiv \pm 1 \bmod 5$.

The other application is we can prove two types of congruences, one is Shimura type and the other Doi-Brumer type of eigen values a_p of Hecke operators by using numerical data for $D \leq 97$:

(1) $a_p \equiv \alpha + \alpha' \bmod l$ *where* $p = \alpha\alpha'$ *in* $\mathbb{Q}(\sqrt{D})$ *and* l *is a factor* $\mathrm{tr}(\varepsilon)$,

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which was proved by Shimura [5].

(2) $a_p \equiv p+1 \bmod \mathfrak{m}$ *where* $\mathfrak{m}$ *is a factor of the generalized Bernoulli number* B_{2,χ_D},

which was proved by Doi-Brumer[2].

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Notation Throughout the paper, D is a positive integer such that $D \equiv 1 \bmod 4$, K the real quadratic field $\mathbb{Q}(\sqrt{D})$, and $\mathcal{O}$ the ring of integers of K. For $x \in K$ x', $\mathrm{N}(x) = xx'$ and $\mathrm{tr}(x) = x+x'$ denote the conjugate, norm and trace of x respectively. We denote by $\chi_D(n)$ the character associated to K, namely $\chi_D(n) = (\frac{n}{D})$. For a complex number z, $\mathbf{e}(z)$ denotes $e^{2\pi iz}$. We let $\mathfrak{H}$ denote the complex upper half plane, and put $q = \mathbf{e}(z)$ for $z \in \mathfrak{H}$. For a positive integer M ,we let $\mathrm{SL}_2(\mathbb{Z})$ the group of 2×2 integral matrices of determinant 1, $\Gamma_0(M)$ the subgroup of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c \equiv 0 \bmod M$, and $\Gamma(M)$ the principal congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ with level M.

1. Statement of the theorem

For an even $k>0$, $M_k(\Gamma_0(D), \chi_D)$ denotes the space of modular forms of weight k, level D and Neben type χ_D, i.e. it is the space of functions $f : \mathfrak{H} \to \mathbf{C}$ satisfying

$$f\left(\frac{az+b}{cz+d}\right) = \chi_D(a)(cz+d)^k f(z) \quad \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(D)\right) \tag{1.1}$$

and which are holomorphic on $\mathfrak{H}$ and the cusps of $\Gamma_0(D)$. A function in $M_k(\Gamma_0(D), \chi_D)$ which vanishes at every cusp of $\Gamma_0(D)$ is called a cusp form, and we denote by $S_k(\Gamma_0(D), \chi_D)$ the subspace of cusp forms in $M_k(\Gamma_0(D), \chi_D)$. The space of the functions f satisfying (1.1) without the holomorphy condition, is denoted by $M_k^*(\Gamma_0(D), \chi_D)$. Such functions are called non-holomorphic modular forms.

For a positive integer n, let $H(n)$ denote the number of equivalence classes of all positive definite binary quadratic forms of discriminant $-n$, where the equivalence classes of $m(x^2+y^2)$ and $m(x^2+xy+y^2)$ are counted with multiplicity $\frac{1}{2}$ and $\frac{1}{3}$ respectively. In other words, $H(n)$ is given by

$$H(n) = \sum_{f^2|n} \frac{h(\Delta)}{w(\Delta)} \tag{1.2}$$

where $h(\Delta)$ (resp. $w(\Delta)$) denotes the class number (resp. half of the number of units) of the order of the discriminant Δ in the imaginary quadratic field $\mathbb{Q}(\sqrt{\Delta})$ with $\Delta = -n/f^2$ and the summation is over all positive integers f such that $-n/f^2 \equiv 0, 1 \bmod 4$. $H(n)$ is called Hurwitz-Kronecker class number. we put $H(0) = -\frac{1}{12}$ and $H(n) = 0$ for negative integers n conventionally. For small n, $H(n)$ is given by

n	0	3	4	7	8	11	12	15	16	19	20	23	24
$H(n)$	$-\frac{1}{12}$	$\frac{1}{3}$	$\frac{1}{2}$	1	1	1	$\frac{4}{3}$	2	$\frac{3}{2}$	1	2	3	2

Now for an integer $N > 0$ we define $J_D(N)$ as

$$J_D(N) = \sum_{\substack{t \in \mathbb{Z} \\ 4N - Dt^2 \geq 0}} H(4N - Dt^2), \tag{1.3}$$

then we obtain

Theorem 1 *For a prime D such that $D \equiv 1 \bmod 4$ the function*

$$\Phi_D(z) = -\frac{1}{12} + \sum_{N=1}^{\infty} J_D(N) q^N + \frac{1}{\mathrm{tr}(\varepsilon)} \sum_{\mathfrak{a}=(\alpha)} \mathrm{tr}(\alpha) q^{\mathrm{N}(\alpha)}$$

belongs to $M_2(\Gamma_0(D), \chi_D)$, where $\varepsilon > 1$ is the fundamental unit of $K = \mathbb{Q}(\sqrt{D})$, $\mathfrak{a}$ runs over all integral principal ideals of K, and α is the unique totally positive generator of $\mathfrak{a}$ such that $\varepsilon'^2 \leq \frac{\alpha}{\alpha'} < \varepsilon^2$.

The proof of this theorem will be completed in §5.

2. The modular form $h_D(z)$

We know by ([2] 2.4) that the function $\mathcal{F}(z)$ defined by

$$\mathcal{F}(z) = \sum_{n=0}^{\infty} H(n) q^n + y^{-1/2} \sum_{u \in \mathbb{Z}} \beta(4\pi u^2 y) q^{-u^2} \quad (y = \mathrm{Im}(z)), \tag{2.1}$$

satisfies

$$\mathcal{F}\left(\frac{az+b}{cz+d}\right) = \left(\frac{-1}{d}\right)\left(\frac{c}{d}\right)(cz+d)^{3/2} \mathcal{F}(z)$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4D) \quad (c \neq 0)$.

Here we mean:

$$\left(\frac{c}{d}\right)=\begin{cases}-\left(\dfrac{c}{|d|}\right) & c<0, d<0\\ \left(\dfrac{c}{|d|}\right) & \text{otherwise,}\end{cases}\qquad \left(\frac{-1}{d}\right)^{1/2}=\begin{cases}1 & d\equiv 1 \bmod 4\\ i & d\equiv 3 \bmod 4.\end{cases}$$

For a complex number w, we take the argument of $w^{1/2}$ so that $-\pi/2<\arg(w)\leq\pi/2$ and $w^{k/2}=(w^{1/2})^k$ $(k\in\mathbb{Z})$. The function $\beta(x)$ is defined by

$$\beta(x)=\frac{1}{16\pi}\int_1^\infty u^{-3/2}e^{-xu}du \qquad (\mathrm{Re}(x)\geq 0). \tag{2.2}$$

Now the function $\theta(z)=\sum_{t\in\mathbb{Z}}q^{t^2}$ satisfies

$$\theta\left(\frac{az+b}{cz+d}\right)=\left(\frac{-1}{d}\right)^{-1/2}\left(\frac{c}{d}\right)(cz+d)^{1/2}\theta(z)$$

for $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma_0(4)$. For a positive integer D, we put

$$f_D(z)=\mathcal{F}(z)\theta(Dz)=\sum_{N=-\infty}^{\infty}c_N(y)q^N. \tag{2.3}$$

then we have

$$\begin{aligned}f_D\left(\frac{az+b}{cz+d}\right)&=\mathcal{F}\left(\frac{az+b}{cz+d}\right)\theta\left(D\cdot\frac{az+b}{cz+d}\right)\\ &=\mathcal{F}\left(\frac{az+b}{cz+d}\right)\theta\left(\frac{aDz+bD}{c/D\cdot Dz+d}\right)\\ &=\left(\frac{-1}{d}\right)\left(\frac{c}{d}\right)(cz+d)^{3/2}\mathcal{F}(z)\\ &\quad\times\left(\frac{-1}{d}\right)^{-1/2}\left(\frac{c/D}{d}\right)(cz+d)^{1/2}\theta(Dz)\\ &=\chi_D(d)(cz+d)^2 f_D(z),\end{aligned}$$

for $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma_0(4D)$ $(c\neq 0)$, and $f_D(z+b)=f_D(z)$ for $b\in\mathbb{Z}$. So $f_D(z)$ belongs to $M_2^*(\Gamma_0(4D),\chi_D)$ and

$$c_N(y)=\sum_{\substack{t\in\mathbb{Z}\\ N-Dt^2\geq 0}}H(N-Dt^2)+y^{-1/2}\sum_{\substack{t,u\in\mathbb{Z}\\ Dt^2-u^2=N}}\beta(4\pi u^2 y). \tag{2.4}$$

Assume $N \equiv 2 \bmod 4$. Since $D \equiv 1 \bmod 4$, we see that $H(N - Dt^2) = 0$ and $Dt^2 - u^2 = N$ has no solution, so we have $c_N(y) = 0$. Hence by [2] Lemma 2, the function

$$\frac{1}{4}\sum_{r=1}^{4} f_D\left(\frac{z+r}{4}\right) = \sum_{N=-\infty}^{\infty} c_{4N}(\frac{1}{4}y)q^N \tag{2.5}$$

belongs to $M_2^*(\Gamma_0(D), \chi_D)$. Here $c_{4N}(\frac{1}{4}y)$ becomes

$$\begin{aligned} c_{4N}(\frac{1}{4}y) &= \sum_{\substack{t\in\mathbb{Z}\\ 4N-Dt^2\geq 0}} H(4N - Dt^2) + 2y^{-1/2} \sum_{\substack{t,u\in\mathbb{Z}\\ Dt^2-u^2=4N}} \beta(\pi u^2 y) \\ &= \sum_{\substack{t\in\mathbb{Z}\\ 4N-Dt^2\geq 0}} H(4N - Dt^2) + 2y^{-1/2} \sum_{\substack{\lambda\in\mathcal{O}\\ \lambda\lambda'=-N}} \beta(\pi \mathrm{tr}(\lambda)^2 y). \end{aligned}$$

Now we define by $h_D(z)$ the function given in (2.5).

Recalling the definition of $J_D(N)$ in (1.3), we obtain

Proposition 2 *The function*

$$\begin{aligned} h_D(z) &= -\frac{1}{12} + \sum_{N=1}^{\infty} J_D(N)q^N \\ &\quad + 2y^{-1/2} \sum_{N=-\infty}^{\infty} \sum_{\substack{\lambda\in\mathcal{O}\\ \lambda\lambda'=-N}} \beta(\pi(\lambda+\lambda')^2 y)q^N \end{aligned}$$

belongs to $M_2^*(\Gamma_0(D), \chi_D)$. *Here* $\beta(x)$ *is given by* (2.2).

3. Non-holomorphic modular form $\mathcal{Z}(z)$

Now we shall investigate the second term of $h_D(z)$ in Proposition 2. For $z \in \mathfrak{H}$, we define complex valued functions $X_z(\lambda, \lambda')$, $Y_z(\lambda, \lambda')$ and $Z_z(\lambda, \lambda')$ on $\mathbb{R}^2$ by

$$\begin{aligned} X_z(\lambda, \lambda') &= 2y^{-1/2}\beta(\pi(\lambda+\lambda')^2 y)\mathbf{e}(-\lambda\lambda' z), \\ Y_z(\lambda, \lambda') &= \begin{cases} \frac{1}{2}\min(|\lambda|, |\lambda'|)\mathbf{e}(-\lambda\lambda' z) & \text{if } \lambda\lambda' < 0, \\ 0 & \text{if } \lambda\lambda' \geq 0, \end{cases} \\ Z_z(\lambda, \lambda') &= X_z(\lambda, \lambda') - Y_z(\lambda, \lambda'). \end{aligned} \tag{3.1}$$

Then

Lemma 3 *We have*

$$\tilde{Z}_z(\mu,\mu)=z^{-2}Z_{-1/z}(\mu,\mu'),$$

where $\tilde{Z}_z$ denotes the Fourier transform of Z_z.

Proof. We denote by $\tilde{X}_z, \tilde{Y}_z$ and $\tilde{Z}_z$ Fourier transforms of X_z, Y_z and Z_z respectively. Now $X_z(\lambda,\lambda')=U_z(\lambda,-\lambda')$, $Y_z(\lambda,\lambda')=V_z(\lambda,-\lambda')$ and $Z_z(\lambda,\lambda')=W_z(\lambda,-\lambda')$, where U_z, V_z and W_z is defined by [4] Proposition 1.1, we have

$$\begin{aligned}\tilde{X}_z(\mu,\mu') &= \tilde{U}_z(\mu,-\mu')\\ &= z^{-2}U_{-1/z}(\mu,\mu')+8w^{-3/2}e(\mu\mu'/z)\beta(\pi(\mu+\mu')^2/w),\end{aligned}$$

$$\begin{aligned}\tilde{Y}_z(\mu,\mu') &= \tilde{V}_z(\mu,-\mu')\\ &= z^{-2}V_{-1/z}(\mu,\mu')+8w^{-3/2}e(\mu\mu'/z)\beta(\pi(\mu+\mu')^2/w),\end{aligned}$$

where we put $w=2z/i$. $(|\arg(w)|<\pi/2)$. The relation between $\tilde{U}_z$ (resp. $\tilde{V}_z$) and U_z (resp. V_z) has been stated in [4] Proposition 1.1. Hence $Z_z=X_z-Y_z$ and its Fourier transform $\tilde{Z}_z$ satisfies

$$\tilde{Z}_z(\mu,\mu')=z^{-2}Z_{-1/z}(\mu,\mu').$$

□

For $\nu\in\mathfrak{d}^{-1}$, we define functions of theta-series type

$$\mathcal{Z}_\nu(z)=\sum_{\lambda\in\mathcal{O}}Z_z(\lambda+\nu,\lambda'+\nu'), \tag{3.2}$$

where $\mathfrak{d}^{-1}$ is the inverse different $(1/\sqrt{D})$ of K. We obtain D distinct functions $\mathcal{Z}_\nu(z)$ with $\mathcal{Z}_0=\mathcal{Z}$. Now

$$\mathcal{Z}_\nu(z+1)=\mathbf{e}(-\nu\nu')\mathcal{Z}_\nu(z),$$

and by the Poisson summation formula

$$\mathcal{Z}_\nu(z)=D^{-1/2}\sum_{\mu\in\mathfrak{d}^{-1}}\tilde{Z}_z(\mu,\mu')\mathbf{e}(\mathrm{tr}(\mu\nu)).$$

Thus we obtain

$$z^{-2}\mathcal{Z}_\nu(-1/z) = z^{-2}D^{-1/2}\sum_{\mu\in\mathfrak{d}^{-1}}\tilde{Z}_z(\mu,\mu')\mathbf{e}(\mathrm{tr}(\mu\nu))$$

$$= D^{-1/2} \sum_{\mu \in \mathfrak{d}^{-1}} Z_z(\mu, \mu')\mathbf{e}(\mathrm{tr}(\mu\nu))$$
$$= D^{-1/2} \sum_{\mu \in \mathfrak{d}^{-1}/\mathcal{O}} \mathcal{Z}_\mu(z)\mathbf{e}(\mathrm{tr}(\mu\nu)).$$

For $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ we have

$$\mathcal{Z}_\nu | T = \mathbf{e}(-\nu\nu'), \quad \mathcal{Z}_\nu | J = D^{-1/2} \sum_{\mu \in \mathfrak{d}^{-1}/\mathcal{O}} \mathcal{Z}_\mu(z)\mathbf{e}(\mathrm{tr}(\mu\nu)), \tag{3.3}$$

where $\mathcal{Z}_\nu | \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ denotes the function $(cz+d)^{-2}\mathcal{Z}_\nu\left(\frac{az+b}{cz+d}\right)$.

For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(D)$, we put $R = T^aJT^dJT^aJ$ then $R \equiv \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ mod D. We can choose $x \in \mathbb{Z}$ so that $A = A_1T^xR$ with $A_1 \in \Gamma(D)$ the principal congruence subgroup.

Proposition 4 *We have*

$$\mathcal{Z}_\nu | R = \chi_D(d)\mathcal{Z}_{-a\nu'}.$$

In particular, $\mathcal{Z}_0(z)$ belongs to $M_2^*(\Gamma_0(D), \chi_D)$.

Proof. From (3.3) we find

$$\mathcal{Z}_\nu | T^aJ = D^{-1/2}\mathbf{e}(-a\mu\mu') \sum_\mu \mathcal{Z}_\mu \mathbf{e}(\mathrm{tr}(\mu\nu)).$$

Therefore

$$\begin{aligned}
\mathcal{Z}_\nu | R &= D^{-3/2} \sum_\mu \sum_\lambda \sum_\kappa \mathbf{e}(\mathrm{tr}(\mu\nu + \lambda\mu + \kappa\lambda) - a\nu\nu' - d\mu\mu' - a\lambda\lambda')\mathcal{Z}_\kappa \\
&= D^{-3/2} \sum_\kappa \mathcal{Z}_\kappa \sum_\lambda \sum_\mu \mathbf{e}(-d\mathrm{N}(\mu - a\lambda' - a\nu') + \mathrm{tr}(a\lambda\nu' + \kappa\lambda)) \\
&= D^{-1}\chi_D(-d) \sum_\kappa \mathcal{Z}_\kappa \sum_\lambda \mathbf{e}(\mathrm{tr}(\lambda(\kappa + a\nu')) \\
&= \chi_D(d)\mathcal{Z}_{-a\nu'}.
\end{aligned}$$

Here we used the well known property of the Gauss sum,

$$\sum_{\mu \in \mathfrak{d}^{-1}/\mathcal{O}} \mathbf{e}(-d\mathrm{N}(\mu - a\lambda' - a\nu')) = D^{1/2}\chi_D(-d).$$

In particular, putting $\nu = 0$ we obtain $\mathcal{Z}_0|R = \chi_D(d)\mathcal{Z}_0$, which means $\mathcal{Z}_0|A = \chi_D(d)\mathcal{Z}_0$ for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(D)$ since we know by [3] that $\mathcal{Z}_\nu|A_1 = \mathcal{Z}_\nu$ for $A_1 \in \Gamma(D)$. □

4. The modular form $\Phi_D(z)$

Recalling definitions (3.1) and (3.2), we have

$$\mathcal{Z}_0(z) = 2y^{-1/2} \sum_{N=-\infty}^{\infty} \sum_{\substack{\lambda \in \mathcal{O} \\ \lambda\lambda' = -N}} \beta(\pi(\lambda + \lambda')^2 y) q^N - \frac{1}{2} \sum_{N=1}^{\infty} \sum_{\substack{\lambda \in \mathcal{O} \\ \lambda\lambda' = -N}} \min(|\lambda|, |\lambda'|) q^N. \tag{4.1}$$

Now we define the function $\Phi_D(z)$ by

$$\Phi_D(z) = h_D(z) - \mathcal{Z}_0(z). \tag{4.2}$$

Then by Proposition 2 and Proposition 4, we see $\Phi_D(z)$ belongs to $M_2^*(\Gamma_0(D), \chi_D)$. Further we can prove

Proposition 5 *The function*

$$\Phi_D(z) = -\frac{1}{12} + \sum_{N=1}^{\infty} J_D(N) q^N + \frac{1}{2} \sum_{N=1}^{\infty} \sum_{\substack{\lambda \in \mathcal{O} \\ \lambda\lambda' = -N}} \min(|\lambda|, |\lambda'|) q^N.$$

belongs to $M_2(\Gamma_0(D), \chi_D)$.

Proof. The N-th Fourier coefficient of $\Phi_D(z)$ is independent of y and is $O(N^r)$ for some $r > 0$, $\Phi_D(z)$ is holomorphic on $\mathfrak{H}$, and also holomorphic at the cusps of $\Gamma_0(D)$, and $\Phi_D(z)$ is $O(y^{-r})$ as $y \to 0$, which implies $\Phi_D(z)$ belongs to $M_2(\Gamma_0(D), \chi_D)$. □

5. Proof of Theorem 1

For a principal integral ideal $\mathfrak{a}$ of $K = \mathbb{Q}(\sqrt{D})$, where we assume D is a prime number, we put

$$r(\mathfrak{a}) = \sum_{\substack{(\lambda) = \mathfrak{a} \\ \lambda\lambda' < 0}} \min(|\lambda|,\ |\lambda'|). \tag{5.1}$$

Then

Lemma 6 *We have*

$$r(\mathfrak{a}) = 2\frac{\mathrm{tr}(\alpha)}{\mathrm{tr}(\varepsilon)},$$

where $\varepsilon > 1$ *is the fundamental unit of* K *and* α *is the unique totally positive generator of* $\mathfrak{a}$ *such that* $\varepsilon'^2 \le \frac{\alpha}{\alpha'} < \varepsilon^2$.

Proof. Since D is a prime number congruent to 1 modulo 4, we have $\mathrm{N}(\varepsilon) = -1$. So we can take $\lambda_0 \in \mathcal{O}$ so that $\mathfrak{a} = (\lambda_0)$, $\lambda_0 > 0$ and $\lambda_0' < 0$ by multiplying a unit to λ_0 suitably. Such λ_0 is unique under the condition $\varepsilon_0^2 \le |\frac{\lambda_0}{\lambda_0'}| < 1$ where ε_0 is the generator of the group of totally positive units which satisfies $0 < \varepsilon_0 < 1$ and $\varepsilon_0' > 1$. For any λ such that $\mathfrak{a} = (\lambda)$, $\lambda > 0$ and $\lambda' < 0$ we can write $\lambda = \lambda_0\varepsilon_0^n$ $(n \in \mathbb{Z})$, and $|\frac{\lambda}{\lambda'}| = |\frac{\lambda_0\varepsilon_0^n}{\lambda_0'\varepsilon_0'^n}| = |\frac{\lambda_0}{\lambda_0'}|\varepsilon_0^{2n}$. Since $\min(|\lambda|, |\lambda'|)$ is equal to $|\lambda|$ or $|\lambda'|$ according as $n \ge 0$ or $n < 0$, we have

$$\begin{aligned}&\{\min(|\lambda|, |\lambda'|) \mid \mathfrak{a} = (\lambda),\ \lambda > 0\ \lambda' < 0\} \\ &\qquad = \{\lambda_0\varepsilon_0^m \mid m \ge 0\} \bigcup \{-\lambda_0'\varepsilon_0^n \mid n > 0\}.\end{aligned}$$

Since we have two choices of the signature of λ in (5.1)

$$\begin{aligned} r(\mathfrak{a}) &= 2\sum_{m=0}^{\infty} \lambda_0\varepsilon_0^m - 2\sum_{n=1}^{\infty} \lambda_0'\epsilon_0^n \\ &= 2\frac{\lambda_0}{1-\varepsilon_0} - 2\frac{\lambda_0'\varepsilon_0}{1-\varepsilon_0} \\ &= 2\frac{\mathrm{tr}(\lambda_0\varepsilon)}{\mathrm{tr}(\varepsilon)} \qquad (\varepsilon_0 = \varepsilon^{-2}).\end{aligned}$$

Therefore $\alpha = \lambda_0\varepsilon$ satisfies the condition in Lemma 6. □

Remark. We can easily verify
(1) $r(\mathfrak{a}) = r(\mathfrak{a}')$ ($\mathfrak{a}'$: the conjugate of $\mathfrak{a}$).
(2) $r((c)) = \frac{2|c|}{\mathrm{tr}(\varepsilon)}$ $(c \in \mathbb{Z})$.
(3) $r((c\sqrt{D})) = \frac{2b|c|D}{a}$ $(\varepsilon = \frac{a+b\sqrt{D}}{2},\quad c \in \mathbb{Z})$.
Now combining Lemma 6 with Proposition 5, we conclude the function $\Phi_D(z)$ has the Fourier expansion stated in Theorem 1. This completes our proof of Theorem 1.

6. Numerical examples

For each positive integer n, we define the Hecke operator on $S_k(\Gamma_0(D), \chi_D)$ by

$$(f|T_n)(z) = n^{k-1} \cdot \sum_{\substack{d>0 \\ ad=n}} \sum_{d=0}^{n-1} \chi_D(a) f((az+b)/d) d^{-k},$$

for $f(z) \in S_k(\Gamma_0(D), \chi_D)$. A common eigen function $f(z) = \sum_{m=1}^{\infty} a_m q^m$ of T_n for all n is called a primitive form if $a_1 = 1$. It is well known that n-th Fouriercoefficient a_n of $f(z)$ is an eigen value of of T_n for a primitive form $f(z)$. For a prime $D \equiv 1 \bmod 4$, the space M_2 $(\Gamma_0(D), \chi_D)$ is the direct sum of $S_2(\Gamma_0(D), \chi_D)$ and the Eisenstein space spanned by two Eisenstein series $E_{D,1}(z)$ and $E_{D,2}(z)$:

$$E_{D,1}(z) = \sum_{n=1}^{\infty} \Big(\sum_{t|n} \chi_D(n/t) t \Big) q^n,$$

and

$$E_{D,2}(z) = -\frac{1}{4} B_{2,\chi_D} + \sum_{n=1}^{\infty} \Big(\sum_{t|n} \chi_D(t) t \Big) q^n,$$

where B_{2,χ_D} is the second generalized Bernoulli number, or

$$B_{2,\chi_D} = \frac{4}{\chi_D(2) - 4} \cdot \sum_{k=1}^{(D-1)/2} \chi_D(k) k.$$

The function $\Phi_D(z)$ is expressed as a linear combination of cusp forms and these Eisenstein series. In this section we shall give two types of numerical applications stated in Introuction.

First, since p-th Fourier coefficients of $\Phi_D(z)$ contain $\mathrm{tr}(\alpha)$ where $p = \mathrm{N}(\alpha)$, we can express $\mathrm{tr}(\alpha)$ by eigen values of Hecke operators. We only state here the simplest case, namely we take $D = 5$. The dimension of $S_2(\Gamma_0(5), \chi_5)$ is 0. Thus $\Phi_5(z)$ is a linear combination of $E_{5,1}(z)$ and $E_{5,2}$. We find easily

$$\Phi_5(z) = \frac{25}{12} E_{5,1}(z) + \frac{5}{12} E_{5,2}(z). \tag{6.1}$$

We know the fundamental unit $\varepsilon > 1$ of $K = \mathbb{Q}(\sqrt{5})$ is $\frac{1+\sqrt{5}}{2}$. Hence comparing p-th coefficients of both sides of (6.1) for primes $\chi_5(p) = 1$, we obtain

$$\mathrm{tr}(\alpha) = \frac{5}{4}(p+1) - \frac{1}{2}J_5(p),$$

where $\alpha = \frac{x+y\sqrt{5}}{2}$ is the generator of an ideal $\mathfrak{a}$ of norm p in K and α satisfies the condition in Theorem 1. In other words,

Proposition 7 *For primes* $p \equiv \pm 1 \bmod 5$, *we put* x *as*

$$x = \frac{5}{4}(p+1) - \frac{1}{2} \sum_{\substack{t \in \mathbb{Z} \\ 4p-5t^2>0}} H(4p-5t^2).$$

Then with a suitable integer y, (x, y) *gives an integral solution of the Pell equation* $x^2 - 5y^2 = 4p$.

Next as second applications, we take $D = 61$. The dimension of $S_2(\Gamma_0(61), \chi_{61})$ is 4. The coefficients a_n of a primitive form $f(z) = \sum_{n=1}^{\infty} a_n q^n$ is contained in $\mathbb{Q}(\sqrt{3})$ or $\mathbb{Q}(\sqrt{-4-\sqrt{3}})$ according as $\chi_{61}(n) = 1$ or not. The space $S_2(\Gamma_0(61), \chi_{61})$ is spanned by f, $\bar{f} = \sum_{n=1}^{\infty} \bar{a_n} q^n$, $f^\sigma = \sum_{n=1}^{\infty} a_n^\sigma q^n$ and $\bar{f}^\sigma = \sum_{n=1}^{\infty} \bar{a_n}^\sigma q^n$ where $\bar{\ }$ denotes complex conjugation and σ denotes the element of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ which is nontrivial on $\mathbb{Q}(\sqrt{3})$ (see the table for $D = 61$ in §7). To avoid the ambiguity of the choice of the conjugates of a_n, we take the form $f(z) = \sum_{n=1}^{\infty} a_n q^n$ such that $a_1 = 1, a_2 = \sqrt{-4-\sqrt{3}}$ and $a_3 = -1-\sqrt{3}$ and fix f hereafter. We put $K = \mathbb{Q}(\sqrt{61})$. Now we have two types of the congruences for a_p.

Proposition 8

(1) *For each prime* p *such that* $\chi_{61}(p) = 1$, *we have*

$$a_p \equiv \alpha_1 + \alpha_1' \bmod \mathfrak{c}_{13}$$

where $\alpha_1 \in \mathcal{O}$ *is totally positive and* $p = \alpha_1 \alpha_1'$, $\mathfrak{c}_{13} = (4+\sqrt{3})$ *is the ideal of norm* 13 *in* $\mathbb{Q}(\sqrt{3})$.

(2) *For primes* $p \neq 61$, *we have*

$$a_p \equiv p+1 \bmod \mathfrak{c}_{11} \qquad \text{for } \chi_{61}(p) = 1$$

and

$$a_p \equiv p-1 \bmod \mathfrak{C}_{11} \quad \text{for } \chi_{61}(p) = -1$$

where $\mathfrak{c}_{11} = (1 - 2\sqrt{3})$ *is the ideal of norm* 11 *in* $\mathbb{Q}(\sqrt{3})$ *and* $\mathfrak{C}_{11}$ *an ideal of norm* 11 *in* $\mathbb{Q}(\sqrt{-4-\sqrt{3}})$ *dividing* $\mathfrak{c}_{11}$.

Proof. Since $f, \bar{f}, f^\sigma$ and $\bar{f}^\sigma$ span the space $S_2(\Gamma_0(61), \chi_{61})$, we see $F_1 = \frac{1}{4}(f+\bar{f}+f^\sigma+\bar{f}^\sigma)$, $F_2 = \frac{1}{4\sqrt{3}}(f+\bar{f}-f^\sigma-\bar{f}^\sigma)$, $F_3 = \frac{1}{4\theta_1}(f-\bar{f})+\frac{1}{4\theta_2}(f^\sigma-\bar{f}^\sigma)$ and $F_4 = \frac{1}{4\theta_1}(f-\bar{f})-\frac{1}{4\theta_2}(f^\sigma-\bar{f}^\sigma)$ ($\theta_1 = \sqrt{-4-\sqrt{3}}, \theta_2 = \sqrt{-4+\sqrt{3}}$) form a basis of $\mathbb{Q}$-rational elements of $S_2(\Gamma_0(61), \chi_{61})$. We can express $\Phi_{61}(z)$ as a $\mathbb{Q}$-linear combination of $E_{61,i}$ and F_j, and we write

$$\Phi_{61}(z) = \gamma_1 E_{61,1} + \gamma_2 E_{61,2} + \sum_{j=1}^{4} \delta_j F_j \qquad \text{for } \gamma_i, \delta_j \in \mathbb{Q}. \tag{6.2}$$

The coefficients of $\Phi_{61}(z) = \sum_{n=0}^{\infty} b(n)q^n$ for small n's are calculated as follows :

$$\begin{aligned}\Phi_{61}(z) \;&=\; -\frac{1}{12} + \frac{43}{78}q^1 + q^2 + \frac{28}{13}q^3 + \frac{125}{78}q^4 + \frac{32}{13}q^5 \\ &\quad + 2q^6 + 2q^7 + 3q^8 + \cdots .\end{aligned}$$

Comparing the coefficients of q^n of both sides of (6.2) for several n, we find $\gamma_1 = \frac{61}{132}$, $\gamma_2 = \frac{1}{132}$, $\delta_1 = \frac{35}{429}$, $\delta_2 = -\frac{51}{143}$, $\delta_3 = \frac{6}{11}$ and $\delta_4 = -\frac{8}{11}$. (It is sufficient to know exact values of $b(n)$ for $n = 1, 2, 3, 5, 7$ to determine γ_i, δ_j.) First we consider a prime p with $\chi_{61}(p) = 1$, then we find an integer $\alpha \in \mathcal{O}$ of norm p, which satisfies the condition stated in Theorem 1. The p-th coefficient of $\Phi_{61}(z)$ is $b(p) = J_{61}(p) + 2\frac{\mathrm{tr}(\alpha)}{\mathrm{tr}(\varepsilon)}$, where ε is the fundamental unit $\varepsilon = \frac{39+5\sqrt{61}}{2}$. Since $\chi_{61}(p) = 1$, a_p is contained in $\mathbb{Q}(\sqrt{3})$ and therefore we can write $a_p = u_p + v_p\sqrt{3}$ ($u_p, v_p \in \mathbb{Z}$). Thus we obtain the equality

$$\begin{aligned}&2 \cdot 3^2 \cdot 11 \cdot 13 J_{61}(p) + 2^2 \cdot 3 \cdot 11 \mathrm{tr}(\alpha) \\ &\qquad = 3 \cdot 13 \cdot 31(p+1) + 2 \cdot 3 \cdot 5 \cdot 7 u_p - 2 \cdot 3^3 \cdot 17 v_p.\end{aligned}$$

Hence we have

$$u_p + 9v_p \equiv \mathrm{tr}(\alpha) \bmod 13,$$

and

$$u_p + 6v_p \equiv p + 1 \bmod 11.$$

These congruences are equivalent to

$$a_p \equiv \mathrm{tr}(\alpha) \bmod \mathfrak{c}_{13},$$

and

$$a_p \equiv p + 1 \bmod \mathfrak{c}_{11}.$$

respectively. For any totally positive element $\alpha_1 \in \mathcal{O}$ of norm p, we can put $\alpha_1 = \alpha\varepsilon^{2n}$ for some $n \in \mathbb{Z}$. Since $\varepsilon^2 \equiv 1 \bmod 13$, this implies (1). Similar arguments are also applicable to primes p satisfying $\chi_{61}(p) = -1$, and we obtains the results. □

Remark 1. Congruences of type (1) were proved by Shimura ([5] 7.7) which was related to the theory of construction of class fields over real quadratic fields. Congruences of the type (2) were proved by Doi-Brumer ([2] 7.5.).

Remark 2. Similarly, we can prove congruences of the same types as in Proposition 8 for $29 \leq D \leq 97$, on which we don't go further.

7. Eigen values of Hecke operators for $S_2(\Gamma_0(D), \chi_D)$

Once we expressed $\Phi_D(z)$ as a linear combination of the Eisenstein series $E_{D,1}(z)$, $E_{D,2}(z)$ and cusp forms in the form of (6.2) by virtue of Eichler-Selberg's trace formula, we are able to give Fourier coeffients a_p of primitive forms $f(z) = \sum_{n=1}^{\infty} a_n q^n$ in $S_2(\Gamma_0(D), \chi_D)$ for large p by calculating Fourier coefficients of $\Phi_D(z)$. We list here values of a_p for all primes $p \leq 97$ including a_D, for the prime level $D \leq 97$ (We note $a_D\bar{a_D} = D$).

$D = 29, \dim S_2(29, \chi_{29}) = 2$

p	χ_{29}	a_p	p	χ_{29}	a_p	p	χ_{29}	a_p
2	$-$	$\sqrt{-5}$	29	0	$-3+2\sqrt{-5}$	67	$+$	8
3	$-$	$-\sqrt{-5}$	31	$-$	$3\sqrt{-5}$	71	$+$	0
5	$+$	-3	37	$-$	0	73	$-$	0
7	$+$	2	41	$-$	$-2\sqrt{-5}$	79	$-$	$-3\sqrt{-5}$
11	$-$	$\sqrt{-5}$	43	$-$	$-3\sqrt{-5}$	83	$+$	-6
13	$+$	-1	47	$-$	$\sqrt{-5}$	89	$-$	$-2\sqrt{-5}$
17	$-$	$-2\sqrt{-5}$	53	$+$	-9	97	$-$	$6\sqrt{-5}$
19	$-$	0	59	$+$	6			
23	$+$	6	61	$-$	$6\sqrt{-5}$			

$D = 37, \dim S_2(37, \chi_{37}) = 2$

p	χ_{37}	a_p	p	χ_{37}	a_p	p	χ_{37}	a_p
2	$-$	$2\sqrt{-1}$	29	$-$	$-4\sqrt{-1}$	67	$+$	-12
3	$+$	-1	31	$-$	0	71	$+$	-3
5	$-$	$-2\sqrt{-1}$	37	0	$-1+6\sqrt{-1}$	73	$+$	9
7	$+$	3	41	$+$	-3	79	$-$	$6\sqrt{-1}$
11	$+$	-3	43	$-$	$-6\sqrt{-1}$	83	$+$	9
13	$-$	$-6\sqrt{-1}$	47	$+$	3	89	$-$	$-14\sqrt{-1}$
17	$-$	$2\sqrt{-1}$	53	$+$	9	97	$-$	$12\sqrt{-1}$
19	$-$	$6\sqrt{-1}$	59	$-$	$-4\sqrt{-1}$			
23	$-$	$4\sqrt{-1}$	61	$-$	0			

$D = 41, \dim S_2(41, \chi_{41}) = 2$

p	χ_{41}	a_p	p	χ_{41}	a_p	p	χ_{41}	a_p
2	$+$	-1	29	$+$	$-4\sqrt{-2}$	67	$-$	$-6\sqrt{-2}$
3	$-$	$2\sqrt{-2}$	31	$+$	-8	71	$-$	$6\sqrt{-2}$
5	$+$	2	37	$+$	2	73	$+$	14
7	$-$	$-2\sqrt{-2}$	41	$-$	$-3+4\sqrt{-2}$	79	$-$	$-2\sqrt{-2}$
11	$-$	$2\sqrt{-2}$	43	$+$	4	83	$+$	-12
13	$-$	$-4\sqrt{-2}$	47	$-$	$-2\sqrt{-2}$	89	$-$	$8\sqrt{-2}$
17	$-$	0	53	$-$	$4\sqrt{-2}$	97	$-$	0
19	$-$	$2\sqrt{-2}$	59	$+$	4			
23	$+$	0	61	$+$	-2			

$D = 53, \dim S_2(53, \chi_{53}) = 4$ (We put $\theta_1 = \sqrt{-3+\sqrt{2}}$.)

p	χ_{53}	a_p	p	χ_{53}	a_p	p	χ_{53}	a_p
2	$-$	θ_1	29	$+$	$-3+3\sqrt{2}$	67	$-$	$-(6+3\sqrt{2})\theta_1$
3	$-$	$(-1-\sqrt{2})\theta_1$	31	$-$	$-3\sqrt{2}\theta_1$	71	$-$	$(3+\sqrt{2})\theta_1$
5	$-$	$\sqrt{2}\theta_1$	37	$+$	$7-\sqrt{2}$	73	$-$	$-3\sqrt{2}\theta_1$
7	$+$	$-2-\sqrt{2}$	41	$-$	$-2\theta_1$	79	$-$	$(3+3\sqrt{2})\theta_1$
11	$+$	$3\sqrt{2}$	43	$+$	$-2+\sqrt{2}$	83	$-$	$(3+4\sqrt{2})\theta_1$
13	$+$	$1-2\sqrt{2}$	47	$+$	6	89	$+$	6
17	$+$	-3	53	0	$-3-3\sqrt{2}+$ $(2-\sqrt{2})\theta_1$	97	$+$	$7+7\sqrt{2}$
19	$-$	$-3\theta_1$	59	$+$	$-6\sqrt{2}$			
23	$-$	θ_1	61	$-$	$(6+3\sqrt{2})\theta_1$			

$D = 61, \dim S_2(61, \chi_{61}) = 4$ (We put $\theta_2 = \sqrt{-4-\sqrt{3}}$.)

p	χ_{61}	a_p	p	χ_{61}	a_p	p	χ_{61}	a_p
2	$-$	θ_2	29	$-$	$(1-\sqrt{3})\theta_2$	67	$-$	$(-6+\sqrt{3})\theta_2$
3	$+$	$-1-\sqrt{3}$	31	$-$	$(3-\sqrt{3})\theta_2$	71	$-$	$-(1+\sqrt{3})\theta_2$
5	$+$	$\sqrt{3}$	37	$-$	$(-3+\sqrt{3})\theta_2$	73	$+$	$4-3\sqrt{3}$
7	$-$	$\sqrt{3}\theta_2$	41	$+$	$-3+2\sqrt{3}$	79	$-$	$(3-4\sqrt{3})\theta_2$
11	$-$	$-\theta_2$	43	$-$	$(3+\sqrt{3})\theta_2$	83	$+$	$6+2\sqrt{3}$
13	$+$	3	47	$+$	$6-4\sqrt{3}$	89	$-$	$(-2+4\sqrt{3})\theta_2$
17	$-$	$(-1+\sqrt{3})\theta_2$	53	$-$	$(-4+2\sqrt{3})\theta_2$	97	$+$	$-2\sqrt{3}$
19	$+$	$3+\sqrt{3}$	59	$-$	$-(2+\sqrt{3})\theta_2$			
23	$-$	$(1-2\sqrt{3})\theta_2$	61	0	$2+3\sqrt{3}+(3-\sqrt{3})\theta_2$			

$D = 73, \dim S_2(73, \chi_{73}) = 4$ (We put $\theta_3 = \sqrt{(-19+\sqrt{5})/2}$.)

p	χ_{73}	a_p	p	χ_{73}	a_p	p	χ_{73}	a_p
2	$+$	$(-1+\sqrt{5})/2$	29	$-$	$-(1+\sqrt{5})/2\cdot\theta_3$	67	$+$	$-3-4\sqrt{5}$
3	$+$	$(1+\sqrt{5})/2$	31	$-$	$2\theta_3$	71	$+$	$(-9+3\sqrt{5})/2$
5	$-$	θ_3	37	$+$	5	73	0	$-4-2\sqrt{5}+(3-\sqrt{5})/2\cdot\theta_3$
7	$-$	$-(1+\sqrt{5})/2\cdot\theta_3$	41	$+$	$5-\sqrt{5}$	79	$+$	$(21+\sqrt{5})/2$
11	$-$	$(1-\sqrt{5})/2\cdot\theta_3$	43	$-$	$-(5+\sqrt{5})/2\cdot\theta_3$	83	$-$	$-\sqrt{5}\theta_3$
13	$-$	$(-1+\sqrt{5})/2\cdot\theta_3$	47	$-$	$(5+\sqrt{5})/2\cdot\theta_3$	89	$+$	$-8+3\sqrt{5}$
17	$-$	$(1+\sqrt{5})/2\cdot\theta_3$	53	$-$	$(-3+\sqrt{5})/2\cdot\theta_3$	97	$+$	$(-1+3\sqrt{5})/2$
19	$+$	$1-2\sqrt{5}$	59	$-$	$-2\theta_3$			
23	$+$	$(5+3\sqrt{5})/2$	61	$+$	$(-7+3\sqrt{5})/2$			

$D = 89, \dim S_2(89, \chi_{89}) = 6$
(η satisfies $\eta^3 + \eta^2 - 3\eta - 1 = 0$, and we put $\mu = \sqrt{-3 + \eta - \eta^2}$.)

p	χ_{89}	a_p	p	χ_{89}	a_p
2	$+$	η	43	$-$	$-(2+\eta)\mu$
3	$-$	μ	47	$+$	$1+4\eta+\eta^2$
5	$+$	$2-\eta^2$	53	$+$	$3+\eta+\eta^2$
7	$-$	$(2-\eta-\eta^2)\mu$	59	$-$	$(-2+3\eta+\eta^2)\mu$
11	$+$	$-3+2\eta+\eta^2$	61	$-$	$(-5+\eta^2)\mu$
13	$-$	$(-3+\eta^2)\mu$	67	$+$	$3-4\eta-3\eta^2$
17	$+$	$6-3\eta-2\eta^2$	71	$+$	$-2+4\eta+4\eta^2$
19	$-$	$(-3+\eta+\eta^2)\mu$	73	$+$	$-10+6\eta+3\eta^2$
23	$-$	$(-1+\eta+\eta^2)\mu$	79	$+$	$4-6\eta-4\eta^2$
29	$-$	$(3-2\eta-\eta^2)\mu$	83	$-$	$(-4-\eta+\eta^2)\mu$
31	$-$	$(2\eta+\eta^2)\mu$	89	0	$-4-4\eta+\eta^2+(5-\eta^2)\mu$
37	$-$	$-(2+2\eta)\mu$	97	$+$	$-8-\eta+2\eta^2$
41	$-$	$(3-\eta^2)\mu$			

$D = 97, \dim S_2(97, \chi_{97}) = 6$
(ξ satisfies $\xi^3 - 3\xi + 1 = 0$, and we put $\nu = \sqrt{-3 - 4\xi - 3\xi^2}$.)

p	χ_{97}	a_p	p	χ_{97}	a_p
2	$+$	$-\xi$	43	$+$	$9+2\xi-3\xi^2$
3	$+$	$-2+\xi+\xi^2$	47	$+$	$-11-2\xi+4\xi^2$
5	$-$	ν	53	$+$	$8+\xi-4\xi^2$
7	$-$	$(2-\xi^2)\nu$	59	$-$	$(7-2\xi^2)\nu$
11	$+$	$-3+\xi+2\xi^2$	61	$+$	$-5+4\xi+5\xi^2$
13	$-$	$(-3+\xi+\xi^2)\nu$	67	$-$	$-\xi^2\nu$
17	$-$	$(1-\xi)\nu$	71	$-$	$\xi^2\nu$
19	$-$	$(2-\xi-\xi^2)\nu$	73	$+$	$9-3\xi-7\xi^2$
23	$-$	$(3-\xi-\xi^2)\nu$	79	$+$	$-8+5\xi+2\xi^2$
29	$-$	$(2+\xi-\xi^2)\nu$	83	$-$	$(-6+2\xi^2)\nu$
31	$+$	$-2\xi+\xi^2$	89	$+$	$8-2\xi-\xi^2$
37	$-$	$(-3-\xi+\xi^2)\nu$	97	0	$6-\xi-5\xi^2+(4-\xi^2)\nu$
41	$-$	$(2-\xi^2)\nu$			

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Division of Mathematics
Department of Fundamental Sciences
Faculty of Integrated Human Studies
Kyoto University
Kyoto 606-01, Japan
E-mail: yamauchi@math.h.kyoto-u.ac.jp