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On asymptotic behaviors for wave equations with a nonlinear dissipative term in $\mathbf{R}^N$

(Dedicated to Professor Kôji Kubota on his 60th birthday)

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Abstract. In this paper we investigate the asymptotic behavior of solutions to the Cauchy problem of the equation

$$w_{tt}(t)-\Delta w(t)+\lambda w(t)+\beta(x,t,w_t)w_t(t)=0$$

for $(x,t)\in\mathbf{R}^N\times(0,\infty)$. Here Δ is the N-demensional Laplacian and $\lambda\geq 0$. $\beta(x,t,w_t)w_t$ represents a dissipative term of the form

$$\beta(x,t,w_t)=|w_t(x,t)|^{\rho-1}\quad(\rho>1),\ \text{or}$$

$$\beta(x,t,w_t)=(V_\gamma*|w_t(t)|^2)(x)=\int_{\mathbf{R}^N}V_\gamma(x-y)|w_t(y,t)|^2dy$$

with $V_\gamma(x)=|x|^{-\gamma}\quad(0<\gamma<N)$. We prove that the solution to above equation behaves like a solution to the free wave equation as $t\to\infty$ when its energy remains positive.

Key words: wave equations, nonlinear dissipative term.

1. Introduction

In our previous paper [9] we are concerned with the energy decay and non-decay problems of solutions to the wave equation

$$w_{tt}(t)-\Delta w(t)+\lambda w(t)+\beta(x,t,w_t)w_t(t)=0 \tag{1.1}$$

for $(x,t)\in\mathbf{R}^N\times(0,\infty)$ with initial data

$$w(x,0)=w_1(x)\quad\text{and}\quad w_t(x,0)=w_2(x),\quad x\in\mathbf{R}^N. \tag{1.2}$$

Here $w(t)=w(x,t)$ is a real valued function, $w_t=\partial w/\partial t$, $w_{tt}=\partial^2 w/\partial t^2$, Δ is the N-demensional Laplacian, $\lambda\geq 0$ and $\beta(x,t,w_t)w_t$ represents a dissipative term of the form

$$\beta(x,t,w_t)=b(x,t)|w_t(x,t)|^{\rho-1} \tag{1.3}$$

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with $b(x,t) \geq 0$ and $\rho > 1$, or

$$\beta(x,t,w_t) = (V_\gamma * |w_t(t)|^2)(x) = \int_{\mathbf{R}^N} V_\gamma(x-y)|w_t(y,t)|^2 dy \quad (1.4)$$

with $V_\gamma(x) = |x|^{-\gamma}$ $(0 < \gamma < N)$. Our purpose is to investigate the asymptotic bvehavior of the energy to equation (1.1) when it does not decay as $t \to \infty$.

In order to state our results, we introduce our notations: L^p $(1 \leq p \leq \infty)$ is the usual space of all L^p-functions in $\mathbf{R}^N$; if X is a Banach space and $I \subset \mathbf{R}$ is an interval, then by C^k $(I;X)$, C_L $(I;X)$ and L^p $(I;X)$ we mean the spaces of all X-valued C^k-functions, locally Lipschitz functions and L^p-functions on I, respectively; $H_p^{k,s}$ is the Sobolev space for $1 \leq p \leq \infty$, $k \in \mathbf{R}$ and $s > -N$ with norm

$$\|u\|_{H_p^{k,s}} = \|\mathfrak{F}^{-1}((1+|\xi|^2)^{k/2}|\xi|^s \hat{u}(\xi))\|_{L^p},$$

were $\hat{}$ denote the Fourier transformation and $\mathfrak{F}^{-1}$ is its inverse; especially we denote by $H^{k,s}$ in case $p=2$, H_p^k in case $s=0$ and H^k in case $p=2$ and $s=0$; E is the space of pairs $f=(f_1,f_2)$ of functions such that

$$\|f\|_E^2 = \frac{1}{2}\{\|\nabla f_1\|_{L^2}^2 + \|f_2\|_{L^2}^2 + \lambda \|f_1\|_{L^2}^2\} < \infty,$$

where $\nabla f_1 = (\partial_1 f_1, \cdots, \partial_N f_1)$ with $\partial_j = \partial/\partial x_j$. This norm is called the energy of the equation (1.1).

We note that the equation (1.1) can be written in the form

$$iW_t(t) - AW(t) + BW(t) = 0, \quad t \in (0,\infty) \quad (1.5)$$

with initial data

$$W(0) = W_0 = {}^t(w_1, w_2), \quad (1.6)$$

where

$$W(t) = \begin{pmatrix} w(t) \\ w_t(t) \end{pmatrix}, \quad A = -i\begin{pmatrix} 0 & -1 \\ -\Delta + \lambda & 0 \end{pmatrix}$$

and

$$BW(t) = i\begin{pmatrix} 0 \\ \beta(x,t,w_t)w_t \end{pmatrix}.$$

Since A is a selfadjoint operator in E, $U_0(t) = \exp(-iAt)$ become a one parameter unitary group in E. And (1.5) and (1.6) can be rewritten in the form of the integral equation

$$W(t) = U_0(t)W_0 - \int_0^t U_0(t-\tau)BW(\tau)d\tau. \tag{1.7}$$

It is known that the initial value problem (1.1) and (1.2) has a global solution.

Theorem 1 *Let $\lambda \geq 0$ and $N \geq 1$.*

(i) *Let $\beta(x,t,w_t)$ satisfy (1.3). Assume that*

$$0 \leq b(x,t) \leq C_1, \quad |b_t(x,t)| + |\nabla b(x,t)| \leq C_2 b(x,t) \tag{1.8}$$

for a.e $(x,t) \in \mathbf{R}^N \times (0,\infty)$ with constants $C_1, C_2 > 0$. And assume that $(w_1(x), w_2(x)) \in H^2 \times (H^1 \cap L^{2\rho})$. Then there exists a unique solution to (1.1) and (1.2) which satisfies the following:

$$w(t) \in \begin{cases} L^\infty([0,\infty); H^{0,2} \cap H^{0,1}) \cap C_L([0,\infty); H^{0,1}), & \text{if } \lambda = 0, \\ L^\infty([0,\infty); H^2) \cap C_L([0,\infty); H^1), & \text{if } \lambda > 0, \end{cases} \tag{1.9}$$

$$w_t(t) \in L^\infty([0,\infty); H^1) \cap C_L([0,\infty); L^2)), \tag{1.10}$$

$$w_{tt}(t) \in L^\infty([0,\infty); L^2), \tag{1.11}$$

$$\beta(x,t,w_t)w_t(t) \in L^\infty([0,\infty); L^2)) \cap L^{r'}([0,\infty); H^1_{q'}), \tag{1.12}$$

where $1/r' = 1/2 + \theta(\rho-1)/2(\rho+1)$ and $1/q' = 1/2 + \theta(\rho-1)/2(\rho+1) + (1-\theta)(\rho-1)/4\rho$ $(0 \leq \theta \leq 1)$. In addition the following integral equation holds:

$$W(t) = U_0(t-s)W(s) - \int_s^t U_0(t-\tau)BW(\tau)d\tau \quad \text{in } E \tag{1.13}$$

for $0 \leq s < t < \infty$.

(ii) *Let $\beta(x,t,w_t)$ satisy (1.4). Assume that $(w_1(x), w_2(x)) \in H^2 \times (H^1 \cap L^{6N/(3N-2\gamma)})$. Then there exists a unique solution to (1.1) and (1.2) which satisfies (1.9) $\sim$ (1.11), (1.13) and*

$$\beta(x,t,w_t)w_t(t) \in L^\infty([0,\infty); L^2)) \cap L^{r'}([0,\infty); H^1_{q'}) \tag{1.14}$$

where $1/r' = 1/2$ and $1/q' = 1/2 + \gamma/2N$.

The theory of monotone operators provides the existence of a global weak solution. (See e.g., Lions-Strauss [4] and Strauss [17].) But our argument in this paper needs the regularity of the nonlinear term, i.e. (1.12) and (1.14), which is deduced from the existence of a strong solution. By a strong solution to (1.1) we mean a solution which satisfies (1.1) in L^2-sense. So we shall sketch a proof of Theorem 1 in Appendix.

Now we state our results.

Theorem 2 *Let $w(t)$ be a solution of the initial value problem* (1.1), (1.2) *and* (1.3) *with $b(x,t) \equiv 1$. We put $\rho_1(N) = \infty (1 \le N \le 6)$, $= N/(N-6)$ $(N \ge 7)$ and $\rho_2(N) = \infty$ $(1 \le N \le 3)$, $= N(N-1)/(N-2)(N-3)$ $(N \ge 4)$.*

(i) *Let $\lambda = 0$ and $N \ge 2$. Then there exists a $W^+ = {}^t(w_1^+, w_2^+) \in E$ for $1 + 2/(N-1) < \rho < \rho_1(N)$ which satisfies the following: in case $1 + 4/(N-1) \le \rho < \rho_1(N)$*

$$\|U_0(-t)W(t) - W^+\|_E \to 0 \quad as \quad t \to \infty; \tag{1.15}$$

in case $1+2/(N-1) < \rho < \rho_2(N)$ there also exists a $0 < 1/p < 1/2$, which depends on ρ and N, such that

$$\|U_0(-t)W(t) - W^+\|_{H_p^{0,1} \times L^p} \to 0 \quad as \quad t \to \infty. \tag{1.16}$$

(ii) *Let $\lambda > 0$ and $N \ge 1$. Then there exists a $W^+ = {}^t(w_1^+, w_2^+) \in E$ for $1 + 2/N < \rho < \rho_1(N)$ which satisfies the following: in case $1 + 4/N \le \rho < \rho_1(N)$*

$$\|U_0(-t)W(t) - W^+\|_E \to 0 \quad as \quad t \to \infty; \tag{1.17}$$

in case $1+2/N < \rho < \rho_2(N)$ there also exists $0 < 1/p < 1/2$, which depends on ρ and N, such that

$$\|U_0(-t)W(t) - W^+\|_{H_p^1 \times L^p} \to 0 \quad as \quad t \to \infty. \tag{1.18}$$

Theorem 3 *Let $w(t)$ be a solution to the initial value problem* (1.1), (1.2) *and* (1.4).

(i) *Let $\lambda = 0$ and $N \ge 4$. Then for $N/(N-1) < \gamma < 2 - 1/(N-1)$ there exists a $W^+ = {}^t(w_1^+, w_2^+) \in E$ and $0 < 1/p < 1/2$, which depends on ρ and N, such that*

$$\|U_0(-t)W(t) - W^+\|_{H_p^{0,1} \times L^p} \to 0 \quad as \quad t \to \infty; \tag{1.19}$$

(ii) *Let $\lambda > 0$ and $N \geq 3$. Then in case $2 < \gamma < 3$ there exists a $W^+ = {}^t(w_1^+, w_2^+) \in E$ such that*

$$\|U_0(-t)W(t) - W^+\|_E \to 0 \quad as \quad t \to \infty. \tag{1.20}$$

In case $1 < \gamma < 2 - 1/(N-1)$ there exists a $W^+ = {}^t(w_1^+, w_2^+) \in E$ and $0 < 1/p < 1/2$, which depends on γ and N, such that

$$\|U_0(-t)W(t) - W^+\|_{H_p^1 \times L^p} \to 0 \quad as \quad t \to \infty. \tag{1.21}$$

Remark. (1) In our previous paper [9] we have already proved that $W^+ \neq 0$ for solutions to (1.1) with sufficiently small initial value (1.2).

(2) We can also treate the nonlinear dissipative term $b(x,t)|w_t|^{\rho-1}w_t$, where $0 \leq b(x,t) \leq C(1+|x|)^{-\delta}$ for some $0 \leq \delta \leq 1$ and $C > 0$. But the weighted Strichartz estimates are needed to deal with this dissipation. So please refer to our forthcoming paper for details.

As stated in [9], there are several works concerning with energy decay for (1.1). However, in case where energy remains positive as $t \to \infty$ there are not many results on the asymptotic behavior of solutions, except low energy scattering. (See e.g. Hidano and Tsutaya [2].) A partial answer to this problem was given in [11] in case $\lambda > 0$. Mochizuki [7] and [8] treated the linear dissipation $b(x,t)w_t$, where $b(x,t) \geq 0$.

In this paper we prove these two theorems by the duality argument which is formulated in the next section. The idea of this formulation is deduced from our previous paper [8] and [11].

2. Semi-abstract formuration

In this section we consider the initial value problem (1.5) and (1.6), that is,

$$iW_t(t) - AW(t) + BW(t) = 0, \quad t \in (0, \infty) \tag{2.1}$$

$$W(0) = W_0 \in E. \tag{2.2}$$

Here E is the energy space with a inner product

$$(f, g)_E = \frac{1}{2}\{(\nabla f_1, \nabla g_1)_{L^2} + (f_2, g_2)_{L^2} + \lambda(f_1, g_1)_{L^2}\},$$

where $f = (f_1, f_2)$, $g = (g_1, g_2)$ and $(\ , \ )_{L^2}$ is a L^2-inner product. We make the following hypotheses in this section.

(H.1) Let $Y = Y_1 \times Y_2$ and $Z = Z_1 \times Z_2$ be product spaces of Sobolev spaces with norms $\|\cdot\|_Y$ and $\|\cdot\|_Z$, respectively. We denote by Y' and Z' the dual space of Y and Z with respect to E, respectively and by Y_2' the dual space of Y_2 with respect to L^2.

(H.2) There exists a real number $r \geq 2$ and a positive constant C such that

$$\|[U_0(\cdot)g]_2\|_{L^r([1,\infty);Y_2)} \leq C\|g\|_{Z'} \tag{2.3}$$

for any $g = {}^t(g_1, g_2) \in C_0^\infty(\mathbf{R}^N) \times C_0^\infty(\mathbf{R}^N)$. Here we denote by $U_0(t)$ a one parameter unitary group in E given in section 1 and by $[U_0(t)g]_2$ the second component of $U_0(t)g$.

(H.3) Let D be a subspace of E. If the initial data $W_0 = {}^t(w_1, w_2) \in D$, there exists a solution to the integral equation

$$W(t) = U_0(t-s)W(s) - \int_s^t U_0(t-\tau)BW(\tau)d\tau \quad \text{in } E \tag{2.4}$$

for $0 \leq s < t < \infty$ such that $W(0) = W_0$,

$$W(t) \in L^\infty([0,\infty); E) \tag{2.5}$$

and

$$\beta(x, t, w_t)w_t(t) \in L^{r'}([1,\infty); Y_2'), \tag{2.6}$$

where $1/r' = 1 - 1/r$.

Then we obtain the following

Proposition 2.1 *Under the hypotheses* (H.1) $\sim$ (H.3) *there exists a* $W^+ \in E$ *such that*

$$\|U_0(-t)W(t) - W^+\|_Z \to 0 \quad as \quad t \to \infty \tag{2.7}$$

Proof. We recall from (2.4)

$$\begin{aligned}(W(t), U_0(t)\Phi)_E \ - \ (U_0(t-s)W_0, U_0(t)\Phi)_E \\ = -\int_s^t (BW(\tau), U_0(\tau)\Phi)_E d\tau\end{aligned} \tag{2.8}$$

for any $\Phi \in C_0^\infty(\mathbf{R}^N) \times C_0^\infty(\mathbf{R}^N)$. By (2.3) and (2.6) we have

$$\begin{aligned} &|(U_0(-t)W(t) - U_0(-s)W(s), \Phi)_E| \\ &\qquad \le \|BW(\cdot)\|_{L^{r'}([s,t];Y')} \|U_0(\cdot)\Phi\|_{L^r([s,t];Y)} \\ &\qquad \le \|\beta(\cdot,\cdot,w_t)w_t\|_{L^{r'}([s,t];Y_2')} \|[U_0(\cdot)\Phi]_2\|_{L^r([s,t];Y_2)} \\ &\qquad \le C\|\beta(\cdot,\cdot,w_t)w_t\|_{L^{r'}([s,t];Y_2')} \|\Phi\|_{Z'} \end{aligned} \tag{2.9}$$

for $1 \le s < t < \infty$. Since $C_0^\infty(\mathbf{R}^N) \times C_0^\infty(\mathbf{R}^N)$ is dense in E', it follows from (2.5), (2.6) and (2.9) that $\{U_0(-t)W(t)\}$ is a weak Cauchy sequence in E. Therefore there exists a $W^+ \in E$ such that

$$U_0(-t)W(t) \to W^+ \text{ weakly in } E \quad \text{as } t \to \infty. \tag{2.10}$$

By (2.8) with $s = 0$ we have

$$(U_0(-t)W(t), \Phi)_E = (W_0, \Phi)_E - \int_0^t (BW(\tau), U_0(\tau)\Phi)_E d\tau. \tag{2.11}$$

So we obtain

$$(W^+, \Phi)_E = (W_0, \Phi)_E - \int_0^\infty (BW(\tau), U_0(\tau)\Phi)_E d\tau \tag{2.12}$$

when $t \to \infty$. It follows from (2.11) and (2.12) that

$$\begin{aligned} |(U_0(-t)W(t) - W^+, \Phi)_E| &= |\int_t^\infty (BW(\tau), U_0(\tau)\Phi)_E d\tau| \\ &\le \|BW(\cdot)\|_{L^{r'}([t,\infty];Y')} \|U_0(\cdot)\Phi\|_{L^r([t,\infty];Y)} \\ &\le C\|\beta(\cdot,\cdot,w_t)w_t\|_{L^{r'}([t,\infty];Y_2')} \|\Phi\|_{Z'}. \end{aligned} \tag{2.13}$$

Since Z is a reflexive Banach space, we assert

$$\|U_0(-t)W(t) - W^+\|_Z \le C\|\beta(\cdot,\cdot,w_t)w_t\|_{L^{r'}([t,\infty];Y_2')}. \tag{2.14}$$

Therefore we obtain (2.7) by (2.14). □

3. Proof of Theorems

We first state some estimates which correspond to (H.2).

Proposition 3.1 (i) *Let $\lambda = 0$ and $N \ge 2$. Suppose that*

$$0 < \frac{1}{r} < \frac{1}{2}, \ \frac{1}{q} = \frac{1}{2} - \frac{2}{(N-1)r}, \ e = 1 - \frac{N+1}{(N-1)r}. \tag{3.1}$$

Then we have

$$\|[U_0(\cdot)g]_2\|_{L^r(\mathbf{R};H_q^{0,e-1})} \leq C\|g\|_E \tag{3.2}$$

for any $g = {}^t(g_1, g_2) \in C_0^\infty(\mathbf{R}^N) \times C_0^\infty(\mathbf{R}^N)$.

(ii) *Let* $\lambda > 0$ *and* $N \geq 1$. *Suppose that*

$$0 < \frac{1}{r} \leq \frac{1}{2}, \ \frac{1}{2} - \frac{2}{(N-1)r} < \frac{1}{q} < \frac{1}{2} - \frac{2}{Nr}, \ e < \frac{1}{2} + \frac{1}{q} - \frac{1}{r}. \tag{3.3}$$

Then we have

$$\|[U_0(\cdot)g]_2\|_{L^r(\mathbf{R};H_q^{e-1})} \leq C\|g\|_E \tag{3.4}$$

for any $g = {}^t(g_1, g_2) \in C_0^\infty(\mathbf{R}^N) \times C_0^\infty(\mathbf{R}^N)$.

Proof.

(i) See Pecher [13] Theorem 1 for a proof.

(ii) See Marshall [5] for a proof. □

Corollary 3.2 (i) *Let* $\lambda = 0$ *and* $N \geq 2$. *Suppose that*

$$0 < \frac{1}{r} < \frac{1}{2} \quad (0 < \frac{1}{r} \leq \frac{1}{3} \ \textit{if}\ N = 2), \tag{3.5}$$

$$e = 1 - \frac{N+1}{(N-1)r}, \frac{1}{q_1} = \frac{1}{2} - \frac{2}{(N-1)r} - \frac{(1-\eta)e}{N} \ (0 \leq \eta \leq 1).$$

Then we have

$$\|[U_0(\cdot)g]_2\|_{L^r(\mathbf{R};H_{q_1}^{0,e\eta-1})} \leq C\|g\|_E \tag{3.6}$$

for any $g = {}^t(g_1, g_2) \in C_0^\infty(\mathbf{R}^N) \times C_0^\infty(\mathbf{R}^N)$.

(ii) *Let* $\lambda > 0$ *and* $N \geq 1$. *Suppose that*

$$0 < \frac{1}{r} \leq \frac{1}{2}, \ 0 \leq e < (\frac{1}{2} + \frac{1}{q_1} - \frac{1}{r})(\frac{N}{N-1+\eta}), \tag{3.7}$$

$$\frac{1}{2} - \frac{2}{(N-1)r} - \frac{(1-\eta)e}{N} < \frac{1}{q_1} < \frac{1}{2} - \frac{2}{Nr} - \frac{(1-\eta)e}{N} \ (0 \leq \eta \leq 1).$$

Then we have

$$\|[U_0(\cdot)g]_2\|_{L^r(\mathbf{R};H_{q_1}^{e\eta-1})} \leq C\|g\|_E \tag{3.8}$$

for any $g = {}^t(g_1, g_2) \in C_0^\infty(\mathbf{R}^N) \times C_0^\infty(\mathbf{R}^N)$.

Proof. The Sobolev embeddings $H_q^{0,e\eta-1+(1-\eta)e} \hookrightarrow H_{q_1}^{0,e\eta-1}$ and $H_q^{e\eta-1+(1-\eta)e} \hookrightarrow H_{q_1}^{e\eta-1}$ hold if $e \geq 0$ and $1/q - (1-\eta)e/N = 1/q_1$. Thus we obtain (3.6) and (3.8) by (3.2) and (3.4), respectively. □

Proposition 3.3 (i) *Let* $\lambda = 0$ *and* $N \geq 2$. *Suppose that* $1/p' = N/q - (N-3)/2$ *for*

$$\frac{1}{2} - \frac{1}{N} \leq \frac{1}{q} \leq \frac{1}{2} - \frac{1}{N+1}, \quad 0 < \frac{1}{r} < (N-1)(\frac{N}{q} - \frac{N-2}{2}) \tag{3.9}$$

and that $1/p' = 1/Nq + (N+1)/2N$ *for*

$$\frac{1}{2} - \frac{1}{N+1} \leq \frac{1}{q} \leq \frac{1}{2}, \quad 0 < \frac{1}{r} < (N-1)(\frac{1}{2} - \frac{1}{q}). \tag{3.10}$$

Then we have

$$\|[U_0(\cdot)g]_2\|_{L^r([1,\infty);H_q^{0,-1})} \leq C\|g\|_{H_{p'}^{0,1}\times L^{p'}} \tag{3.11}$$

for any $g = {}^t(g_1, g_2) \in C_0^\infty(\mathbf{R}^N) \times C_0^\infty(\mathbf{R}^N)$.

(ii) *Let* $\lambda > 0$ *and* $N \geq 1$. *Suppose that* $1/p' = N/q - (N-3)/2$ *for*

$$\frac{1}{2} - \frac{1}{N} \leq \frac{1}{q} \leq \frac{1}{2} - \frac{1}{N+1}, \quad 0 < \frac{1}{r} < (N-1)(\frac{1}{q} - \frac{N-2}{2}) \tag{3.12}$$

and that $1/p' = 1 - 1/q$ *for*

$$\frac{1}{2} - \frac{1}{N+1} \leq \frac{1}{q} \leq \frac{1}{2}, \quad 0 < \frac{1}{r} < \mathrm{Min}\{\frac{2}{q},\ N(\frac{1}{2} - \frac{1}{q})\}. \tag{3.13}$$

Then we have

$$\|[U_0(\cdot)g]_2\|_{L^r([1,\infty);H_q^{-1})} \leq C\|g\|_{H_{p'}^{1}\times L^{p'}} \tag{3.14}$$

for any $g = {}^t(g_1, g_2) \in C_0^\infty(\mathbf{R}^N) \times C_0^\infty(\mathbf{R}^N)$.

Proof. Let $\lambda = 0$ and $N \geq 2$. It follows from Marshall-Strauss-Wainger [6] that

$$\|[U_0(\cdot)g]_2\|_{H_q^{0,-1}} \leq C|t|^{-(N-1)(\frac{N}{q}-\frac{N-2}{2})}\|g\|_{H_{p'}^{0,1}\times L^{p'}} \tag{3.15}$$

for $1/2 - 1/N \leq 1/q \leq 1/2 - 1/(N+1)$ and $1/p' = N/q - (N-3)/2$. So we have (3.11) for any r which satisfies

$$r(N-1)(\frac{N}{q}-\frac{N-2}{2})>1.$$

Summarizing these conditions, we obtain (3.9).

In other cases we can prove exactly in the same way as above. □

Proof of Theorem 2. First we prove (i). We consider the case $1+4/(N-1)<\rho<\rho_1(N)$. In order to apply Proposition 2.1, we put $1/r=1/2-\theta(\rho-1)/2(\rho+1)$,

$$Y=H_q^{-h,h}\times H_q^{-h,h-1},\ Z=E,\ D=H^2\times(H^1\cap L^{2\rho}),$$

where $1/q=1/2-\theta(\rho-1)/2(\rho+1)-(1-\theta)(\rho-1)/4\rho$ $(0\le\theta\le 1)$, $h=\eta e$ $(0\le\eta\le 1)$ and $e=1-(N+1)/(N-1)r$. Then we have

$$Y'=H_{q'}^{2+h,-h}\times H_{q'}^{h,1-h},\ Z'=E,$$

where $1/q'=1-1/q$. Since (H.3) is obvious, we have only to verify (H.2). Let $N\ge 3$. By Corollary 3.2 (i) we have (2.3) if q and r satisfy (3.5). Thus, we have the condition

$$\begin{aligned}&[2\{(N+1)\theta+(N-3)\}\eta+(N-1)\{(N+2)\theta+(N-6)\}]\rho^2\\&\quad-2[\{(N+1)\theta-(N-3)\}\eta+(N-1)\{3+(N+1)\theta]\rho\\&\qquad-N(N-1)(1-\theta)=0\end{aligned}\tag{3.16}$$

with $0\le\eta\le 1$ and $0<\theta\le 1$. We can solve this quadratic equation if

$$2\{(N+1)\theta+(N-3)\}\eta+(N-1)\{(N+2)\theta+(N-6)\}>0.\tag{3.17}$$

In case $N\ge 7$ (3.17) holds for $0\le\eta\le 1$ and in case $3\le N\le 6$ (3.17) holds for $0\le\theta\le(6-N)/(N+2)$. Let $\alpha_N(\theta,\eta)$ be its positive solution of (3.16). The supreme value of $\alpha_N(\theta,\eta)$ is $\alpha_N(0,0)=\rho_1(N)$ in case $N\ge 7$ and $\alpha_N(\theta,0)\uparrow\infty$ as $\theta\downarrow(N-6)/(N+2)$ in case $3\le N\le 6$. On the other hand, the minimal value of ρ is given by $\alpha_N(1,1)$. Therefore we obtain the range of ρ if we note the continuity of $\alpha_N(\theta,\eta)$.

In case $N=2$ the condition $0<1/r\le 1/3$ implies

$$\rho\ge(3\theta+1)/(3\theta-1).\tag{3.18}$$

We denote by $g(\rho,\theta)$ the lefthand side of (3.16) with $\eta=1$. Then we have $g((3\theta+1)/(3\theta-1),\theta) = -96\theta^2/(3\theta-1)^2 < 0$. This implies that the positive solution $\alpha_N(\theta,1)$ satisfies (3.18). The minimal value of ρ is given by $\alpha_N(1,1)$. Noting $\alpha_N(\theta,1) \uparrow \infty$ as $\theta \downarrow 3/5$, which follows from (3.17), we get the range of ρ.

Next we consider the case $1+2/(N-1) < \rho < \rho_2(N)$. We put

$$Y = L^q \times H_q^{0,1}, \quad Z = E, \quad D = H^2 \times (H^1 \cap L^{2\rho}),$$

where $1/q = 1/2 - \theta(\rho-1)/2(\rho+1) - (1-\theta)(\rho-1)/4\rho$ and $1/r = 1/2 - \theta(\rho-1)/2(\rho+1)$ $(0 \le \theta \le 1)$. As in the previous case, it is enough to verify (H.2). Let q and r satisfy (3.10). Then obviously we have (2.3). Moreover, substituting q and r to (3.10), we obtain

$$\begin{aligned}\{(N+1)\theta+(N-3)\}\rho^2 \;&-\; 2\{(N+1)\theta+2\}\rho \\ &-\; (N+1)(1-\theta) \le 0,\end{aligned} \tag{3.19}$$

$$\{(N+1)\theta+(N-3)\}\rho^2 - 2(1+N\theta)\rho - (N-1)(1-\theta) > 0. \tag{3.20}$$

By (3.20) we need $(3-N)/(N+1) < \theta \le 1$ if $N = 2,3$. We denote by $g_1(\rho,\theta)$ and $g_2(\rho,\theta)$ the lefthand side of (3.19) and (3.20), respectively. Then we have $g_2(\rho,\theta) > g_1(\rho,\theta)$. So it follows from (3.19) and (3.20) that $\beta_N(\theta) < \rho \le \alpha_N(\theta)$, where $\alpha_N(\theta)$ and $\beta_N(\theta)$ are positive solutions of $g_1(\rho,\theta)$ and $g_2(\rho,\theta)$, respectively. Since $\beta_N(1) = 1+2/(N-1)$, $\alpha_N(\theta) \uparrow \infty$ as $\theta \downarrow (3-N)/(N+1)$ if $N = 2,3$ and $\alpha_N(0) = (N+1)/(N-3)$ if $N \ge 4$, we have

$$1 + \frac{2}{(N-1)} < \rho \begin{cases} < \infty & (N = 2,3), \\ \le \dfrac{(N+1)}{(N-3)} & (N \ge 4). \end{cases} \tag{3.21}$$

This proves (1.16) for $N = 2,3$.

Let $N \ge 4$, and let q and r be as above. Substituting q and r to (3.9) with $\theta \approx 0$, we have

$$(N-3)\rho^2 - 4\rho - (N+1) > 0, \tag{3.22}$$

$$(N-4)\rho^2 - 4\rho - N < 0, \tag{3.23}$$

$$(N-2)(N-3)\rho^2 - 2(2N-3)\rho - N(N-1) < 0. \tag{3.24}$$

Thus, we obtain

$$\frac{N+1}{N-3} < \rho < \frac{N(N-1)}{(N-2)(N-3)}. \tag{3.25}$$

and this completes the proof of Theorem 2 (i).

We can prove (ii) exactly in the same way as above. So we may omit its proof. □

Proof of Theorem 3. The proof of Theorem 3 is the same as that of Theorem 2. So we only prove (ii) of Theorem 3. In order to apply Proposition 2.1, we put $1/q = 1/2 - \gamma/2N$, $1/r = 1/2$ and

$$Y = L^q \times H_q^{-1},\ Z = E,\ D = H^2 \times (H^1 \cap L^{6N/(3N-2\gamma)}).$$

On the other hand, we have (2.3) by Corollary 3.2 (ii) and the embedding $H_q^{e\eta-1} \hookrightarrow H_q^{-1}$ if q and r satisfy

$$\frac{1}{2} - \frac{1}{N} - \frac{1}{Nr} < \frac{1}{q} < \frac{1}{2} - \frac{2}{Nr}. \tag{3.26}$$

(3.26) follows from (3.7) if we note the range of η and e. Thus, substituting q qnd r to (3.26), we get the range $2 < \gamma < 3$. Using Proposition 3.3 (ii), we have the range $1 < \gamma < 2 - 1/(N-1)$ exactly in the same way. Thus Theorem3 (ii) is proved. □

Appendix

In this appendix we sketch a proof of Theorem 1. The method employed here is the compactness argument based on uniform energy estimates which is due to Segal [15], Lions [3] and Strauss [16]. (See also Reed [14] and Ginibre-Velo [1].) A detailed proof of Theorem 1 can be found in Motai [12].

We approximate the nonlinear dissipative term by the double convolution molifier due to Segal [13] and Ginibre-Velo [1]. We choose an even nonnegative function $h(x) \in C_0^\infty(\mathbf{R}^N)$ such that $\|h\|_{L^1} = 1$. For any natural numbers j we put

$$\mathfrak{B}_j(x,t,u) = h_j * (\beta(x,t,h_j * u)h_j * u), \tag{1}$$

where $h_j(x) = j^N(jx)$. Coresponding (1.1) and (1.2), we consider the Cauchy problem;

$$\begin{cases} w_{jtt}(t) - \Delta w_j(t) + \lambda w_j(t) + \mathfrak{B}_j(x,t,w_{jt}) = 0, \\ w_j(x,0) = h_j * w_1(x), \quad w_{jt}(x,0) = h_j * w_2(x). \end{cases} \tag{2}$$

Lemma A.1 *Let j be any natural number. Under the same assumptions of Theorem 1 there exists a unique solution of (2) which satisfies*

$$w_j(t) \in \cap_{i=0}^k C^i([0,\infty); H^{k-i}) \quad \textit{for any natural number } k. \tag{3}$$

And the following energy inequalities hold : in case (1.3)

$$\|W_j(t)\|_E^2 + \int_0^t \int_{\mathbf{R}^N} b(x,\tau)|h_j * w_t(\tau)|^{\rho+1} dx d\tau \le \|W_0\|_E^2, \tag{4}$$

where $W_j(t) = {}^t(w_j(t), w_{jt}(t))$ and $W_0 = {}^t(w_1, w_2)$,

$$\begin{aligned} \|\Delta w_j(t)\|_{L^2}^2 \; &+ \; \|\nabla w_{jt}(t)\|_{L^2}^2 + \lambda \|\nabla w_j(t)\|_{L^2}^2 \\ &+ \int_0^t \int_{\mathbf{R}^N} b(x,\tau)|h_j * w_{jt}(\tau)|^{\rho-1} |\nabla(h_j * w_{jt}(\tau))|^2 dx d\tau \\ &\qquad \le C_3(\|w_1\|_{H^2}^2 + \|w_2\|_{H^1}^2) \end{aligned} \tag{5}$$

$$\begin{aligned} \|\nabla w_{tj}(t)\|_{L^2}^2 \; &+ \; \|w_{jtt}(t)\|_{L^2}^2 + \lambda \|w_{jt}(t)\|_{L^2}^2 \\ &+ \int_0^t \int_{\mathbf{R}^N} b(x,\tau)|h_j * w_{jt}(\tau)|^{\rho-1} |h_j * w_{jtt}(\tau)|^2 dx d\tau \\ &\qquad \le C_4(\|w_1\|_{H^2}^2 + \|w_2\|_{H^1}^2 + \|w_2\|_{L^{2\rho}}^{2\rho}) \end{aligned} \tag{6}$$

for $0 \leq t < \infty$ *with some constants* $C_3, C_4 > 0$; *and in case* (1.4)

$$\|W_j(t)\|_E^2 + \int_0^t \|V_{(N+\gamma)/2} * |h_j * w_t(\tau)|^2\|_{L^2}^2 d\tau \leq \|W_0\|_E^2, \tag{7}$$

$$\begin{aligned}\|\Delta w_j(t)\|_{L^2}^2 &+ \|\nabla w_{jt}(t)\|_{L^2}^2 + \lambda \|\nabla w_j(t)\|_{L^2}^2 \\ &+ \int_0^t \int_{\mathbf{R}^N} (V_\gamma * |h_j * w_{jt}(\tau)|^2 |\nabla (h_j * w_{jt}(\tau))|^2 dx d\tau \\ &\leq C_5(\|w_1\|_{H^2}^2 + \|w_2\|_{H^1}^2)\end{aligned} \tag{8}$$

$$\begin{aligned}\|\nabla w_{tj}(t)\|_{L^2}^2 &+ \|w_{jtt}(t)\|_{L^2}^2 + \lambda \|w_{jt}(t)\|_{L^2}^2 \\ &+ \int_0^t \int_{\mathbf{R}^N} (V_\gamma * |h_j * w_{jt}(\tau)|^2) |h_j * w_{jtt}(\tau)|^2 dx d\tau \\ &+ \int_0^t \|\partial_t (V_{(N+\gamma)/2} * |h_j * w_{jt}(\tau)|^2)\|_{L^2}^2 d\tau \\ &\leq C_6(\|w_1\|_{H^2}^2 + \|w_2\|_{H^1}^2 + \|w_2\|_{L^{6N/(3N-2\gamma)}}^6)\end{aligned} \tag{9}$$

for $0 \leq t < \infty$ *with some constants* C_5, $C_6 > 0$. *Futhermore, the following integral equation holds:*

$$W_j(t) = U_0(t-s)W_j(s) - \int_s^t U_0(t-\tau) B_j W_j(\tau) d\tau \quad in\ E \tag{10}$$

for $0 \leq s < t < \infty$, *where* $B_j W_j(t) = {}^t(0, \mathfrak{B}_j(t))$.

Proof. Applying Reed [14, Theorem 2] to (2), we can show the existence of a unique global solution. The double convolution molifier implies the regularity of solutions. And noting (1.8) and the Schwartz inequality, (4) $\sim$ (9) are obtained by the standard energy method. □

Lemma A.2 *Let* $\{w_j(t)\} = \{w_j(t)\}_{j=0}^\infty$ *be a sequence of solutions to* (15) *obtained by Lemma* A.1. *Then* $\{h_j * w_j(t)\}$ *has a convergent subsequence (again denoted by* $\{h_j * w_j(t)\}$*) as follows : there exists a* $w(t)$ *which satisfies* (1.9) $\sim$ (1.11), *and*

$$h_j * w_j(t) \to w(t) \quad in\ C(I; H^1(\Omega_R)) \quad as\ j \to \infty, \tag{11}$$

$$h_j * w_{jt}(t) \to w_t(t) \quad in\ C(I; L^2(\Omega_R)) \quad as\ j \to \infty \tag{12}$$

for any closed interbal $I \subset \mathbf{R}$ *and any open ball* $\Omega_R = \{x \in \mathbf{R}^N; |x| < R\}$

Proof. We put $I_n = [0, n]$. By energy inequalities which are obtained in Lemma A.1 we have $\{h_j * w_j(t)\}, \{h_j * w_{jt}(t)\}$ and $\{h_j * w_{jtt}(t)\}$ are unioformly bounded with respect to j and $t \in I_n$ in $H^2(\Omega_n), H^1(\Omega_n)$ and $L^2(\Omega_n)$, respectively. The Rellich theorem and the Ascoli-Arzela theorem tell us that $\{h_j * w_j(t)\}$ has a convergent subsequence (which is denoted by $\{h_j * w_j(t)\}$ again) such that

$$h_j * w_j(t) \to w(t) \quad \text{in } C(I_n; H^1(\Omega_n)) \quad \text{as } j \to \infty, \tag{13}$$

$$h_j * w_{jt}(t) \to w_t(t) \quad \text{in } C(I_n; L^2(\Omega_n)) \quad \text{as } j \to \infty. \tag{14}$$

By the diagnal argument we can have a subsequence so that (13) and (14) hold for each n. It follows from (13) that

$$h_j * w_j(t) \to w(t) \quad \text{in } \mathfrak{D}'(\mathbf{R}^N \times [0, \infty)) \quad \text{as } j \to \infty, \tag{15}$$

and

$$h_j * w_j(t) \to w(t) \quad \text{in } \mathfrak{D}'(\mathbf{R}^N) \quad \text{as } j \to \infty \tag{16}$$

uniformly on any interval $I \subset \mathbf{R}$. Here $\mathfrak{D}'$ means the space of distributions. So noting (15), (16) and energy inequalities, we can prove that $w(t)$ satisfy (1.9) $\sim$ (1.11). □

Under these preliminary arrangements we sketch the proof of Theorem 1.

Proof of Theorem 1. It follows from (10) that

$$\begin{aligned}(W_j(t), f)_E &= (U_0(0)W_j(0), f)_E \\ &\quad - \int_0^t (B_j W_j(\tau), U_0(\tau - t)f)_E d\tau\end{aligned} \tag{17}$$

for any $f \in C_0^\infty(\mathbf{R}^N) \times C_0^\infty(\mathbf{R}^N)$ and $0 \leq s < t < \infty$. Suppose that we can show that

$$\mathfrak{B}_j(x, t, w_{jt}) \to \beta(x, t, w_t)w_t \quad \text{weakly in } L^{r'}([0, \infty); L^{q'}) \tag{18}$$

as $j \to \infty$ for suitable $0 < r', q' \leq 2$. Since it follows from the Hausdorff-Young inequality that $\chi_t(\cdot)[U_0(\cdot - t)f]_2 \in L^r([0, \infty); L^q)$, where $\chi_t(\tau) = 1 (0 \leq \tau \leq t), = 0$(otherwise), $1/r = 1 - 1/r'$ and $1/q = 1 - 1/q'$, we can take limit in (17) to get

$$(W(t), f)_E = (U_0(0)W(0), f)_E - \int_0^t (BW(\tau), U_0(\tau - t)f)_E d\tau. \quad (19)$$

Combining (1.9)~(1.11) and (19), we have $\beta(x, t, w_t)w_t(t) \in L^\infty([0, \infty); L^2))$. (19) also asserts (1.13) because of the unitarity of $U_0(t)$. So we prove (18) by showing

$$\mathfrak{B}_j(x, t, w_{jt}) \to \beta(x, t, w_t)w_t \quad \text{in } \mathfrak{D}'(\mathbf{R}^N \times [0, \infty)) \quad (20)$$

as $j \to \infty$, and

$$\|\mathfrak{B}_j(\cdot, \cdot, w_{jt})\|_{L^{r'}([0,\infty); L^{q'})} \leq C_7, \quad (21)$$

where C_7 is independent on j. (20) is proved by the same way employed by e,g. Reed [14] (pp. 61 ~ 62) in case (1.3) and by Motai [10] in case (1.4). (21) is proved by showing more extended result

$$\|\mathfrak{B}_j(\cdot, \cdot, w_{jt})\|_{L^{r'}([0,\infty); H^1_{q'})} \leq C_8, \quad (22)$$

where r' and q' are as in Theorem 1. By means of (4) ~ (5) and (7) ~ (8), (22) is proved by the duality argeument.

Obviously (20) and (22) imply (1.12) and (1.14).

The uniquness follows from the property

$$\{\beta(x, t, u_1)u_1 - \beta(x, t, u_2)u_2\}(u_1 - u_2) \geq 0. \quad (23)$$

Thus this completes the sketch of the proof of Theorem 1. □

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