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A note on the decomposition of the Burnside rings of finite groups

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Abstract. The Burnside ring $\Omega(G)$ of a finite group G has, as an abelian group, a decomposition $\Omega(G)=\Omega(G,\mathcal{X})\oplus K(\mathcal{X})$ where $K(\mathcal{X})$ is an ideal and $\Omega(G,\mathcal{X})$ is the generalized Burnside ring with respect to a family $\mathcal{X}$ of subgroups of G.

Key words: Burnside ring, p-locally determined function, Alperin's conjecture.

1. Introduction

In 7.2 of [Th], J. Thévenaz showed the following theorem for the Burnside ring $\Omega(G)$ of a finite group G.

Theorem (7.2 [Th]) *Let $\mathcal{X}$ be the family of subgroup H of G such that $O_p(H)$ is not trivial. Then*

$$\Omega(G)=B_{\mathcal{X}}\oplus K_{\mathcal{X}},$$

where

$$\begin{aligned} B_{\mathcal{X}} &= \Big\{\sum_{S\in\mathcal{X}}\lambda_S[G/S]\mid\lambda_S\in\boldsymbol{Z}\Big\},\\ K_{\mathcal{X}} &= \bigcap_{S\in\mathcal{X}}\operatorname{Ker}(\varphi_S). \end{aligned}$$

Here $\varphi_S:\Omega(G)\to\boldsymbol{Z}$ is defined for each G-set X by $\varphi_S(X)=|X^S|$, the number of S-fixed points in X. The above is equivalent to 3.1 of [Th], which is a key result for showing that Alperin's conjecture [Al] and the following assertion on p-locally determined functions are equivalent.

Conjecture (Thévenaz [Th]) *If we write*

$$k(H)-z(H)=\frac{1}{|H|}\sum_{S\leq H}f_1(S)$$

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for all $H \leq G$, then the function f_1 vanishes on subgroups H such that $O_p(H) = 1$.

Here $k(G)$ is the number of irreducible complex representation of a finite group G and $z(G)$ is the number of those representations whose degrees are multiples of $|G|_p$.

Let $S(G)$ be the poset of all subgroups of G and let $\mathcal{F}(G, \boldsymbol{Z})$ be the set of all functions $f : S(G) \to \boldsymbol{Z}$ such that $f(gHg^{-1}) = f(H)$ for all $g \in G$ and $H \in S(G)$. A function $f \in \mathcal{F}(G, \boldsymbol{Z})$ is called a *p-locally determined function* if $\hat{f}(H) = 0$ whenever $O_p(H) = 1$. Here the function $\hat{f}$ is defined by a formula:

$$\hat{f}(H) = \sum_{S \leq H} \mu(S, H)|S|f(S) \quad \text{for all} \quad H \in S(G),$$

where μ denotes the Möbius function of the poset $S(G)$.

For the proof of the above theorem, he used the argument of simplicial complexes of non-trivial p-subgroups introduced by K.S. Brown.

The purpose of this paper is to establish, in a slightly general way, a decomposition of the Burnside ring in this nature, using the notion of a generalized Burnside ring instead of Brown's simplicial complex, and thus to give a new proof to 7.2 of [Th].

The result is stated in Theorem 1 below, in which we prove that a generalized Burnside ring is a direct summand of $\Omega(G)$ under some conditions on $\mathcal{X}$ weaker than Thévenaz'.

2. A decomposition of $\Omega(G)$

Details and basic results of Burnside rings are found in [Yo]. Throughout this paper, G denotes a finite group and p denotes a prime (or ∞).

The Burnside ring $\Omega(G)$ is the Grothendieck ring of the category of G-sets and G-maps with respect to disjoint unions and cartesian products. So this ring is as additive groups free abelian group on the set $\{[G/H] \mid (H) \in C(G)\}$, where $C(G)$ is the set of the G-conjugacy classes (H) of subgroups H of G. Let $\mathcal{X}$ be a family of additive subgroup of G such that if $H \in \mathcal{X}$, then ${}^gH := gHg^{-1} \in \mathcal{X}$ for any $g \in G$. Let $\Omega(G, \mathcal{X})$ be the subgroups of $\Omega(G)$ generated by elements $[G/H]$ for $H \in \mathcal{X}$. For a prime p, let $\boldsymbol{Z}_{(p)}$ be the localization of $\boldsymbol{Z}$ at p:

$$\boldsymbol{Z}_{(p)} := \{a/b | a \in \boldsymbol{Z}, b \in \boldsymbol{Z} - p\boldsymbol{Z}\} \subseteq \boldsymbol{Q}.$$

Put

$$\Omega(G)_{(p)} := \boldsymbol{Z}_{(p)} \otimes z\Omega(G), \qquad \Omega(G,\mathcal{X})_{(p)} := \boldsymbol{Z}_{(p)} \otimes z\Omega(G,\mathcal{X}).$$

Furthermore, if there is no confusion, it is we extend the above notation to $p = \infty$ as follows:

$$\Omega(G)_{(\infty)} := \Omega(G), \quad \Omega(G,\mathcal{X})_{(\infty)} := \Omega(G,\mathcal{X}).$$

Let $K(\mathcal{X})$ be the ideal of the ordinary Burnside ring $\Omega(G)$ defined by

$$K(\mathcal{X}) := \{x \in \Omega(G) \mid x(S) = 0 \quad \text{for all} \quad S \in \mathcal{X}\}.$$

Moreover, for $\Omega(G,\mathcal{X})_{(p)}$

$$K(\mathcal{X})_{(p)} := \{x \in \Omega(G)_{(p)} \mid x(S) = 0 \quad \text{for all} \quad S \in \mathcal{X}\},$$

and we put $K(\mathcal{X})_{(\infty)} = K(\mathcal{X})$ for $p = \infty$.

For a subgroup S of G, we use the following symbols for the normalizer and the Weyl group:

$$NS := N_G(S), \quad WS := NS/S.$$

Let G_p be a Sylow p-subgroup of a finite group G, and $|G|_p$ the order of G_p.

Theorem 1 *Let $\mathcal{X}$ be a family of subgroups of G such that if $H \in \mathcal{X}$ then $gHg^{-1} \in \mathcal{X}$ for any $g \in G$. Assume that the family satisfies the following condition:*

$$S \in \mathcal{X}, \quad gS \in (WS)_p \quad \Rightarrow \quad \langle g\rangle S \in \mathcal{X}.$$

Then

$$\Omega(G)_{(p)} = \Omega(G,\mathcal{X})_{(p)} \oplus K(\mathcal{X})_{(p)}.$$

Proof. In the argument in 3.15 (c) of [Yo], Yoshida defines an abelian group homomorphism $\rho : \Omega(G)_{(p)} \to \Omega(G,\mathcal{X})_{(p)}$, and pointed out that $\mathrm{Ker}\rho = K(\mathcal{X})_{(p)}$. (See also 6.3 and 6.4 of [Yo].) Moreover, it is clear from the definition that ρ is identity on $\Omega(G,\mathcal{X})_{(p)}$, and hence it is a split epimorphism. Thus we have the desired decomposition. $\square$

Corollary 1 (7.2 [Th]) *Let $\mathcal{X}$ be the family of subgroup H of G such* *that $O_p(H)$ is not trivial. Then*

$$\Omega(G) = \Omega(G, \mathcal{X}) \oplus K(\mathcal{X}).$$

Proof. In order to establish this corollary, we need only to see that the family $\mathcal{X}$ satisfies the assumption of the Theorem 1. Let S be an element of $\mathcal{X}$. Since $O_p(S)$ is a characteristic subgroup of S, we have that $O_p(S)$ is the non-trivial normal p-subgroup of NS, and so $\langle g \rangle S$ is in $\mathcal{X}$ where $g \in NS$, proving the corollary. $\square$

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