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The Schrödinger operator with random vector potentials

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Abstract. We consider a non-Gaussian probability measure on $(\mathcal{S}'(\mathbf{R}^d; \mathbf{R}^d), \mathcal{B}(\mathcal{S}'(\mathbf{R}^d; \mathbf{R}^d))$ whose characteristic functional is given by Lévy-Khinchine formula. We construct a one parameter semi-group whose generator is expressed by "$(\partial - iA)^2$", $A \in \mathcal{S}'(\mathbf{R}^d; \mathbf{R}^d)$ formally. It is also shown that the generator is self-adjoint.

Key words: electromagnetic fields, semi-group process, gauge covariance.

1. Introduction

Nonlinear electromagnetism is a theory of generalized random fields in four dimensional Euclidean space-time obtained by solving a system of coupled stochastic partial differential equations [AHK1, 2, AHKI, AIK1]. The fields are homogeneous with respect to the Euclidean group. It has been shown in [I, AIK3] that the fields have the sharp global Markov property. In [AIK2], relativistic time ordered functions are constructed in the model. The theory includes the usual theory of electromagnetic fields as a special case. As other special cases, it describes a class of models where the fields confine charges in Wilson's sense, and it has mild ultraviolet behavior [Ta]. Recently, in [AT], they consider a coupled theory of the confining nonlinear electromagnetic field and a charged scalar field within the quenched approximation. And they consider the propagator of matter field in approximation. The propagator is the expectation value of the resolvent kernel of the Schrödinger operator with the vector potential of the electromagnetic field. They define it as an element of $L^p(\mathcal{S}'(\mathbf{R}^d; \mathbf{R}^d), \mu)$ where μ is the probability measure which characterizes the nonlinear electromagnetic field. And they examined the asymptotic behavior of the propagator and showed that the correlation length is zero. However, they did not define the operator.

In this paper we consider the semi-group with random vector potential formally represented by $e^{t/2(\partial - iA)^2}$ on $L^2(\mathbf{R}^d)$, $d \geq 3$, $A \in \mathcal{S}'(\mathbf{R}^d; \mathbf{R}^d)$. The word "formally" reflects the fact we have not given a simultaneous definition of the operator $e^{t/2(\partial - iA)^2}$ for every A in the support of the measure μ on

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$\mathcal{S}'(\mathbf{R}^d;\mathbf{R}^d)$. We define the semi-group on $L^2(\mathbf{R}^d)$ μ-almost surely and show it is a strongly continuous symmetric contraction semi-group. Then there exists a generator of the semigroup that is self-adjoint. And we can show this operator is gauge covariant.

In section 2, we give the definition of the measure μ. In section 3 we shall show that measure μ has the property of ergodicity. In section 4 we shall show some lemmas which are needed in the following sections. In section 5 we define the semi-group $e^{t/2(\partial-iA)^2}$ on $L^2(\mathbf{R}^d)$ μ-almost surely, which is measurable in the sense of [CL]. We shall show that the semi-group is a strongly-continuous symmetric contraction semi-group μ-almost surely $A \in \mathcal{S}'(\mathbf{R}^d;\mathbf{R}^d)$ (We call it a semi-group process). Then, we can define the self-adjoint operator "$(\partial - iA)^2$" μ-almost surely as the generator of the semi-group. In section 6 we shall show for $\lambda \in L^1_{loc}(\mathbf{R}^d)$, $e^{i\lambda}e^{t/2(\partial-iA)^2}e^{-i\lambda} = e^{t/2(\partial-i(A+\partial\lambda))^2}$ μ-almost surely. That is to say, $(\partial-iA)^2$ is gauge covariant. In section 7 we shall note a property of the spectrum of the generator.

2. Notations

In this paper we assume that $d \geq 3$. Let μ be the Borel probability measure on $\mathcal{S}'(\mathbf{R}^d;\mathbf{R}^d)$, the dual of the Schwartz space $\mathcal{S}(\mathbf{R}^d;\mathbf{R}^d)$ of all $\mathbf{R}^d$ valued rapidly decreasing C^∞ functions on $\mathbf{R}^d$, characterized by

$$\int_{\mathcal{S}'(\mathbf{R}^d;\mathbf{R}^d)} e^{i\langle f,A\rangle} d\mu(A) = \exp\Big(-\int_{\mathbf{R}^d} \psi(Sf(x))dx\Big), \tag{2.1}$$

in the sense of Minlos' theorem (see e.g. [GV]). Here $f = (f_0, f_1, ...f_{d-1}) \in \mathcal{S}(\mathbf{R}^d;\mathbf{R}^d)$, and Sf is defined by

$$(Sf)_\nu(x) = \sum_{\rho=0}^{d-1} \int_{\mathbf{R}^d} S_{\nu\rho}(x-y) f_\rho(y) dy, \tag{2.2}$$

with

$$\begin{aligned} S_{0\nu}(x-y) &= \partial_\nu(-\Delta)^{-1}(x,y) = -\frac{\Gamma(d/2)}{2\pi^{d/2}}\frac{x_\nu - y_\nu}{|x-y|^d}, \\ S_{j0}(x-y) &= -\partial_j(-\Delta)^{-1}(x,y) = \frac{\Gamma(d/2)}{2\pi^{d/2}}\frac{x_j - y_j}{|x-y|^d}, \\ S_{ij}(x-y) &= (\delta_{ij}\partial_0 + \sum_{k=1}^{d-1} a_{ijk}\partial_k)(-\Delta)^{-1}(x,y), \\ &= -\frac{\Gamma(d/2)}{2\pi^{d/2}}\frac{(x_0-y_0)\delta_{ij} + \sum_{k=1}^{d-1} a_{ijk}(x_k - y_k)}{|x-y|^d}, \end{aligned} \tag{2.3}$$

for $x = (x_0, \cdots, x_{d-1})$, $0 \leq \nu \leq d-1$ and $1 \leq i,\ j \leq d-1$. Here δ_{ij} is Kronecker's delta and a_{ijk} is a real constant. ψ is a function on $\mathbf{R}^d$, represented by the Lévy-Khinchine formula

$$\psi(\lambda) = \int_{\mathbf{R}^d} \left(1 - e^{i\alpha\cdot\lambda} + \frac{i\alpha\cdot\lambda}{1+|\alpha|^2}\right) d\nu(\alpha) + \frac{1}{2}\langle\lambda, \mathbf{A}\lambda\rangle, \tag{2.4}$$

where $\mathbf{A}$ is a non-negative definite $d \times d$ matrix and ν is a measure on $\mathbf{R}^d$ such that $\nu(\{0\}) = 0$, $\int_{\mathbf{R}^d}(|\alpha|^2 \wedge 1)d\nu(\alpha) < \infty$.

In this paper we assume that ψ satisfies the following conditions:

(A.1): The function ψ is continuous, non-negative and bounded, i.e. there exists a constant c_0 such that

$$0 \leq \psi \leq c_0. \tag{2.5}$$

(A.2): There exist constants $c_1 > 0$ and $\frac{d}{d-1} < \eta \leq 2$ such that

$$|\psi(\lambda)| \leq c_1|\lambda|^\eta, \qquad \text{for} \quad |\lambda| \leq 1. \tag{2.6}$$

The conditions (A.1) (A.2) are slightly weak compared with those in [AT].

Remark 2.1. Under (A.1), $\mathbf{A}$ must be 0 in the representation (2.4).

Remark 2.2. The function ψ which satisfies these conditions may grow as $|\lambda|^\eta$ for large $|\lambda|$; i.e. there exists a constant $c > 0$,

$$\psi(\lambda) \leq c|\lambda|^\eta, \qquad \text{for} \quad \lambda \in \mathbf{R}^d. \tag{2.7}$$

We give here two examples of ψ which satisfies the above conditions for the case $d = 4$.

Example 2.3. The function

$$\psi(\lambda) = \frac{|\lambda|^\eta}{1+|\lambda|^\eta} \qquad \lambda \in \mathbf{R}^4,$$

satisfies (A.1), (A.2).

Example 2.4. Let $\alpha = (\alpha_0, \alpha') = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathbf{R}^4$. The function ψ which is defined by the equation (2.4) with the matrix $\mathbf{A} = 0$ and measure

$\nu(\alpha) = \delta(\alpha_0)\nu'(\alpha'),$

$$d\nu'(\alpha') = \frac{H(|\alpha'|-1)}{|\alpha'|^{3+\eta}} d\alpha',$$

where $d\alpha'$ is the Lebesgue measure on $\mathbf{R}^3$ and H is the Heviside function satisfies (A.1), (A.2).

The proofs are in [Ta].

Let $I : \mathbf{R} \to [0,1]$ be a C_0^∞ function such that

$$I(s) = \begin{cases} 1, & |s| \leq \frac{1}{4}, \\ 0, & |s| \geq \frac{1}{2}, \end{cases}$$

and set

$$I_l(s) = \begin{cases} 1, & |s| \leq l-1, \\ I(|s|-l+1), & l-1 \leq |s| \leq l, \\ 0, & |s| \geq l, \end{cases} \qquad \text{for} \quad l \in \mathbf{N}.$$

Then the function

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon^d \|I\|_{L^1}^d} \prod_{\nu=0}^{d-1} I\left(\frac{x_\nu}{\varepsilon}\right) \qquad (\varepsilon > 0) \tag{2.8}$$

satisfies

$$\rho_\varepsilon \in \mathcal{D}(\mathbf{R}^d), \quad \mathrm{supp}\rho_\varepsilon \subset \left\{x \in \mathbf{R}^d \mid |x| = \Big(\sum_{\nu=0}^{d-1} |x_\nu|^2\Big)^{1/2} \leq \frac{\sqrt{d}}{2}\varepsilon\right\},$$

$$\int_{\mathbf{R}^d} \rho_\varepsilon(x)dx = 1, \qquad \rho_\varepsilon \geq 0,$$

and hence

$$\lim_{\varepsilon \to 0} \rho_\varepsilon = \delta, \qquad \text{in } \mathcal{S}'(\mathbf{R}^d), \tag{2.9}$$

where $\|\cdot\|_{L^1}$ is the norm of $L^1(\mathbf{R}^d, dx)$ with Lebesgue measure dx and $\mathcal{D}(\mathbf{R}^d)$ is a family of C^∞ functions with compact support. The function

$$w_l(x) = \prod_{\nu=0}^{d-1} I_l(x_\nu),$$

satisfies

$$w_l \in \mathcal{D}(\mathbf{R}^d), \qquad 0 \le w_l \le 1,$$

and

$$w_l \uparrow 1 \qquad \text{as} \quad l \to \infty.$$

For each $A \in \mathcal{S}'(\mathbf{R}^d; \mathbf{R}^d)$, we define A^n $(n \in \mathbf{N})$ by

$$A_\nu^n(x) = w_n(x)\langle \rho_{1/n}(x - \cdot), A_\nu \rangle, \qquad \nu = 0, 1, 2, ..., d-1. \tag{2.10}$$

Obviously $A^n \in \mathcal{D}(\mathbf{R}^d; \mathbf{R}^d)$ and

$$\lim_{n \to \infty} A^n = A, \qquad \text{in} \quad \mathcal{S}'(\mathbf{R}^d; \mathbf{R}^d). \tag{2.11}$$

Theorem 2.5 (*Feynman-Kac-Itô*) *Let $A \in \mathcal{D}(\mathbf{R}^d; \mathbf{R}^d)$. For any $f \in \mathcal{D}(\mathbf{R}^d)$,*

$$\begin{aligned} e^{t/2(\partial - iA)^2} f(x) \; = \; E\Big[f(x + b(t)) \exp\Big(-i \sum_{\nu=0}^{d-1} \int_0^t A_\nu(x + b(s)) db_\nu(s) \\ - \frac{i}{2} \sum_{\nu=0}^{d-1} \int_0^t \partial_\nu A_\nu(x + b(s)) ds\Big)\Big], \end{aligned} \tag{2.12}$$

where E denotes the expectation with respect to a d dimensional standard Brownian motion $(C([0, \infty); \mathbf{R}^d)$ $\mathcal{B}(C([0, \infty); \mathbf{R}^d))$, $\{b(s)\}_{s \in [0,\infty)}$, $\{\mathcal{F}_s^b\}_{s \in [0,\infty)}$, $P)$.

Proof. See [Si]. □

The operators $\{e^{t/2(\partial - iA)^2}\}_{t \ge 0}$ on $L^2(\mathbf{R}^d)$ form a strongly continuous symmetric contraction semi-group. (See [Si], [F]).

Lemma 2.6 *For any $t > 0$, $f \in \mathcal{D}(\mathbf{R}^d \times \mathbf{R}^d)$ and $x \in \mathbf{R}^d$, we can take a suitable version of $\int_0^t f(x + b(s), \cdot) db_\nu(s)$ such that*

$$\int_0^t f(x + b(s), \cdot) db_\nu(s) \in \mathcal{D}(\mathbf{R}^d), \qquad P\text{-}a.s. \tag{2.13}$$

and

$$\langle \int_0^t f(x + b(s), \cdot) db_\nu(s), \varphi \rangle \; = \; \int_0^t \langle f(x + b(s), \cdot), \varphi \rangle db_\nu(s), \qquad P\text{-}a.s. \tag{2.14}$$

holds for every $\varphi \in \mathcal{D}'(\mathbf{R}^d)$, $0 \le \nu \le d-1$. *The equality*

$$\langle \int_0^t g(x+b(s),\cdot)ds, \varphi\rangle = \int_0^t \langle g(x+b(s),\cdot),\varphi\rangle ds, \qquad P\text{-a.s.} \quad (2.15)$$

also holds for any $g \in \mathcal{D}(\mathbf{R}^d \times \mathbf{R}^d)$, *and* $\varphi \in \mathcal{D}'(\mathbf{R}^d)$.

Proof. We can show this lemma in the same way as in [AT Lemma 2]. So we omit here the proof. □

3. Ergodicity

In this section, we shall show the probability measure μ on $(\mathcal{S}'(\mathbf{R}^d;\mathbf{R}^d), \mathcal{B}(\mathcal{S}'(\mathbf{R}^d;\mathbf{R}^d)))$ defined in section 2 has the ergodic property. Here we assume that φ is a function on $\mathbf{R}^d$, represented by the Lévy-Khinchine formula (2.3) and satisfies the condition (A). We define the transformations $\{\theta_z\}_{z\in\mathbf{R}^d}$ on $\mathcal{S}'(\mathbf{R}^d;\mathbf{R}^d)$ by

$$\theta_z A \equiv A(\cdot - z), \qquad A \in \mathcal{S}'(\mathbf{R}^d;\mathbf{R}^d).$$

It is obvious that

$$\theta_0 = \text{ Identity}, \quad \theta_x \circ \theta_y = \theta_{x+y}, \quad x,y \in \mathbf{R}^d.$$

In the following, we shall show the family of transformations $\{\theta_z\}_{z\in\mathbf{R}^d}$ is a family of measure preserving transformations on $\mathcal{S}'(\mathbf{R}^d;\mathbf{R}^d)$ and it is an ergodic family.

Lemma 3.1 $\{\theta_z\}_{z\in\mathbf{R}^d}$ *is a family of measure preserving transformations on* $\mathcal{S}'(\mathbf{R}^d;\mathbf{R}^d)$; *i.e.*

$$\mu(B) = \mu(\theta_z B), \qquad B \in \mathcal{B}(\mathcal{S}'(\mathbf{R}^d;\mathbf{R}^d)).$$

Proof. To show this lemma, it is sufficient that the characteristic functional defined by (2.1) is invariant under this transformation. By the definition of the characteristic functional,

$$\begin{aligned}\int_{\mathcal{S}'(\mathbf{R}^d;\mathbf{R}^d)} \exp\, i\langle f, \theta_z A\rangle d\mu &= \int_{\mathcal{S}'(\mathbf{R}^d;\mathbf{R}^d)} \exp\, i\langle f(\cdot+z), A\rangle d\mu \\ &= \exp\left(-\int_{\mathbf{R}^d} \varphi(Sf(x+z))dx\right) \\ &= \exp\left(-\int_{\mathbf{R}^d} \varphi(Sf(x))dx\right)\end{aligned}$$

$$= \int_{\mathcal{S}'(\mathbf{R}^d;\mathbf{R}^d)} \exp\ i\langle f, A\rangle d\mu.$$

□

Moreover, we can show that $\{\theta_z\}_{z\in\mathbf{R}^d}$ is an ergodic family.

Theorem 3.2 *The family of measure preserving transformations $\{\theta_z\}_{z\in\mathbf{R}^d}$ on $\mathcal{S}'(\mathbf{R}^d;\mathbf{R}^d)$ is an ergodic family.*

To show this theorem, we shall show the measure μ satisfies the mixing property. Before the proof of Theorem 3.2, we shall show two lemmas.

Lemma 3.3 *Suppose that V is a finite dimensional vector space over $\mathbf{R}$ and $\mathcal{A}$ is a $\mathbf{C}$-module functions of $\mathcal{S}'(\mathbf{R}^d;V)$ generated by $\{e^{i\langle\xi,\cdot\rangle};\xi\in\mathcal{D}(\mathbf{R}^d;V)\}$. Then $\mathcal{A}$ is a $\mathbf{C}$-algebra. If μ is a probability measure on $\mathcal{S}'(\mathbf{R}^d;V)$ then $\mathcal{A}$ is dense in $L^2(\mathcal{S}'(\mathbf{R}^d;V),\mu)$.*

Proof. We can show this lemma by slightly modifying the proof of Theorem 4.1 [Hi]. □

Lemma 3.4 *Suppose that V is a finite dimensional vector space over $\mathbf{R}$ and $\varphi : V\to\mathbf{R}$ is a continuous function such that $\varphi(\lambda) = O(|\lambda|^\eta)$ as $\lambda\to 0$ for some $\eta\in[1,\infty)$. If $f,g:\mathbf{R}^d\to V$ are continuous functions vanishing at ∞ and $|f|^\eta, |g|^\eta$ are integrable with respect to the Lebesgue measure, then*

$$\lim_{x\to\infty}\int_{\mathbf{R}^d}\varphi(f(x+y)+g(y))dy = \int_{\mathbf{R}^d}\varphi(f(y))dy + \int_{\mathbf{R}^d}\varphi(g(y))dy.$$

Proof. Because of the assumption, $\varphi(f(x+\cdot)+g(\cdot))$, $\varphi(f(x+\cdot))$ and $\varphi(g(\cdot))$ are integrable. For any $\varepsilon>0$, there exists $N>0$ such that for any $x\in\mathbf{R}$,

$$\int_{(B_N(0)\cup B_N(-x))^c}|\varphi(f(x+y)+g(y))-\varphi(f(x+y))-\varphi(g(y))|dy < \varepsilon,$$

where $B_N(\cdot)$ is a closed ball with radius N and center $\cdot$. We take x sufficiently large and we can assume that $B_N(0)\cap B_N(-x)=\emptyset$. We have

$$\begin{aligned}&\int_{(B_N(0)\cup B_N(-x))}|\varphi(f(x+y)+g(y))-\varphi(f(x+y))-\varphi(g(y))|dy\\ &= \int_{B_N(0)}|\varphi(f(x+y)+g(y))-\varphi(f(x+y))-\varphi(g(y))|dy\end{aligned}$$

$$
\begin{aligned}
&+\int_{B_N(-x)} |\varphi(f(x+y)+g(y))-\varphi(f(x+y))-\varphi(g(y))|dy \\
&\leq (\max_{y\in B_N(0)} |\varphi(f(x+y))|+\max_{y\in B_N(-x)} |\varphi(g(y))|)\mathrm{Vol}B_N(0) \\
&+\int_{B_N(0)} |\varphi(f(x+y)+g(y))-\varphi(g(y))|dy \\
&+\int_{B_N(-x)} |\varphi(f(x+y)+g(y))-\varphi(f(x+y))|dy. \qquad (3.1)
\end{aligned}
$$

Then we get

$$
\begin{aligned}
(3.1) = &(\max_{y\in B_N(0)} |\varphi(f(x+y))|+\max_{y\in B_N(0)} |\varphi(g(y-x))|)\mathrm{Vol}B_N(0) \\
&+\int_{B_N(0)} |\varphi(f(x+y)+g(y))-\varphi(g(y))|dy \\
&+\int_{B_N(0)} |\varphi(f(y)+g(y-x))-\varphi(f(y))|dy. \qquad (3.2)
\end{aligned}
$$

By the assumption of φ, f and g, we have

$$
\begin{aligned}
&\max_{y\in B_N(0)} |\varphi(f(x+y)+g(y))-\varphi(g(y))| \to 0, \\
&\max_{y\in B_N(0)} |\varphi(f(y)+g(y-x))-\varphi(f(y))| \to 0, \qquad \text{as} \quad |x|\to\infty.
\end{aligned}
$$

So we can conclude that (3.2) tends to 0 as $x\to\infty$. □

Proof of Theorem 3.2. We only show, for any $\xi_1, \xi_2 \in \mathcal{D}(\mathbf{R}^d;\mathbf{R}^d)$,

$$
\begin{aligned}
&\lim_{z\to\infty} \int_{\mathcal{S}'(\mathbf{R}^d;\mathbf{R}^d)} e^{i\langle \xi_1+\theta_z\xi_2,\cdot\rangle} d\mu \\
&= \int_{\mathcal{S}'(\mathbf{R}^d;\mathbf{R}^d)} e^{i\langle \xi_1,\cdot\rangle} d\mu \int_{\mathcal{S}'(\mathbf{R}^d;\mathbf{R}^d)} e^{i\langle \xi_2,\cdot\rangle} d\mu. \qquad (3.3)
\end{aligned}
$$

We have

$$
\begin{aligned}
\int_{\mathcal{S}'(\mathbf{R}^d;\mathbf{R}^d)} e^{i\langle \xi_1+\theta_z\xi_2,\cdot\rangle} d\mu &= \exp\left(-\int_{\mathbf{R}^d} \varphi(S(\xi_1+\theta_z\xi_2))dx\right) \\
\int_{\mathcal{S}'(\mathbf{R}^d;\mathbf{R}^d)} e^{i\langle \xi_1,\cdot\rangle} d\mu &= \exp\left(-\int_{\mathbf{R}^d} \varphi(S(\xi_1))dx\right) \\
\int_{\mathcal{S}'(\mathbf{R}^d;\mathbf{R}^d)} e^{i\langle \xi_2,\cdot\rangle} d\mu &= \exp\left(-\int_{\mathbf{R}^d} \varphi(S(\xi_2))dx\right).
\end{aligned}
$$

Since $S(\xi_1)$ and $S(\xi_2)$ are continuous and vanishing at ∞, $|S(\xi_1)|^\eta$ and

$|S(\xi_2)|^\eta$ are integrable, then, by Lemma 3.4, we get (3.3). □

4. Preliminaries

In this section, we will prove some important lemmas. For any $t_0 > 0$, we set $\mathcal{T} = (0, t_0]$. We set for any $t \in \mathcal{T}$,

$$\begin{aligned} T_t^{A,n} f(x) &\equiv e^{t/2\,(\partial - iA^n)^2} f(x) \\ &= E[f(x+b(t)) \exp(-i \sum_{\nu=0}^{d-1} \int_0^t A_\nu^n(x+b(s)) db_\nu(s) \\ &\quad - \frac{i}{2} \sum_{\nu=0}^{d-1} \int_0^t \partial_\nu A_\nu^n(x+b(s)) ds)], \text{ for } f \in \mathcal{D}(\mathbf{R}^d), \quad (4.1) \end{aligned}$$

where $A^n \in \mathcal{D}(\mathbf{R}^d; \mathbf{R}^d)$ in (2.10).

Lemma 4.1 *For any $d > 2$ and $T > 0$, we set*

$$f(s,x) = |x|^{2-d} (2T - 2s)^{(d-4)/2} \exp\left(\frac{-|x|^2}{2(T-s)}\right). \quad (4.2)$$

Then $f(s,x)$ is a solution of the partial differential equation

$$\frac{\partial f}{\partial s} + \frac{1}{2} \Delta f = 0, \quad (4.3)$$

at $(s,x) \in (-\infty, T) \times \mathbf{R}^d \backslash \{0\}$. Moreover, $f(t,x) \geq f(t,y) \Leftrightarrow |x| \leq |y|$ for each $t < T$.

Proof. We can show this by a direct calculation. □

Lemma 4.2 *Let $0 < t < T$ and $\varepsilon > 0$. Then, for any $d > 2$,*

$$E[\mathrm{Vol}(\{z \mid \inf_{s \in [0,t]} |b(s) - z| \leq \varepsilon\})] \leq \varepsilon^{d-2} \cdot K(d,t,T,\varepsilon), \quad (4.4)$$

where

$$\begin{aligned} K(d,t,T,\varepsilon) &= \frac{\pi^{d/2}}{\Gamma(d/2+1)} \varepsilon^2 + \frac{2\pi^{d/2}}{\Gamma(d/2)} \left(\frac{T}{T-t}\right)^{(d-4)/2 \vee 0} T \\ &\quad \exp\left(\frac{\varepsilon^2}{2(T-t)} - \frac{\varepsilon^2}{2T}\right). \end{aligned}$$

Proof. Let $\varepsilon > 0$ and $0 < t < T$. We define a stopping time $\tau_\varepsilon(a)$ by

$$\tau_\varepsilon(a) \equiv \inf\{s \in [0,\infty) \mid |b(s)+a| = \varepsilon\} \wedge t$$

for $a \in \mathbf{R}^d \backslash \{0\}$ and $\varepsilon < |a|$. Let $X(s)$ be the process

$$X(s) = b(s \wedge \tau_\varepsilon(a)) + a = \int_0^s I_{u \le \tau_\varepsilon(a)} db(u) + a.$$

By Lemma 4.1, the process $f(s \wedge \tau_\varepsilon(a), X(s))$ is a martingale. In fact

$$\begin{aligned} & f(s \wedge \tau_\varepsilon(a), X(s)) \\ &= f(0,a) + \sum_{\nu=0}^{d-1} \int_0^s I_{u \le \tau_\varepsilon(a)} \frac{\partial f}{\partial x_\nu}(u, X(u)) dX_\nu(u) \\ &\qquad + \int_0^s I_{u \le \tau_\varepsilon(a)} \{\frac{\partial f}{\partial u}(u, X(u)) + \frac{1}{2}\Delta f(u, X(u))\} du \\ &= f(0,a) + \sum_{\nu=0}^{d-1} \int_0^{s \wedge \tau_\varepsilon(a)} \frac{\partial f}{\partial x_\nu}(u, X(u)) dX_\nu(u). \end{aligned}$$

And by the Doob martingale inequality, we have

$$\begin{aligned} & P[\{\omega \mid \inf_{s \in [0,t]} |X(s,\omega)| \le \varepsilon\}] \\ &\le P[\{\omega \mid \sup_{s \in [0,t]} f(s \wedge \tau_\varepsilon(a), X(s,\omega)) \\ &\qquad \ge \varepsilon^{2-d} \min_{s \in [0,t]} \{(2T-2s)^{(d-4)/2}\} \exp \frac{-\varepsilon^2}{2(T-t)}\}] \\ &\le \varepsilon^{d-2} \max_{s \in [0,t]} \{(2T-2s)^{(4-d)/2}\} \exp \frac{\varepsilon^2}{2(T-t)} E[f(t \wedge \tau_\varepsilon(a), X(t,\omega))] \\ &= \varepsilon^{d-2} \max_{s \in [0,t]} \{(2T-2s)^{(4-d)/2}\} \exp \frac{\varepsilon^2}{2(T-t)} E[f(0, X(0,\omega))] \\ &= \left(\frac{\varepsilon}{|a|}\right)^{d-2} (2T)^{(d-4)/2} \max_{s \in [0,t]} \{(2T-2s)^{(4-d)/2}\} \\ &\qquad \exp\left(\frac{\varepsilon^2}{2(T-t)} - \frac{|a|^2}{2T}\right). \end{aligned} \tag{4.5}$$

The estimate (4.5) leads to

$$E[\mathrm{Vol}\{z \mid \inf_{s \in [0,t]} |b(s) - z| \le \varepsilon\}]$$

$$
\begin{aligned}
&= \int_{\mathbf{R}^d} P[\{\omega \mid \inf_{s\in[0,t]} |b(s) - z| \le \varepsilon\}]dz \\
&= \int_{\mathbf{R}^d} \chi_{\{|z|\le\varepsilon\}} + \chi_{\{|z|>\varepsilon\}} P[\{\omega \mid \inf_{s\in[0,t]} |b(s) - z| \le \varepsilon\}]dz \\
&\le \frac{\pi^{d/2}}{\Gamma(d/2+1)}\varepsilon^d + \int_\varepsilon^\infty \left(\frac{\varepsilon}{r}\right)^{d-2} (2T)^{(d-4)/2} \max_{s\in[0,t]}\{(2T-2s)^{(4-d)/2}\} \\
&\qquad \exp\left(\frac{\varepsilon^2}{2(T-t)} - \frac{r^2}{2T}\right) \frac{2\pi^{d/2}}{\Gamma(d/2)} r^{d-1} dr \\
&= \frac{\pi^{d/2}}{\Gamma(d/2+1)}\varepsilon^d + \frac{2\pi^{d/2}}{\Gamma(d/2)}\varepsilon^{d-2}(2T)^{(d-4)/2} \max_{s\in[0,t]}\{(2T-2s)^{(4-d)/2}\} \\
&\qquad \cdot T \cdot \exp\left(\frac{\varepsilon^2}{2(T-t)} - \frac{\varepsilon^2}{2T}\right),
\end{aligned}
$$

where χ_K is the indicator function of the set K. We readily conclude (4.4). □

For each $f \in \mathcal{D}(\mathbf{R}^d)$ and $x \in \mathbf{R}^d$, it is obvious that $T_t^{A,n} f(x)$ is $\mathcal{B}(\mathcal{S}'(\mathbf{R}^d;\mathbf{R}^d))$ measurable [AT]. By using Lemma 2.5, we have for any $t \in \mathcal{T}$,

$$
\begin{aligned}
&F_{m,n}^t(x) \equiv \|T_t^{A,m} f(x) - T_t^{A,n} f(x)\|^2_{L^2(\mathcal{S}'(\mathbf{R}^d;\mathbf{R}^d))} \\
&\le \int_{\mathcal{S}'(\mathbf{R}^d;\mathbf{R}^d)} E[|f(x+b(t))| \\
&\times |1 - \exp\{-i(\sum_{\nu=0}^{d-1} \int_0^t (A_\nu^m - A_\nu^n) db_\nu(s) \\
&\qquad + \frac{1}{2}\sum_{\nu=0}^{d-1} \int_0^t (\partial_\nu A_\nu^m - \partial_\nu A_\nu^n) ds)\}|]^2 d\mu \\
&\le \int_{\mathcal{S}'(\mathbf{R}^d;\mathbf{R}^d)} E[|f(x+b(t))|^2 \\
&\times 2\{1 - \cos(\sum_{\nu=0}^{d-1} \langle \int_0^t f_{m,n}(x+b(s),\cdot) db_\nu(s) \\
&\qquad + \int_0^t \frac{1}{2} g_{m,n;\nu}(x+b(s),\cdot) ds, A_\nu\rangle)\}] d\mu,
\end{aligned}
$$

where

$$
f_{m,n}(x,y) \equiv w_m(x)\rho_{1/m}(x-y) - w_n(x)\rho_{1/n}(x-y)
$$

$$g_{m,n;\nu}(x,y) \equiv \frac{\partial}{\partial x_\nu} f_{m,n}(x,y).$$

By (2.4) and Fubini's theorem, we have

$$F^t_{n,m}(x) \le 2E[|f(x+b(t))|^2(1-\exp(-\int_{\mathbf{R}^d}\psi_{m,n}(t,z,x)dz))], \quad (4.6)$$

where

$$\begin{aligned}\psi_{m,n}(t,z,x) \;&=\; \psi\Big(\sum_{\nu=0}^{d-1}\int_0^t db_\nu(s)\int_{\mathbf{R}^d} dv S_{\cdot\nu}(z-v) f_{m,n}(x+b(s),v)\\ &\quad + \frac{1}{2}\sum_{\nu=0}^{d-1}\int_0^t ds \int_{\mathbf{R}^d} dv S_{\cdot\nu}(z-v) g_{m,n;\nu}(x+b(s),v)\Big).\end{aligned}$$

Lemma 4.3 *For each $x \in \mathbf{R}^d$, $E[\exp(-\int_{\mathbf{R}^d}\psi_{m,n}(t,z,x)dz)]$ converges to 1 uniformly in $t \in \mathcal{T}$ as $m,n \to \infty$.*

Proof. By Jensen's inequality,

$$\begin{aligned}0 \;&\le\; E[1-\exp(-\int_{\mathbf{R}^d}\psi_{m,n}(t,z,x)dz)]\\ &\le\; 1-\exp\,(-E[\int_{\mathbf{R}^d}\psi_{m,n}(t,z,x)\,dz]).\end{aligned}$$

So it is sufficient to show that

$$G_{m,n}(x) \equiv \sup_{t\in\mathcal{T}} E[\int_{\mathbf{R}^d}\psi_{m,n}(t,z,x)\,dz] \to 0 \quad \text{as} \quad n,m\to\infty.$$

Put

$$V(t,x+b,\varepsilon) \equiv \{z\in\mathbf{R}^d \mid \inf_{s\in[0,t]}|x+b(s)-z| \le \varepsilon\}, \quad (4.7)$$

then by (A),

$$\begin{aligned}G_{m,n}(x) \;&\le\; \sup_{t\in\mathcal{T}} E[\int_{V(t,x+b,\varepsilon)}\psi_{m,n}(t,z,x)\,dz]\\ &\quad + \sup_{t\in\mathcal{T}} E[\int_{V^c(t,x+b,\varepsilon)}\psi_{m,n}(t,z,x)\,dz]\\ &\le\; \sup_{t\in\mathcal{T}} c_0 E[\mathrm{Vol}\,V(t,x+b,\varepsilon)]\\ &\quad + \sup_{t\in\mathcal{T}} E[\int_{V^c(t,x+b,\varepsilon)}\psi_{m,n}(t,z,x)\,dz]\end{aligned}$$

$$= c_0 E[\mathrm{Vol}\, V(t_0, x+b, \varepsilon)] + \sup_{t\in\mathcal{T}} E[\int_{V^c(t,x+b,\varepsilon)} \psi_{m,n}(t,z,x)\, dz], \tag{4.8}$$

because $V(t, x+b, \varepsilon)$ is increasing in t. By Lemma 4.2, the first term of (4.8) is dominated as follows:

$$E[\mathrm{Vol}\, V(t_0, x+b, \varepsilon)] = \int_{\mathbf{R}^d} P[\{\omega \mid \inf_{s\in[0,t_0]} |x+b(s)-z| \le \varepsilon\}] dz \le \varepsilon^{d-2} \cdot K(d, t_0, t_0+1, \varepsilon).$$

We shall show in the following the second term tends to 0 as $n, m \to \infty$ for arbitrary but fixed $\varepsilon > 0$. We may assume that n and m are so large that $1/n, 1/m < \varepsilon/\sqrt{d}$ hold. By (2.6), we have

$$\sup_{t\in\mathcal{T}} E[\int_{V^c(t,x+b,\varepsilon)} \psi_{m,n}(t,z,x)\, dz]$$
$$\le \sup_{t\in\mathcal{T}} c_1 E[\int_{\mathbf{R}^d} \chi_{V^c(t,x+b,\varepsilon)}(z) dz (\sum_{j=1}^{d} \mid \sum_{\nu=0}^{d-1} \int_0^t db_\nu(s) \int_{\mathbf{R}^d} dv S_{j\nu}(z-v) f_{m,n}(x+b(s), v) + \frac{1}{2}\sum_{\nu=0}^{d-1} \int_0^t ds \int_{\mathbf{R}^d} dv S_{j\nu}(z-v) g_{m,n;\nu}(x+b(s), v) \mid^2)^{\eta/2}]. \tag{4.9}$$

Since the following inequality

$$E[(\sum_{j=1}^{d} \mid \sum_{\nu=0}^{d-1} a_{j\nu}(\omega) \mid^2)^{\eta/2}] \le E[(d\sum_{j,\nu} a_{j\nu}^2)^{\eta/2}] \le d^{\eta/2}(\sum_{j,\nu} E[a_{j\nu}^2])^{\eta/2} \le d^{\eta/2} \sum_{j,\nu} (E[a_{j\nu}^2])^{\eta/2}$$

holds for real $a_{j\nu}$'s, which is a consequence of Jensen's inequality (recall $\eta/2 \le 1$). We get

$$\sup_{t\in\mathcal{T}} E[\int_{V^c(t,x+b,\varepsilon)} \psi_{m,n}(t,z,x)\, dz]$$
$$\le \sup_{t\in\mathcal{T}} d^{\eta/2} c_1 \sum_{j,\nu} \int_{\mathbf{R}^d} dz (E[\chi_{V^c(t,x+b,\varepsilon)}(z) \mid \int_0^t db_\nu(s) \int_{\mathbf{R}^d} dv S_{j\nu}(z-v) f_{m,n}(x+b(s), v)$$

$$
\begin{aligned}
&+\frac{1}{2}\int_0^t ds\int_{\mathbf{R}^d} dv S_{j\nu}(z-v)g_{m,n;\nu}(x+b(s),v)\ |^2])^{\eta/2},\\
&\leq \sup_{t\in\mathcal{T}} d^{\eta/2}c_1\sum_{j,\nu}\int_{\mathbf{R}^d} dz(2E[\chi_{V^c(t,x+b,\varepsilon)}(z)\\
&\quad |\int_0^t db_\nu\int_{\mathbf{R}^d} dv S_{j\nu}(z-v)f_{m,n}(x+b(s),v)(s)\ |^2]\\
&\quad +\frac{1}{2}E[\chi_{V^c(t,x+b,\varepsilon)}(z)\\
&\quad |\int_0^t ds\int_{\mathbf{R}^d} dv S_{j\nu}(z-v)g_{m,n;\nu}(x+b(s),v)\ |^2])^{\eta/2}\\
&= \sup_{t\in\mathcal{T}} d^{\eta/2}c_1\sum_{j,\nu}\int_{\mathbf{R}^d} dz(2E[\chi_{|x-z|>\varepsilon}(z)\\
&\quad |\int_0^{t\wedge\tau_\varepsilon(x-z)} db_\nu(s)\int_{\mathbf{R}^d} dv S_{j\nu}(z-v)f_{m,n}(x+b(s),v)\ |^2]\\
&\quad +\frac{1}{2}E[\chi_{|x-z|>\varepsilon}(z)\\
&\quad |\int_0^{t\wedge\tau_\varepsilon(x-z)} ds\int_{\mathbf{R}^d} dv S_{j\nu}(z-v)g_{m,n;\nu}(x+b(s),v)\ |^2])^{\eta/2}\\
&= \sup_{t\in\mathcal{T}} d^{\eta/2}c_1\sum_{j,\nu}\int_{\mathbf{R}^d} dz(2E[|\int_0^{t\wedge\tau_\varepsilon(x-z)} db_\nu(s)\\
&\quad \int_{\mathbf{R}^d} dv\chi_{|x-z|>\varepsilon}(z)S_{j\nu}(z-v)f_{m,n}(x+b(s),v)\ |^2]\\
&\quad +\frac{1}{2}E[|\int_0^{t\wedge\tau_\varepsilon(x-z)} ds\\
&\quad \int_{\mathbf{R}^d} dv\chi_{|x-z|>\varepsilon}(z)S_{j\nu}(z-v)g_{m,n;\nu}(x+b(s),v)\ |^2])^{\eta/2}\\
&= \sup_{t\in\mathcal{T}} d^{\eta/2}c_1\sum_{j,\nu}\int_{\mathbf{R}^d} dz(2E[\int_0^{t\wedge\tau_\varepsilon(x-z)} ds\\
&\quad |\int_{\mathbf{R}^d} dv\chi_{|x-z|>\varepsilon}(z)S_{j\nu}(z-v)f_{m,n}(x+b(s),v)\ |^2]\\
&\quad +\frac{1}{2}E[|\int_0^{t\wedge\tau_\varepsilon(x-z)} ds\\
&\quad \int_{\mathbf{R}^d} dv\chi_{|x-z|>\varepsilon}(z)S_{j\nu}(z-v)g_{m,n;\nu}(x+b(s),v)\ |^2])^{\eta/2}
\end{aligned}
$$

$$\le d^{\eta/2} c_1 \sum_{j,\nu} \int_{\mathbf{R}^d} dz (2E[\int_0^{t_0} ds$$

$$| \int_{\mathbf{R}^d} dv \chi_{V^c(s,x+b,\varepsilon)}(z) S_{j\nu}(z-v) f_{m,n}(x+b(s),v) |^2]$$

$$+ \frac{1}{2} E[t_0 \int_0^{t_0} ds \, | \int_{\mathbf{R}^d} dv \chi_{V^c(s,x+b,\varepsilon)}(z)$$

$$S_{j\nu}(z-v) g_{m,n;\nu}(x+b(s),v) |^2])^{\eta/2}. \tag{4.10}$$

Since $\operatorname{supp}\rho_{1/n}, \operatorname{supp}\rho_{1/m} \subset \{z \in \mathbf{R}^d \mid |z| \le \frac{\varepsilon}{2}\}$,

$$| \int_{\mathbf{R}^d} dv \chi_{V^c(s,x+b,\varepsilon)}(z) S_{j\nu}(z-v) f_{m,n}(x+b(s),v) |$$
$$\le c_2 \sup_{|x+b(s)-v| \le \varepsilon/2} \left(\chi_{V^c(s,x+b,\varepsilon)}(z) \frac{1}{|z-v|^{d-1}} \right)$$
$$\le H(x+b(s),z), \tag{4.11}$$

and

$$| \int_{\mathbf{R}^d} dv \chi_{V^c(s,x+b,\varepsilon)}(z) S_{j\nu}(z-v) g_{m,n;\nu}(x+b(s),v) |$$
$$\le c_2 \sup_{|x+b(s)-v| \le \varepsilon/2} \left(\chi_{V^c(s,x+b,\varepsilon)}(z) \frac{1}{|z-v|^{d-1}} \right)$$
$$+ c_3 \sup_{|x+b(s)-v| \le \varepsilon/2} \left(\chi_{V^c(s,x+b,\varepsilon)}(z) \partial_\nu \frac{1}{|z-v|^{d-1}} \right)$$
$$\le H(x+b(s),z), \tag{4.12}$$

hold, where

$$H(x+y,z) = \begin{cases} C_\varepsilon, & |x+y| \ge \frac{|z|}{4} \\ C_\varepsilon (1 \wedge \frac{1}{|z|^{d-1}}), & |x+y| < \frac{|z|}{4}, \end{cases}$$

with constants c_2, c_3 and C_ε. Also, by the definition of $f_{m,n}$ and $g_{m,n;\nu}$,

$$| \int_{\mathbf{R}^d} dv \chi_{V^c(s,x+b,\varepsilon)}(z) S_{j\nu}(z-v) f_{m,n}(x+b(s),v) | \to 0$$
$$| \int_{\mathbf{R}^d} dv \chi_{V^c(s,x+b,\varepsilon)}(z) S_{j\nu}(z-v) g_{m,n;\nu}(x+b(s),v) | \to 0,$$
$$\text{as} \quad m,n \to \infty$$

hold. Thus if we show

$$\int_{\mathbf{R}^d} E[\int_0^{t_0} H^2(x+b(s),z)ds]^{\eta/2}dz < \infty, \tag{4.13}$$

then we have by Lebesgue's dominated convergence theorem that the second term of (4.8) converges to 0 as $n, m \to \infty$, because the left hand sides of (4.11) and (4.12) tend to 0 as $m, n \to \infty$. Finally we shall show (4.13). Since

$$P[\sup_{s\in\mathcal{T}} |x+b(s)| \geq \frac{|z|}{4}] \leq 2d \exp\left(-\frac{1}{2dt_0}(|z|/4-|x|)_+^2\right),$$

we have

$$\begin{aligned} E[H^2(x+b(s),z)] &= E[H^2(x+b(s),z); \sup_{s\in\mathcal{T}} |x+b(s)| \geq \frac{|z|}{4}] \\ &\quad + E[H^2(x+b(s),z); \sup_{s\in\mathcal{T}} |x+b(s)| < \frac{|z|}{4}] \\ &\leq C_\varepsilon^2 (P[\sup_{s\in\mathcal{T}} |x+b(s)| \geq \frac{|z|}{4}] + (1 \wedge \frac{1}{|z|^{2(d-1)}})), \end{aligned} \tag{4.14}$$

the right hand side in (4.14) is integrable. □

Lemma 4.4 *For each $t \in \mathcal{T}$, $f \in \mathcal{D}(\mathbf{R}^d)$, $\{T_t^{A,n} f\}_{n=1}^\infty$ is a Cauchy sequence in $L^2(\mathcal{S}'(\mathbf{R}^d;\mathbf{R}^d) \times \mathbf{R}^d, \mu \times dx)$, where dx denotes the Lebesgue measure in $\mathbf{R}^d$. Moreover, this convergence is uniform in $t \in \mathcal{T}$,*

$$\sup_{t\in\mathcal{T}} \|T_t^{A,n} f - T_t^{A,m} f\|_{L^2(\mathcal{S}'(\mathbf{R}^d;\mathbf{R}^d)\times\mathbf{R}^d)} \to 0 \quad \text{as} \quad m,n \to \infty.$$

Proof. Since $T_t^{A,n} f(x)$ is continuous with respect to $x \in \mathbf{R}^d$, $T_t^{A,n} f$ is $\mathcal{B}(\mathcal{S}'(\mathbf{R}^d;\mathbf{R}^d) \times \mathbf{R}^d)$ measurable.

By (4.6) and Fubini's theorem,

$$\begin{aligned} &\|T_t^{A,n} f - T_t^{A,m} f\|^2_{L^2(\mathcal{S}'(\mathbf{R}^d;\mathbf{R}^d)\times\mathbf{R}^d)} \\ &= \int_{\mathbf{R}^d} F_{m,n}^t(x)dx \\ &\leq 2\int_{\mathbf{R}^d} E[|f(x+b(t))|^2\{1-\exp(-\int_{\mathbf{R}^d} \psi_{m,n}(t,z,x)\,dz)\}]dx \\ &= 2\int_{B_k(0)} E[|f(x+b(t))|^2\{1-\exp(-\int_{\mathbf{R}^d} \psi_{m,n}(t,z,x)\,dz)\}]dx \end{aligned}$$

$$+2\int_{B_k(0)^c} E[|f(x+b(t))|^2\{1-\exp(-\int_{\mathbf{R}^d}\psi_{m,n}(t,z,x)\,dz)\}]dx, \tag{4.15}$$

where $B_k(0)$ is a closed ball with radius k and center 0. The second term of (4.15) is dominated as follows:

$$\begin{aligned}
&\int_{B_k(0)^c} E[|f(x+b(t))|^2\{1-\exp(-\int_{\mathbf{R}^d}\psi_{m,n}(t,z,x)\,dz)\}]dx,\\
&= \int_{B_k(0)^c} E[1_{\{\inf_{0\le s\le t}|x+b(s)|\le r\}}|f(x+b(t))|^2\\
&\qquad \{1-\exp(-\int_{\mathbf{R}^d}\psi_{m,n}(t,z,x)\,dz)\}]dx\\
&\le M^2\int_{B_k(0)^c} P[\{\omega \mid \inf_{0\le s\le t_0}|x+b(s)|\le r\}]dx,
\end{aligned} \tag{4.16}$$

where $r>0$ and $M>0$ are constants such that

$$\mathrm{supp} f\subset\{|x|\le r\}, \qquad \max_{\mathbf{R}^d}|f|\le M.$$

Since $P[\{\omega \mid \inf_{0\le s\le t_0}|x+b(s)|\le r\}]$ is integrable with respect to dx in $\mathbf{R}^d$, the second term of (4.15) converges to 0 uniformly in $t\in\mathcal{T}$ and m,n as $k\to\infty$.

Next, the first term of (4.15) is dominated as follows: For any $k>0$,

$$\begin{aligned}
&\sup_{t\in\mathcal{T}}\int_{B_k(0)} E[|f(x+b(t))|^2(1-\exp(-\int_{\mathbf{R}^d}\psi_{m,n}(t,z,x)\,dz))]dx\\
&\le M^2\int_{B_k(0)}\sup_{t\in\mathcal{T}} E[1-\exp(-\int_{\mathbf{R}^d}\psi_{m,n}(t,z,x)\,dz)]dx.
\end{aligned} \tag{4.17}$$

By Lemma 4.3, the integrand in the right hand side of (4.17) tends to 0 as $n,m\to\infty$. Since $B_k(0)$ is compact, we have by the bounded convergence theorem that (4.17) converges to 0 as $n,m\to\infty$. □

5. Construction of the Generator

First of all, we introduce a notion of a semi-group process. Let $(\Omega,\mathcal{F},P)$ be a probability space. Let X be a Hilbert space. We denote the space of all continuous endomorphism of X by $\mathcal{L}(X)$.

Definition 5.1 If an $\mathcal{L}(X)$-valued continuous process $\{T_t\}_{t\ge 0}$ satisfies that

$$P(\{T_t\}_{t\geq 0} \text{ is a (strongly continuous) symmetric contraction semi-group}) = 1,$$

we call it a (strongly continuous) semi-group process on X with respect to P.

Remark 5.2. In this case, by contractivity and symmetricity, $\{T_t\}_{t\geq 0}$ is a self-adjoint semi-group P-a.s.

In this section, we shall construct a semi-group process on $L^2(\mathbf{R}^d)$ whose generator is formally expressed as "$(\partial - iA)^2$" and show it is self-adjoint. First, let us consider the problem in a general setting.

Theorem 5.3 *Let $\{T_t^n\}_{t\geq 0}(n = 1, 2, \cdots)$ be strongly continuous semi-group processes. If there exists a countable dense subset $\mathcal{D}$ of X such that for any $f \in \mathcal{D}$, $\varepsilon > 0$ and $T > 0$,*

$$\lim_{m,n\to\infty} \sup_{t\in[0,T]} P(\|T_t^n f - T_t^m f\|_X > \varepsilon) = 0. \tag{5.1}$$

Then there exists a strongly continuous semi-group process $\{T_t\}_{t\geq 0}$ such that

$$\lim_{n\to\infty} \sup_{t\in[0,T]} P(\|T_t^n f - T_t f\|_X > \varepsilon) = 0 \qquad \textit{for } f \in X. \tag{5.2}$$

Proof. Without loss of generality, we may assume that for any $\omega \in \Omega$, $\{T_t^n(\omega)\}_{t\geq 0}$ is a strongly continuous symmetric contraction semi-group for any $n \in \mathbf{N}$. By (5.1) and diagonal method, we can select a subsequence $\{n_k\}_{k=1}^{\infty}$ such that, for each $f \in \mathcal{D}$,

$$\sum_{l=1}^{\infty} \frac{1}{2^l} \sup_{0\leq t\leq l} P(\|T_t^{n_k} f - T_t^{n_{k+1}} f\|_X > \frac{1}{2^k}) < \frac{1}{2^k},$$

except for a finite number of k's which may depend on f. We set,

$$\Lambda_{k,t,f} \equiv \{\|T_t^{n_k} f - T_t^{n_{k+1}} f\|_X > \frac{1}{2^k}\}.$$

By using Borel-Cantelli's lemma, $P(\liminf_{k\to\infty} \Lambda_{k,t,f}^c) = 1$, for $f \in \mathcal{D}$. For any $\omega \in \liminf_{k\to\infty} \Lambda_{k,t,f}^c$, $\{T_t^{n_k} f\}_{k=1}^{\infty}$ is a Cauchy sequence in X. Since $\mathcal{D}$ is contained in X densely and $\mathrm{Card}(\mathcal{D}) = \aleph_0$, for any $f \in X$, $\{T_t^{n_k} f\}_{k=1}^{\infty}$ is a Cauchy sequence in X. We denote the limit of $\{T_t^{n_k} f\}_{k=1}^{\infty}$ by $T_t f$. It is obvious that T_t is a linear operator and $\|T_t\| \leq 1$. Then we have

$$P(T_t^{n_k} f \to T_t f \quad \text{in} \quad X, \quad \text{for } f \in X) = 1. \tag{5.3}$$

In the following, we shall construct semi-group process $\{T_t\}_{t\geq 0}$. □

Step 1 (strong continuity)

Let $\{t_n\}_{n=1}^{\infty} \subset \mathbf{Q} \cap [0,1]$ be a decreasing sequence which converges to 0. Then

$$P(\lim_{n\to\infty} \|T_{t_n} f - f\|_X = 0, \qquad \text{for } f \in X) = 1 \tag{5.4}$$

holds.

Proof. Because $\{T_t^n\}_{t\geq 0}$ are contraction semi-groups and (5.3), we can show that $\|T_t f\|_X \leq \|f\|_X$ P-a.s for any $t \geq 0$. For any $k \in \mathbf{N}$, $f \in \mathcal{D}$, $\|T_t^{n_k} f - f\|_X$ is an increasing function with respect to $t \geq 0$. In fact, by the spectrum representation of the self-adjoint operator,

$$\|T_t^{n_k} f - f\|_X^2 = \int_0^{\infty} (1 - e^{-t\lambda})^2 d\|E_\lambda^{n_k} f\|^2. \tag{5.5}$$

$(1 - e^{-t\lambda})^2$ is an increasing function of t, then we can show this.

Since $\|T_t f - f\|_X$ is the limit of the sequence of the increasing functions $\|T_t^{n_k} f - f\|_X$, $\|T_t f - f\|_X$ is also increasing. For any $\varepsilon > 0$ and $m \in \mathbf{N}$,

$$\begin{aligned} P(\lim_{n\to\infty} \|T_{t_n} f - f\|_X > \varepsilon) &\leq P(\|T_{t_m} f - f\|_X > \varepsilon) \\ &\leq P(\|T_{t_m}^{n_k} f - T_{t_m} f\|_X > \varepsilon/2) + P(\|T_{t_m}^{n_k} f - f\|_X > \varepsilon/2). \end{aligned} \tag{5.6}$$

By the definition of $T_t f$, for any $T > 1$,

$$\begin{aligned} &\sup_{t\in\mathbf{Q}\cap[0,T]} P(\|T_t^n f - T_t f\|_X > \varepsilon) \\ &\quad = \sup_{t\in\mathbf{Q}\cap[0,T]} P(\lim_{k\to\infty} \|T_t^n f - T_t^{n_k} f\|_X > \varepsilon) \\ &\quad \leq \sup_{t\in\mathbf{Q}\cap[0,T]} \liminf_{k\to\infty} P(\|T_t^n f - T_t^{n_k} f\|_X > \varepsilon) \\ &\quad \leq \liminf_{k\to\infty} \sup_{t\in\mathbf{Q}\cap[0,T]} P(\|T_t^n f - T_t^{n_k} f\|_X > \varepsilon). \end{aligned} \tag{5.7}$$

Then, the first term of (5.6) can be arbitrarily small independently of $t \in \mathbf{Q} \cap [0,T]$, the second term tends to 0 as $m \to \infty$. Since $\mathcal{D}$ is contained in X densely and Card $(\mathcal{D}) = \aleph_0$, we have (5.4). □

Now we set

$$\Omega_t \equiv \{\omega \in \Omega : T_t^{n_k} f \to T_t f \quad \text{in} \quad X, \quad \text{for } f \in X\} \tag{5.8}$$

$$\Omega_0 \equiv \{\omega \in \Omega : \lim_{n\to\infty} \|T_{t_n} f - f\|_X = 0, \qquad \text{for } f \in X\} \tag{5.9}$$

and

$$\Omega_1 \equiv \cap_{t \in \mathbf{Q}\cap[0,\infty)} \Omega_t \bigcap \Omega_0. \tag{5.10}$$

We readily see $P(\Omega_1) = 1$.

Remark 5.4. Ω_1 does not depend on the decreasing sequence $\{t_n\}_{n=1}^{\infty}$, because $\|T_t f - f\|_X$ is increasing with respect to $t \geq 0$.

Step 2 (semi-group property)

For any $\omega \in \Omega_1$, $t \in \mathbf{Q} \cap [0, \infty)$,

$$T_s \circ T_t = T_t \circ T_s = T_{t+s}. \tag{5.11}$$

Proof. For any $f \in X$,

$$\begin{aligned} &\|T_s \circ T_t f - T_{s+t} f\|_X \\ &\quad \leq \|T_s \circ T_t f - T_s^{n_k} \circ T_t f\|_X + \|T_s^{n_k} \circ T_t f - T_s^{n_k} \circ T_t^{n_k} f\|_X \\ &\quad + \|T_s^{n_k} \circ T_t^{n_k} f - T_{t+s}^{n_k} f\|_X + \|T_{t+s}^{n_k} f - T_{t+s} f\|_X. \end{aligned} \tag{5.12}$$

The third term of the right-hand side of (5.12) is 0, because of the semi-group property of $\{T_t^{n_k}\}_{t\geq 0}$. By (5.8), the first and fourth terms tend to 0 as $k \to \infty$. Finally, for the second term, we have

$$\begin{aligned} \|T_s^{n_k} \circ T_t f - T_s^{n_k} \circ T_t^{n_k} f\|_X &\leq \|T_s^{n_k}\| \cdot \|T_t f - T_t^{n_k} f\|_X \\ &\leq \|T_t f - T_t^{n_k} f\|_X. \end{aligned}$$

Then we get (5.11). □

Step 3 (construction)

For any $s \in [0, \infty)$, we take $\{s_n\} \subset \mathbf{Q} \cap [0, \infty)$ such that $s_n \downarrow s$. By the semi-group property, contraction property and Step 1, we can show that for any $f \in X$, $\{T_{s_n} f\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Therefore we define the operator $\{T_t\}_{t\geq 0}$ as follows.

For $\omega \in \Omega_1$ and $f \in X$,

$$T_t f \equiv \lim_{n\to\infty} T_{t_n} f. \tag{5.13}$$

Then for any $\omega \in \Omega_1$, $\{T_t\}_{t\geq 0}$ is a strongly continuous symmetric contraction semi-group. In fact, for any $\omega \in \Omega_1$, by Step 1 and the definition of $\{T_t\}_{t\geq 0}$,

$$T_t f \to f \quad \text{as} \quad t \to 0 \quad \text{in} \quad X. \tag{5.14}$$

Hence the semi-group $\{T_t\}_{t\geq 0}$ is strongly continuous. By the definition of $\{T_t\}_{t\geq 0}$, it is contractive. Because $\{T_t^{n_k}\}_{t\geq 0}$ is symmetric and $T_t^{n_k} \to T_t$ strongly, it is obvious that $\{T_t\}_{t\geq 0}$ is a symmetric semi-group. Since $\{T_t^{n_k}\}_{t\geq 0}$ and $\{T_t\}_{t\geq 0}$ are strongly continuous, we have (5.2) by (5.7) easily. Then we can conclude this theorem.

Remark 5.5. By (5.14), we may denote

$$\Omega_0 = \{\omega \in \Omega : \lim_{t\to 0} \|T_t f - f\|_X = 0, \qquad \text{for } f \in X\}. \tag{5.15}$$

In our case, $\{T_t^{A,n}\}_{t\geq 0}(n = 1, 2, \cdots)$ $A \in \mathcal{S}'(\mathbf{R}^d; \mathbf{R}^d)$ are strongly continuous semi-group processes on $L^2(\mathbf{R}^d)$. By Lemma 4.4, we can easily check they satisfy the condition of Theorem 5.3. Therefore there is a strongly continuous symmetric contraction semi-group $\{T_t^A\}_{t\geq 0}$ for μ-almost surely. Hence there exists a positive self-adjoint operator $\mathcal{G}_A$ for μ-almost surely which generates the semi-group $\{T_t^A\}_{t\geq 0}$.

6. Gauge Covariance

Our purpose in this section is to show the gauge covariance of the generator $\mathcal{G}_A$. For any $\lambda \in L^1_{loc}(\mathbf{R}^d)$, we set

$$\lambda^n(x) \equiv w_n(x)\langle \rho_{1/n}(x - \cdot), \lambda\rangle.$$

Lemma 6.1 *Let $\lambda \in L^1_{loc}(\mathbf{R}^d)$. For any $f \in L^2(\mathbf{R}^d)$, we have*

$$e^{i\lambda^n} f \to e^{i\lambda} f \quad in \quad L^2(\mathbf{R}^d) \tag{6.1}$$

$$e^{-i\lambda^n} f \to e^{-i\lambda} f \quad in \quad L^2(\mathbf{R}^d). \tag{6.2}$$

Proof. We only show here (6.1). For any $f \in \mathcal{D}(\mathbf{R}^d)$, we have

$$\begin{aligned}\|e^{i\lambda^n} f - e^{i\lambda} f\|^2_{L^2(\mathbf{R}^d)} &= \|f - e^{i(\lambda-\lambda^n)} f\|^2_{L^2(\mathbf{R}^d)} \\ &= `2 \int_{\mathbf{R}^d} |f|^2 |1 - \cos(\lambda - \lambda^n)| dx\end{aligned}$$

$$\leq 2M^2 \int_{|x|\leq r} |1-\cos(\lambda-\lambda^n)|dx, \tag{6.3}$$

where M, r are constants such that $M = \sup|f|$, $\mathrm{supp} f \subset \{|x| \leq r\}$. Because of the inequality $|1-\cos x| \leq |x|$, we have

$$\begin{aligned}(6.3) &\leq 2M^2 \int_{|x|\leq r} |\lambda-\lambda^n|dx \\ &\leq 2M^2 \int_{|x|\leq r} |1-w_n(x)|\cdot|\lambda|dx \\ &\quad + \|w_n\|_\infty \int_{|x|\leq r} |\lambda - \langle\rho_{1/n},\lambda\rangle|dx. \end{aligned}\tag{6.4}$$

The last term of (6.4) tends to 0 as $n \to \infty$, because of the definition of w_n and $\lambda \in L^1_{loc}(\mathbf{R}^d)$.

Since $e^{i\lambda^n}$ and $e^{i\lambda}$ are contraction operators on $L^2(\mathbf{R}^d)$ and $\mathcal{D}(\mathbf{R}^d)$ is dense in $L^2(\mathbf{R}^d)$, $e^{i\lambda^n}f$ is L^2 convergent to $e^{i\lambda}f$ for any $f \in L^2(\mathbf{R}^d)$. □

Theorem 6.2 *Let $\lambda \in L^1_{loc}(\mathbf{R}^d)$. We have*

$$\begin{aligned}\mu(e^{i\lambda^n}T_t^{A,n}e^{-i\lambda^n}f \to e^{i\lambda}T_t^A e^{-i\lambda}f \quad &in \quad L^2(\mathbf{R}^d) \\ for\ f \in L^2(\mathbf{R}^d)) = 1,\end{aligned}\tag{6.5}$$

where $e^{i\lambda^n}$, $e^{-i\lambda^n}$ and $e^{i\lambda}$ are multiplication operators on $L^2(\mathbf{R}^d)$.

Proof. Let Ω_1 be the subset of $\mathcal{S}'(\mathbf{R}^d;\mathbf{R}^d)$ which was defined in (5.10) and let $\{n_k\}_{k=1}^\infty$ be an associated subsequence with Ω_1. For any $A \in \Omega_1$ and $f \in L^2(\mathbf{R}^d)$,

$$\begin{aligned}&\|e^{i\lambda}T_t^A e^{-i\lambda}f - e^{i\lambda^{n_k}}T_t^{A,n_k}e^{-i\lambda^{n_k}}f\|_{L^2(\mathbf{R}^d)} \\ &\leq \|e^{i\lambda}T_t^A e^{-i\lambda}f - e^{i\lambda^{n_k}}T_t^A e^{-i\lambda}f\|_{L^2(\mathbf{R}^d)} \\ &\qquad + \|e^{i\lambda^{n_k}}T_t^A e^{-i\lambda}f - e^{i\lambda^{n_k}}T_t^{A,n_k}e^{-i\lambda}f\|_{L^2(\mathbf{R}^d)} \\ &\qquad + \|e^{i\lambda^{n_k}}T_t^{A,n_k}e^{-i\lambda}f - e^{i\lambda^{n_k}}T_t^{A,n_k}e^{-i\lambda^{n_k}}f\|_{L^2(\mathbf{R}^d)} \\ &\leq \|e^{i\lambda}T_t^A e^{-i\lambda}f - e^{i\lambda^{n_k}}T_t^A e^{-i\lambda}f\|_{L^2(\mathbf{R}^d)} \\ &\qquad + \|T_t^A e^{-i\lambda}f - T_t^{A,n_k}e^{-i\lambda}f\|_{L^2(\mathbf{R}^d)} \\ &\qquad + \|e^{-i\lambda}f - e^{-i\lambda^n}f\|_{L^2(\mathbf{R}^d)}.\end{aligned}\tag{6.6}$$

By using Lemma 6.1, the first and third terms in (6.6) tend to 0 as $k \to \infty$. By (5,8) the second term in (6.6) also tends to 0 as $k \to \infty$. □

We shall define

$$(A + \partial\lambda)^n(x) \equiv A^n(x) + w_n(x)\langle \rho_{1/n}(x - \cdot), \partial\lambda \rangle.$$

Then we have, by the definition of the semi-groups $\{T_t^{A,n}\}_{t\geq 0}$ and $\{T_t^{(A+\partial\lambda),n}\}_{t\geq 0}$, for any $A \in \mathcal{S}'(\mathbf{R}^d; \mathbf{R}^d)$, $f \in L^2(\mathbf{R}^d)$ and $T > 0$,

$$\|e^{i\lambda^n} T_t^{A,n} e^{-i\lambda^n} f - T_t^{(A+\partial\lambda),n} f\|_{L^2(\mathbf{R}^d)} \to 0 \qquad \text{as } n \to \infty. \tag{6.7}$$

Moreover this convergence is uniform in $t \in [0, T]$. Since $\{T_t^{A,n}\}_{t\geq 0}$ and $\{T_t^A\}_{t\geq 0}$ are strongly continuous semi-group processes, we have for any $T > 0$, the convergence (6.6) is also uniform in $t \in [0, T]$. Then we can obtain, for any $f \in \mathcal{D}(\mathbf{R}^d)$, $\varepsilon > 0$ and $T > 0$,

$$\lim_{m,n\to\infty} \sup_{t\in[0,T]} \mu(\|T_t^{A+\partial\lambda,n} f - T_t^{A+\partial\lambda,m} f\|_{L^2(\mathbf{R}^d)} > \varepsilon) = 0. \tag{6.8}$$

By Theorem 5.3, we have a strongly continuous symmetric semi-group process $\{T_t^{A+\partial\lambda}\}_{t\geq 0}$ as the limit of $\{T_t^{A+\partial\lambda,n}\}_{t\geq 0}$. We denote its generator by $\mathcal{G}_{A+\partial\lambda}$. (6.7) implies that the semi-group process $\{T_t^{A+\partial\lambda}\}_{t\geq 0}$ coincides with the semi-group process $\{e^{i\lambda} T_t^A e^{-i\lambda}\}_{t\geq 0}$. Thus, the following theorem holds.

Theorem 6.3 *Let $\lambda \in L^1_{loc}(\mathbf{R}^d)$. We have*

$$\mu(e^{i\lambda} \mathcal{G}_A e^{-i\lambda} = \mathcal{G}_{A+\partial\lambda}) = 1.$$

7. The Spectra of the Generator

This section deals with the study of spectra of $\mathcal{G}_A$, $A \in \mathcal{S}'(\mathbf{R}^d; \mathbf{R}^d)$. Let $\{\theta_z\}_{z\in\mathbf{R}^d}$ be an ergodic family of measure preserving transformations defined in section 3. Let $\{\tau_z\}_{z\in\mathbf{R}^d}$ be an unitary operator

$$\tau_z f = f(\cdot - z) \qquad f \in L^2(\mathbf{R}^d).$$

We define $\{T_t^{\theta_z A}\}_{t>0}$ as follows. For any $f \in L^2(\mathbf{R}^d)$,

$$T_t^{\theta_z A} f \equiv \lim_{k\to\infty} T_t^{\theta_z A, n_k} f \qquad \text{in} \quad L^2(\mathbf{R}^d).$$

Lemma 7.1 *For any $z \in \mathbf{R}^d$, there exists a semi-group process $\{T_t^{\theta_z A}\}_{t\geq 0}$ μ-almost surely. Moreover for any $t > 0$, we have*

$$T_t^{\theta_z A} = \tau_z^* T_t^A \tau_z, \tag{7.1}$$

where τ_z^* is an adjoint operator of τ_z.

Proof. For any $f \in L^2(\mathbf{R}^d)$ and $t \in \mathbf{Q} \cap [0, \infty)$,

$$\begin{aligned} \|T_t^{\theta_z A, n_k} f - T_t^{\theta_z A, n_l} f\|_{L^2(\mathbf{R}^d)} &= \|\tau_z^* T_t^{A, n_k} \tau_z f - \tau_z^* T_t^{A, n_l} \tau_z f\|_{L^2(\mathbf{R}^d)} \\ &= \|T_t^{A, n_k} \tau_z f - T_t^{A, n_l} \tau_z f\|_{L^2(\mathbf{R}^d)}. \quad (7.2) \end{aligned}$$

By (5.3), the right hand side of (7.2) converges to 0 as $k, l \to \infty$. Then we get $\{T_t^{\theta_z A, n_k} f\}_{k=1}^{\infty}$ is a Cauchy sequence in $L^2(\mathbf{R}^d)$ μ-almost surely, because $\{T_t^{A, n_k} f\}_{k=1}^{\infty}$ is a Cauchy sequence. And we have

$$\begin{aligned} \|T_t^{\theta_z A} f - \tau_z^* T_t^A \tau_z f\|_{L^2(\mathbf{R}^d)} \leq\ & \|T_t^{\theta_z A} f - T_t^{\theta_z A, n_k} f\|_{L^2(\mathbf{R}^d)} \\ &+ \|T_t^{\theta_z A, n_k} f - \tau_z^* T_t^{A, n_k} \tau_z f\|_{L^2(\mathbf{R}^d)} \\ &+ \|\tau_z^* T_t^{A, n_k} \tau_z f - \tau_z^* T_t^A \tau_z f\|_{L^2(\mathbf{R}^d)}. \end{aligned}$$

The second term is 0. The first and third terms tend to 0 as k to infinity. For any $t \geq 0$, $T_t^{\theta_z A}$ was defined by (5.13), then we obtain (7.1). □

Let $\sigma(A)$, $\sigma_{pp}(A)$, $\sigma_{ac}(A)$ and $\sigma_{sc}(A)$ be the spectrum, point spectrum, absolutely continuous spectrum and singular continuous spectrum of $\mathcal{G}_A$ respectively.

Theorem 7.2 *There exist closed subsets σ, σ_{pp}, σ_{ac} and σ_{sc} of $\mathbf{R}$ such that $\sigma = \sigma(A)$, $\sigma_{pp} = \sigma_{pp}(A)$, $\sigma_{ac} = \sigma_{ac}(A)$ and $\sigma_{sc} = \sigma_{sc}(A)$ for almost μ-surely.*

Proof. Let $E(\Lambda, A)$, $\Lambda \in \mathcal{B}(\mathbf{R}^d)$, be the resolution of the identity of the operator $\mathcal{G}_A$ and let

$$E(\cdot, A) = E_{pp}(\cdot, A) + E_{ac}(\cdot, A) + E_{sc}(\cdot, A)$$

be its Lebesgue decomposition. By Lemma 7.1, we have

$$E_{\#}(\cdot, A) = \tau_z^* E_{\#}(\cdot, A) \tau_z,$$

where $\# = pp, ac$ and sc. Then there exists a closed subset $\sigma_{\#}$ of $\mathbf{R}$ such that topological support of $E_{\#}(\cdot, A)$ is $\sigma_{\#}$ for μ-almost surely. (cf. [C] Proposition V.2.4). □

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