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Equilibrium vector potentials in $\mathbb{R}^3$

(Dedicated to Professor Makoto Ohtsuka on his 70th birthday)

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Abstract. In the potential theory it is well known that the notion of equilibrium potentials for a bounded domain D with smooth boundary surfaces Σ in $\mathbb{R}^3$ is rested on the basis of the electric condenser. In this paper we introduce the notion of equilibrium vector potentials for D based on the electric solenoid. We then find that such vector potentials are related to harmonic 2-forms on $\overline{D}$ whose normal component with respect to Σ vanishes at any point of Σ.

Key words: vector potential, solenoid, harmonic forms, newton kernel, weyls' orthogonal decomposition.

Introduction

Let D be an electric condenser with smooth boundary surfaces Σ in $\mathbb{R}^3$. Then D carries the equilibrium charge distribution $\rho\, dS_x$ on Σ, where dS_x is the surface area element of Σ, which induces the electric field $E(x)$ in $\mathbb{R}^3 \setminus \Sigma$ being identically 0 in D:

$$\begin{aligned} u(x) &= \frac{1}{4\pi}\int_\Sigma \frac{\rho(y)}{\|x-y\|} dS_y && \text{for } x \in \mathbb{R}^3, \\ E(x) &= \text{grad } u(x) = \frac{-1}{4\pi}\int_\Sigma \rho(y)\frac{x-y}{\|x-y\|^3} dS_y && \text{for } x \in \mathbb{R}^3 \setminus \Sigma. \end{aligned}$$

The function $u(x)$ is called the equilibrium potential for D. We consider the total energy $\mu = \int_{\mathbb{R}^3} \|E(x)\|^2\, dv_x$ of the electric field $E(x)$. Now assume that the condenser D_t varies smoothly with real parameter t. Then the total energy $\mu(t)$ varies with parameter t. In [Y1, §2] and [LY, §9], we formed the variation formula of second order $\mu''(t)$ for $\mu(t)$ with respect to t. We intend to make the corresponding studies in the magnetic fields' version. In this paper, motivated by the electric solenoid (see the beginning of §8) we introduce the notion of equilibrium current density $J dS_x$ on Σ, the magnetic field $B(x)$ induced by $J dS_x$ and the equilibrium vector potential $A(x)$, and study their properties. We consider the total energy $\nu = \int_{\mathbb{R}^3} \|B(x)\|^2 dv_x$ of

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the magnetic field $B(x)$. In [Y3], we shall construct the variation formula of second order $\nu''(t)$ for $\nu(t)$, when the domain D_t varies smoothly with real parameter t.

Let $J(x)$ be a C_0^∞ vector field in $\mathbb{R}^3$ such that div $J(x) = 0$. Then Jdv_x, where dv_x denotes the volume element of $\mathbb{R}^3$, is called a volume current density in $\mathbb{R}^3$. For a 1-cycle γ in $\mathbb{R}^3$ with $\gamma = \partial Q$, we put $J[\gamma] = \int_Q J(x) \cdot n_x dS_x$, where n_x and dS_x denote the unit outer normal vector and the surface area element of Q at x, respectively. $J[\gamma]$ is called the total current of Jdv_x through $[\gamma]$, or through Q. Now let D be a bounded domain with real analytic smooth boundary surfaces Σ, and put $D' = \mathbb{R}^3 \setminus (D \cup \Sigma)$. Let $J(x)$ be a C^∞ vector field on Σ. If there exists a sequence of volume current densities $\{J_n dv_x\}_n$ in $\mathbb{R}^3$ such that $J_n dv_x \to JdS_x$ on Σ in the sense of distribution, then we say that JdS_x is a surface current density on Σ. For a 1-cycle γ in $\mathbb{R}^3 \setminus \Sigma$, we put $J[\gamma] = \lim_{n\to\infty} J_n[\gamma]$, which is called the total current of JdS_x through $[\gamma]$. We consider the following vector-valued integrals:

$$A(x) = \frac{1}{4\pi} \int_\Sigma \frac{J(y)}{\|x-y\|} dS_y \qquad \text{for } x \in \mathbb{R}^3,$$

$$B(x) = \text{rot}\, A(x) = \frac{1}{4\pi} \int_\Sigma J(y) \times \frac{x-y}{\|x-y\|^3} dS_y \quad \text{for } x \in \mathbb{R}^3 \setminus \Sigma \ .$$

Following Biot-Savart we say that the surface current density JdS_x on Σ induces the magnetic field $B(x)$ in $\mathbb{R}^3 \setminus \Sigma$. If a surface current density $J_0 dS_x$ on Σ induces a magnetic field $B(x)$ which is identically 0 in D', then $J_0 dS_x$ is called an *equilibrium current density on* Σ. We say that $A(x)$ for $J_0 dS_x$ is an *equilibrium vector potential for* D. Now let $\{\gamma_j\}_{j=1,\dots,q}$ be a base of the 1-dimensional homology group of D. Then we shall prove

Main Theorem *There exist q linearly independent equilibrium current densities $\{\mathbf{J}_i dS_x\}_{i=1,\dots,q}$ on Σ such that $\mathbf{J}_i[\gamma_j] = \delta_{ij}$ $(1 \le j \le q)$ (Kronecker's delta). Further, any equilibrium current density on Σ can be written by a linear combination of $\{\mathbf{J}_i dS_x\}_{i=1,\dots,q}$.*

Let γ be a 1-cycle in D. By H. Weyl [Wy], there exists a unique harmonic 2-form Ω_γ on $D \cup \Sigma$ such that $\int_\gamma \omega = (\omega, *\Omega_\gamma)_D$ for all C^∞ square integrable closed 1-forms ω in D. $*\Omega_\gamma$ is called the reproducing 1-form for (D, γ). We write $\Omega_\gamma = \alpha dy \wedge dz + \beta dz \wedge dx + \gamma dx \wedge dy$ on $D \cup \Sigma$, and consider artificially the following vector field $B_\gamma(x)$ in $\mathbb{R}^3 \setminus \Sigma$:

$$B_\gamma(x) = \begin{cases} (\alpha, \beta, \gamma) & \text{in } D \\ (0,0,0) & \text{in } D'. \end{cases}$$

We put $J_\gamma(x) = n_x \times (\alpha, \beta, \gamma)$ on Σ. Then the essential part of the proof of the main theorem is to show the fact that $J_\gamma \, dS_x$ is an equilibrium current density on Σ which induces $B_\gamma(x)$ as a magnetic field. In §§5 $\sim$ 6 we canonically construct a sequence of volume current densities $\{J_n dv_x\}_n$ in $\mathbb{R}^3$ which converges $J_\gamma dS_x$ on Σ in the sense of distribution. Such a construction of $\{J_n dv_x\}_n$ is useful not only for the proof of the above fact but also for the studies in §§7, 8 and Appendix. In §§1 $\sim$ 4 we give physical and mathematical preparations for the theorem. In §7 we study extremal properties of equilibrium current densities and equilibrium vector potentials. We then find that an equilibrium current density induces the magnetic field with minimum total magnetic energy among all surface current densities on Σ with given total currents through $[\gamma_j]$ $(j = 1, ..., q)$, while an equilibrium vector potential is regarded as a magnetic field induced by a (generalized) volume current density in $\mathbb{R}^3$ with minimum total current energy among all volume current densities with given total currents through $Q_i (i = 1, ..., q)$, where Q_i is a 2-chain in D such that $\partial Q_i \subset \Sigma$ and $Q_i \times \gamma_j = \delta_{ij}$ $(j = 1, \ldots, q)$. In §8, we study examples of (D, γ_i) such that D is a z-axially symmetric domain in $\mathbb{R}^3$, and show the explicit formulas for $\mathbf{J}_i dS_x$ in the main theorem and for its vector potential $\mathbf{A}_i$ and magnetic field $\mathbf{B}_i$. They will be written by use of functions $u(x, z)$ which satisfy the Stokes-Beltrami partial differential equations:

$$\Delta^{\pm} u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} \pm \frac{1}{x}\frac{\partial u}{\partial x} = f.$$

Such equations are classical and have been studied by E. Beltrami [B], A. Weinstein [Wi], R.P. Gilbert [G], etc.. Our research gives a different view on those equations. In Appendix, we show the electromagnetic meaning of the fundamental solutions for $\Delta^{\pm} u = 0$ obtained by A. Weinstein in order to apply them to our study.

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The main results in this paper have been announced in [$Y2$].

Contents

1. Surface current densities

We use the simple notation: $x = (x, y, z) = (x_1, x_2, x_3) \in \mathbb{R}^3$. We recall some notions in the static electromagnetism. Let $J(x)=(f_1(x), f_2(x), f_3(x))$ be a vector field in $\mathbb{R}^3$ such that

$$\text{(i)} \quad f_i(x) \in C_0^\infty(\mathbb{R}^3), \qquad \text{(ii)} \quad \operatorname{div} J(x) = \sum_{i=1}^{3} \frac{\partial f_i}{\partial x_i} = 0 \quad \text{in } \mathbb{R}^3.$$

Then Jdv_x, where dv_x is a volume element in $\mathbb{R}^3$, is called a *volume current density* in $\mathbb{R}^3$. Let γ be a 1-cycle in $\mathbb{R}^3$ which bounds a 2-chain Q, namely, $\partial Q = \gamma$. We set

$$J[\gamma] = \int_Q J(x) \cdot n_x dS_x, \tag{1.1}$$

which is called the *total current of Jdv_x through* $[\gamma]$. We consider the following vector-valued integrals:

$$A(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{J(y)}{\|x-y\|} dv_y \quad \text{for } x \in \mathbb{R}^3, \tag{1.2}$$

$$B(x) = \operatorname{rot} A(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} J(y) \times \frac{x-y}{\|x-y\|^3} dv_y \quad \text{for } x \in \mathbb{R}^3. \tag{1.3}$$

Then $A(x)$ is called the *vector potential for Jdv_x*, and $B(x)$ the *magnetic field induced by Jdv_x*.

Let $D \subset\subset \mathbb{R}^3$ be a domain bounded by a finite number of real analytic smooth closed surfaces $\Sigma(=\partial D)$. We denote by dS_x the surface area element of Σ. Throughout this paper, we put $D' = \mathbb{R}^3 \setminus \overline{D}$, where $\overline{D} = D \cup \Sigma$. Let $J(x) = (f_1(x), f_2(x), f_3(x))$ be a vector field on Σ such that

(i′) $f_i(x) \in C^\infty(\Sigma)$,

(ii′) There exists a sequence of volume current densities $\{J_n dv_x\}_{n=1,2,\dots}$ in $\mathbb{R}^3$ such that $J_n dv_x \to J dS_x$ $(n \to \infty)$ on Σ in the sense of distribution.

Precisely, (ii′) means that $\{\mathrm{Supp}\, J_n\}_{n=1,2,\dots}$ are uniformly bounded in $\mathbb{R}^3$ and $\lim_{n\to\infty} \int_{\mathbb{R}^3} \psi J_n dv_x = \int_\Sigma \psi J dS_x$ for any $\psi \in C_0^\infty(\mathbb{R}^3)$. Then JdS_x is called a *surface current density on* Σ. For any 1-cycle γ in $\mathbb{R}^3 \setminus \Sigma$, we set

$$J[\gamma] = \lim_{n\to\infty} J_n[\gamma], \tag{1.4}$$

which is called the *total current of* JdS_x *through* $[\gamma]$. In Corollary 3.1 we shall represent $J[\gamma]$ by JdS_x itself (without using $\{J_n(x)dv_x\}_n$). We set

$$A(x) = \frac{1}{4\pi}\int_\Sigma \frac{J(y)}{\|x-y\|} dS_y \quad \text{for } x \in \mathbb{R}^3, \tag{1.5}$$

$$B(x) = \mathrm{rot}\, A(x) = \frac{1}{4\pi}\int_\Sigma J(y) \times \frac{x-y}{\|x-y\|^3} dS_y \quad \text{for } x \in \mathbb{R}^3 \setminus \Sigma. \tag{1.6}$$

Then $A(x)$ is called the *vector potential for* JdS_x, and $B(x)$ the *magnetic field induced by* JdS_x.

Theorem 1.1 *Let* $J(x) = (f_1(x), f_2(x), f_3(x))$ *be a* C^∞ *vector field on* Σ. *We put* $n_x \times J(x) = (g_1(x), g_2(x), g_3(x))$ *for* $x \in \Sigma$ *and*

$$b_J(x) = g_1 dx + g_2 dy + g_3 dz \quad \text{on } \Sigma. \tag{1.7}$$

Then JdS_x *is a surface current density on* Σ, *if and only if*

(1) $J(x)$ *is a tangent vector of* Σ *at* x,

(2) $b_J(x)$ *is a closed* 1*-form on* Σ.

To prove this, we shall prepare Lemma 1.1 concerning the signed distance function $R(x)$ for Σ defined as follows: Given $x \in \mathbb{R}^3$ sufficiently close to Σ, we find a unique point $y = y(x) \in \Sigma$ such that

$$x - y = R(x)\, n_y \quad \text{where} \quad R(x) \in \mathbb{R}, \tag{1.8}$$

where n_y is the unit outer normal vector of Σ at y. Then $R(x)$ is a C^ω function in a neighborhood U of Σ in $\mathbb{R}^3$ such that $n_x = \mathrm{grad}\, R(x)$ on Σ

and

$$U \cap D\,(\text{resp.}\,\Sigma, U \cap D') = \{x \in U \mid R(x) < (\text{resp.} =, >)\ 0\}.$$

We define a sequence of C^∞ functions $\{\chi_n(R)\}_{n=1,2,\dots}$ on $(-\infty, \infty)$ such that

$$0 \le \chi_n(R) \le 1, \qquad \chi_n(R) = \begin{cases} 1 & \text{on } (-\infty, -\frac{1}{n}\,] \\ 0 & \text{on } [-\frac{1}{2n}, \infty). \end{cases} \tag{1.9}$$

Then, $\chi_n(R(x))$ is a C^∞ function in U. We take an integer n_0 such that

$$\Gamma_n := \{x \in U \mid -\frac{1}{n} \le R(x) \le -\frac{1}{2n}\} \subset\subset U \quad \text{for all } n \ge n_0. \tag{1.10}$$

Thus $\Gamma_n \to \Sigma\,(n \to \infty)$ and Supp $\chi_n'(R(x)) \subset \Gamma_n$ where $\chi_n'(R) = \frac{\partial \chi_n}{\partial R}$. By putting $\chi_n'(R(x)) \equiv 0$ in $\mathbb{R}^3 \setminus \Gamma_n$, we may consider $\chi_n'(R(x)) \in C_0^\infty(\mathbb{R}^3)$. Similarly, $\chi_n''(R(x)) \in C_0^\infty(\mathbb{R}^3)$.

Lemma 1.1 *Let $f \in C_0^\infty(\mathbb{R}^3)$. Then*

(1) $\chi_n'(R(x))f(x)dv_x \to -f(x)dS_x$ $(n \to \infty)$ *on Σ in the sense of distribution.*

(2) $\{\chi_n''(R(x))f(x)dv_x\}_{n \ge n_0}$ *is convergent on Σ in the sense of distribution, if and only if $f(x) = 0$ on Σ. In this case, the limit is $\frac{\partial f}{\partial n_x} dS_x$ on Σ.*

Proof. Let $x \in \Sigma$. By a Euclidean motion of $\mathbb{R}^3$, we assume $x = 0$ and $n_x = (0,0,1)$, so that Σ near 0 is of the form: $\zeta = \phi(\xi, \eta)$ where $\phi(\xi, \eta) = O(\xi^2 + \eta^2)$ at $(0,0)$. We put $H(x) = \left(\frac{\partial^2 \phi}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial \eta^2}\right)(0,0)$, which is called the *mean curvature of* Σ *at* x. Now fix $0 < \delta \ll 1$ such that

$$\Sigma \subset U(\delta) := \{x \in U \mid -\delta < R(x) < \delta\} \subset\subset U.$$

We divide $U(\delta)$ into a finite number of disjoint piecewise smooth domains $\{U_j\}_{j=1,\dots,N}$ such that $U(\delta) = \bigcup_{j=1}^N \overline{U_j}$ and we can write, under a certain Euclidean motion T_j of $\mathbb{R}^3$,

(a) $U_j \cap \Sigma$ is of the form $\zeta = \phi_j\ (\xi, \eta)$ where $(\xi, \eta) \in K_j :=$ a domain bounded by a finite number of piecewise smooth arcs in the (ξ, η)-plane and $\phi_j(\xi, \eta) = O(\xi^2 + \eta^2)$ at $(0,0)$,

(b) $U_j = \{x = (\xi, \eta, \phi_j(\xi, \eta)) + Rn_y \mid (\xi, \eta, R) \in V_j\}$, where $V_j = K_j \times (-\delta, \delta)$ and $y = y(x) = (\xi, \eta, \phi_j(\xi, \eta))$.

We thus have, for each j $(1 \le j \le N)$,

$$\begin{aligned} &dv_x = \left|\frac{\partial(x,y,z)}{\partial(\xi,\eta,R)}\right| d\xi d\eta dR \equiv J_j(\xi,\eta,R) d\xi d\eta dR \quad \text{in } U_j, \\ &J_j(\xi,\eta,0) d\xi d\eta = dS_x \quad \text{on } \Sigma \cap U_j, \\ &\frac{\partial J_j}{\partial R}(\xi,\eta,0) d\xi d\eta = H(x) dS_x \quad \text{on } \Sigma \cap U_j. \end{aligned} \tag{1.11}$$

We only give the proof of (2) of Lemma 1.1, since the proof of (1) is similar. We first note that $\operatorname{Supp}\chi_n''(R(x)) \subset \Gamma_n \subset U$ for $n \ge n_0$. Next let $\psi \in C_0^\infty(\mathbb{R}^3)$. Since $\chi_n''(R(x)) = 0$ in $\mathbb{R}^3 \setminus U(\delta)$ for sufficiently large n, it follows that

$$\begin{aligned} I_n : &= \int_{\mathbb{R}^3} \chi_n''(R(x)) f(x) \psi(x) dv_x \\ &= \sum_{j=1}^N \left\{ \int_{K_j} \left(\int_{\frac{-1}{n}}^{\frac{-1}{2n}} \chi_n''(R) f \psi J_j dR \right) d\xi d\eta \right\}. \end{aligned}$$

From $\chi_n'(-1/n) = \chi_n'(-1/2n) = \chi_n(-1/2n) = 0$ and $\chi_n(-1/n) = 1$, we have, by the integration by parts twice,

$$\begin{aligned} &\int_{\frac{-1}{n}}^{\frac{-1}{2n}} \chi_n''(R) f \psi J_j dR \\ &\qquad = \frac{\partial(f\psi J_j)}{\partial R}\bigg]_{(\xi,\eta,-1/n)} + \int_{\frac{-1}{n}}^{\frac{-1}{2n}} \chi_n \frac{\partial^2(f\psi J_j)}{\partial R^2} dR \\ &\qquad \to \frac{\partial(f\psi J_j)}{\partial R}\bigg]_{(\xi,\eta,0)} \qquad \text{as } n \to \infty. \end{aligned}$$

The last limiting formula follows, since $0 \le \chi_n(R) \le 1$ and $\partial^2(f\psi J_j)/\partial R^2$ is bounded in V_j. By (1.11), we have

$$\begin{aligned} \lim_{n\to\infty} I_n &= \sum_{j=1}^N \left\{ \int_{K_j} \left(\frac{\partial f}{\partial R}\psi J_j + f\frac{\partial \psi}{\partial R} J_j + f\psi \frac{\partial J_j}{\partial R} \right)_{(\xi,\eta,0)} d\xi d\eta \right\} \\ &= \int_\Sigma \left\{ \left(\frac{\partial f}{\partial n_x} + fH \right)\psi + f\frac{\partial \psi}{\partial n_x} \right\} dS_x, \end{aligned}$$

by which (2) follows. □

Proof of Theorem 1.1. Assume that JdS_x is a surface current density on Σ. We take a sequence of volume current densities $\{J_n dv_x\}_n = \{(f_{1n}, f_{2n},$

$f_{3n})dv_x\}_n$ in $\mathbb{R}^3$ which converges to JdS_x on Σ in the sense of distribution. For any $\rho \in C^\infty(\mathbb{R}^3)$ and $\chi \in C_0^\infty(\mathbb{R}^3)$, it holds

$$\begin{aligned}\int_\Sigma \Big(\sum_{i=1}^3 f_i \frac{\partial(\rho\chi)}{\partial x_i}\Big) dS_x &= \lim_{n\to\infty}\int_{\mathbb{R}^3}\sum_{i=1}^3 \Big(f_{in}\frac{\partial(\rho\chi)}{\partial x_i}\Big) dv_x \\ &= -\lim_{n\to\infty}\int_{\mathbb{R}^3}(\mathrm{div}\ J_n)\rho\chi dv_x = 0.\end{aligned} \tag{1.12}$$

As ρ we take a defining function ψ of D in $\mathbb{R}^3$ such that $\|\mathrm{grad}\ \psi(x)\| = 1$ on Σ. Then (1.12) is reduced to $\int_\Sigma \chi\cdot(\sum_{i=1}^3 f_i\partial\psi/\partial x_i)dS_x = 0$ for any $\chi \in C_0^\infty(\mathbb{R}^3)$, so that $J(x)\perp n_x$ at any $x\in\Sigma$. (1) is proved. By taking $\chi\equiv 1$ in a neighborhood of Σ in $\mathbb{R}^3$ for (1.12), we obtain

$$\int_\Sigma \Big(\sum_{i=1}^3 f_i\frac{\partial\rho}{\partial x_i}\Big)dS_x = 0 \quad \text{for any } \rho\in C^\infty(\mathbb{R}^3). \tag{1.13}$$

We note that (2) is a local property and that our argument is invariant under the Euclidean motions of R^3. Let $x_0\in\Sigma$. We may assume $x_0 = 0$ and $n_0 = (0,0,1)$. Thus, Σ in a neighborhood $V\subset\mathbb{R}^3$ of 0 is of the form: $z = \phi(x,y)$ for $(x,y)\in K := \{x^2+y^2<r^2\}$ such that $\phi(x,y) = O(x^2+y^2)$ at $(0,0)$. For any $x = (x,y,\phi(x,y))\in V\cap\Sigma$, we have, by (1),

$$dz = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy, \qquad f_3 = f_1\frac{\partial\phi}{\partial x} + f_2\frac{\partial\phi}{\partial y}.$$

Hence, by use of local parameter (x,y) of $\Sigma\cap V$, the 1-form $b_J(x)$ is written as

$$b_J(x) = \left[1+\Big(\frac{\partial\phi}{\partial x}\Big)^2+\Big(\frac{\partial\phi}{\partial y}\Big)^2\right]^{1/2}(-f_2dx+f_1dy) \text{ on } \Sigma\cap V.$$

Given any $h(x,y)\in C_0^\infty(K)$, we consider a function $\rho\in C^\infty(\mathbb{R}^3)$ such that $\rho(x,y,z) = h(x,y)$ in V, and $\rho = 0$ in a neighborhood of $\Sigma\setminus(V\cap\Sigma)$. Then (1.13) gives

$$0 = \int_{V\cap\Sigma}\Big(f_1\frac{\partial h}{\partial x}+f_2\frac{\partial h}{\partial y}\Big)dS_x = \int_K (dh)\wedge b_J = -\int_K h\,(db_J),$$

so that $b_J(x)$ is closed on $\Sigma\cap V$. (2) is proved.

Conversely, let $J = (f_1,f_2,f_3), f_i\in C^\infty(\Sigma)$ satisfy (1) and (2). By (1.7) and (1), we have (1′) $(g_1,g_2,g_3)\times n_x = (f_1,f_2,f_3)$ for $x\in\Sigma$. By (2) we can construct a C^∞ closed 1-form $\widetilde{b}_J(x) = \widetilde{g}_1dx+\widetilde{g}_2dy+\widetilde{g}_3dz$ in a neighborhood $U_0 \subset U$ of Σ in $\mathbb{R}^3$ such that $\widetilde{b}_J(x) = b_J(x)$ as 1-forms on Σ, namely, (2′) $(\widetilde{g}_1, \widetilde{g}_2, \widetilde{g}_3) \times n_x = (g_1, g_2, g_3) \times n_x$ for $x \in \Sigma$. We may assume that $U_0 \supset \Gamma_n$ for sufficiently large $n\,(\geq n_1)$, where Γ_n is defined in (1.10). Using $R(x)$ of (1.8) and $\chi_n(R)$ of (1.9), we define

$$\widetilde{\chi}_n(x) = \begin{cases} 1 & \text{in } D \setminus U \\ \chi_n(R(x)) & \text{in } U \\ 0 & \text{in } D' \setminus U, \end{cases} \tag{1.14}$$

$$\begin{aligned} \eta_n &= *\,[(d\widetilde{\chi}_n(x)) \wedge \widetilde{b}_J(x)] \\ &\equiv f_{1n}dx + f_{2n}dy + f_{3n}dz \quad \text{in } \mathbb{R}^3. \end{aligned}$$

That is, $f_{1n} = \chi_n'(R(x))(\widetilde{g}_3 \frac{\partial R}{\partial y} - \widetilde{g}_2 \frac{\partial R}{\partial z})$ etc.. We put

$$J_n dv_x = (f_{1n}, f_{2n}, f_{3n}) dv_x \quad \text{in } \mathbb{R}^3. \tag{1.15}$$

Then $\widetilde{\chi}_n(x)$ is a C^∞ function in $\mathbb{R}^3$ with Supp $\widetilde{\chi}_n \subset D$, and η_n is a C^∞ co-closed 1-form in $\mathbb{R}^3$ with Supp $\eta_n \subset \Gamma_n$. Hence $J_n dv_x$ is a volume current density in $\mathbb{R}^3$ such that Supp $J_n \to \Sigma$ $(n \to \infty)$. It suffices for the converse to prove that $\{J_n dv_x\}_n$ converges to the given JdS_x on Σ in the sense of distribution. For any $\psi \in C_0^\infty(\mathbb{R}^3)$, we have from (1) of Lemma 1.1,

$$\begin{aligned} \lim_{n\to\infty} \int_{\mathbb{R}^3} \psi f_{1n} dv_x &= \lim_{n\to\infty} \int_{\mathbb{R}^3} \psi \chi_n'(R(x)) \Big(\widetilde{g}_3 \frac{\partial R}{\partial y} - \widetilde{g}_2 \frac{\partial R}{\partial z} \Big) dv_x \\ &= \int_\Sigma \psi \Big(\widetilde{g}_2 \frac{\partial R}{\partial z} - \widetilde{g}_3 \frac{\partial R}{\partial y} \Big) dS_x \\ &= \int_\Sigma \psi f_1 dS_x \qquad \text{by } (1') \text{ and } (2'). \end{aligned}$$

Similar formulas for i=2, 3 hold. Theorem 1.1 is completely proved. □

Corollary 1.1 *Let JdS_x be a surface current density on Σ. Then there exists a sequence of volume current densities $\{J_n dv_x\}_n$ in $\mathbb{R}^3$ which converges JdS_x on Σ in the sense of distribution such that, if we denote by A_n or A the vector potential for $J_n dv_x$ or JdS_x, and B_n or B the magnetic field induced by $J_n dv_x$ or JdS_x, respectively, then $A_n(x) \to A(x)$ and $B_n(x) \to B(x)$ uniformly on any compact set in $\mathbb{R}^3 \setminus \Sigma$.*

Proof. In the proof of the converse of Theorem 1.1, we considered C^∞ functions $\chi_n(R)$ on $(-\infty, \infty)$ with (1.9). We here use $\chi_n(R)$ with the additional property: There exists an $M > 0$ such that

$$\begin{aligned} &|\chi_n'(R)| \le nM, \\ &|\chi_n''(R)| \le n^2 M \quad \text{for } R \in (-\infty, \infty) \text{ and } n \ge 1 . \end{aligned} \tag{1.16}$$

We analogously define $J_n dv_x = (f_{1n}, f_{2n}, f_{3n})dv_x$ by (1.15). Then we can show that $\{J_n dv_x\}_n$ satisfies Corollary 1.1. In fact, we already proved that $J_n dS_x \to J dS_x$ $(n \to \infty)$ on Σ in the sense of distribution. Hence $A_n(x) \to A(x)$ $(n \to \infty)$ pointwise in $\mathbb{R}^3 \setminus \Sigma$. It follows from $\int_{\Gamma_n} dv_x = O(1/n)$ that there exists a constant $c > 0$ (independent of n) such that

$$\int_{\mathbb{R}^3} |f_{1n}| dv_x = \int_{\Gamma_n} \left| \chi_n'(R(x)) \left(\tilde{g}_3 \frac{\partial R}{\partial y} - \tilde{g}_2 \frac{\partial R}{\partial z} \right) \right| dv_x \le c.$$

We may assume that c satisfies similar inequality for f_{2n} and f_{3n}. Let $K \subset\subset \mathbb{R}^3 \setminus \Sigma$. If we take a large n_1 such that $m(K) = \operatorname{dist}(K, \bigcup_{n=n_1}^{\infty} \Gamma_n) > 0$, then we have, for any $n \ge n_1$ and $x \in K$,

$$\|A_n(x)\| = \frac{1}{4\pi} \left\| \int_{\Gamma n} \frac{(f_{1n}(y), f_{2n}(y), f_{3n}(y))}{\|x - y\|} dv_x \right\| \le \frac{\sqrt{3}c}{4\pi m(K)},$$

so that $\{A_n(x)\}_n$ is uniformly bounded in K. Since each component of $A_n(x)$ is a harmonic function in $\mathbb{R}^3 \setminus \Gamma_n$, $\{A_n(x)\}_n$ is a normal family in K. Hence, $A_n(x) \to A(x)$ uniformly on K, by which $B_n(x) \to B(x)$ uniformly on K. □

The vector field $A(x)$ is continuous in $\mathbb{R}^3$, while $B(x)$ has the following jump property along Σ (cf. [FLS]):

Proposition 1.1 *Let $J dS_x$ be a surface current density on Σ and denote by $B(x)$ its magnetic field in $\mathbb{R}^3 \setminus \Sigma$. Then, for any $\zeta \in \Sigma$, the limits $B^+(\zeta) = \lim_{D \ni x \to \zeta} B(x)$ and $B^-(\zeta) = \lim_{D' \ni x \to \zeta} B(x)$ exist such that*

$$B^+(\zeta) - B^-(\zeta) = n_\zeta \times J(\zeta).$$

Proof. We may assume that $\zeta = 0$ and $n_0 = (0, 0, 1)$, so that the tangent plane of Σ at 0 is the (x, y)-plane. If we put $J = (f_1, f_2, f_3)$ on Σ, then Theorem 1.1 implies that $J(0) = (f_1(0), f_2(0), 0)$. We consider the Newton potential $u_i(x)$ of $f_i(x)$ $(i = 1, 2, 3)$ defined by

$$u_i(x) = \frac{1}{4\pi} \int_\Sigma \frac{f_i(y)}{\|x - y\|} dS_y \qquad \text{for } x \in \mathbb{R}^3.$$

Then the following theorem is well-known:

$$\lim_{D\ni x\to 0} \frac{\partial u_i}{\partial r}(x) - \lim_{D'\ni x\to 0} \frac{\partial u_i}{\partial r}(x) = \begin{cases} 0 & \text{when} \quad r = x, y \\ f_i(0) & \text{when} \quad r = z. \end{cases}$$

Hence, by definition (1.6) of $B(x)$, we see that $B^+(0)$ and $B^-(0)$ exist such that $B^+(0) - B^-(0) = (-f_2(0), f_1(0), 0) = n_0 \times J(0)$. $\square$

2. Statement of the main theorem

Let $D \subset\subset \mathbb{R}^3$ be a domain bounded by C^ω smooth surfaces Σ and put $D' = \mathbb{R}^3 \setminus \overline{D}$. Let $\{\gamma_j\}_{j=1,\dots,q}$ be a 1-dimensional homology base of D. A surface current density JdS_x on Σ is called an *equilibrium current density on* Σ, if the magnetic field $B(x)$ in $\mathbb{R}^3 \setminus \Sigma$ induced by JdS_x is identically 0 in D'. $A(x)$ defined by (1.5) for such JdS_x is called the *equilibrium vector potential for* JdS_x. Then we shall prove

Main Theorem

(1) *For a fixed* i $(1 \le i \le q)$, *there exists a unique equilibrium current density* $\mathbf{J}_i dS_x$ *on* Σ *such that* $\mathbf{J}_i[\gamma_j] = \delta_{ij}$ $(1 \le j \le q)$.

(2) *Any equilibrium current density* JdS_x *on* Σ *is written by a linear combination of* $\{\mathbf{J}_i dS_x\}_{i=1,\dots,q}$.

We denote by $\mathbf{A}_i$ and $\mathbf{B}_i$ the vector potential and the magnetic field induced by the above $\mathbf{J}_i dS_x$. The proof of the main theorem will be given in §§3 $\sim$ 6.

Proposition 2.1 (FLEMING'S LAW). *For a magnetic field* $B(x)$ *in* $\mathbb{R}^3 \setminus \Sigma$ *induced by an equilibrium current density* JdS_x *on* Σ, *we have* $J(x) \perp n_x$ *and* $B^+(x) = n_x \times J(x)$ *for* $x \in \Sigma$.

Proof. Since $B^-(x) = 0$ on Σ, this proposition follows by (1) of Theorem 1.1 and Proposition 1.1. $\square$

3. Correspondences

We regard volume current densities or magnetic fields as co-closed 1-forms or closed 2-forms in $\mathbb{R}^3$ (cf. [Fl], [H]). In this section we shall show results concerning C^∞ 1- or 2-forms in $\mathbb{R}^3$, some of which correspond to theorems in the theory of Maxwell's equations in the time independent case (cf. [FLS]). Given a C^∞ 1-form $\sigma = \sum_{i=1}^3 f_i dx_i$ in $\mathbb{R}^3$, we put $\|\sigma\|(x) =$

$\left(\sum_{i=1}^3 f_i(x)^2\right)^{1/2} \geq 0$ and $\Delta\sigma = \sum_{i=1}^3 (\Delta f_i) dx_i$. In the case of σ such that $\sigma(x) = O(1/\|x\|^3)$ near ∞, we also put

$$\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\sigma(y)}{\|x-y\|} dv_y = \sum_{i=1}^3 \left(\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{f_i(y)}{\|x-y\|} dv_y\right) dx_i.$$

This as well as $\Delta\sigma$ is a C^∞ 1-form in $\mathbb{R}^3$. We analogously define corresponding ones for any C^∞ i-form $\sigma_i\,(i = 0, 1, 2, 3)$. From the property of the Newton kernel $1/\|x-y\|$, we see that, for any C_0^∞ i-form σ in $\mathbb{R}^3$,

$$d\left(\int_{\mathbb{R}^3} \frac{\sigma(y)}{\|x-y\|}\right) dv_y = \int_{\mathbb{R}^3} \frac{(d\sigma)(y)}{\|x-y\|} dv_y,$$
$$*\left(\int_{\mathbb{R}^3} \frac{\sigma(y)}{\|x-y\|}\right) dv_y = \int_{\mathbb{R}^3} \frac{*\sigma(y)}{\|x-y\|} dv_y.$$

We often use

$$\Delta\left(\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\sigma(y)}{\|x-y\|}\, dv_y\right) = -\sigma(x) \qquad \text{(Poisson's equation)},$$
$$\Delta\sigma = (-1)^i(\delta d - d\delta)\sigma \qquad \text{where} \quad \delta = *d*.$$

Lemma 3.1 *Let $\eta = f_1 dx + f_2 dy + f_3 dz$ be a C_0^∞ co-closed 1-form in $\mathbb{R}^3$, namely, $f_i \in C_0^\infty(\mathbb{R}^3)$ $(i = 1, 2, 3)$ and $\delta\eta = 0$. We set*

$$p(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\eta(y)}{\|x-y\|} dv_y \qquad \textit{for} \;\; x \in \mathbb{R}^3, \tag{3.1}$$

$$\omega(x) = dp(x) \qquad \textit{for} \;\; x \in \mathbb{R}^3. \tag{3.2}$$

Then
(1) *p is a C^∞ co-closed 1-form in $\mathbb{R}^3$ such that $\Delta p = -\eta$.*
(2) *$\eta = \delta\omega$ in $\mathbb{R}^3$.*

Proof. Assume that η is a C_0^∞ co-closed 1-form in $\mathbb{R}^3$. Then we have $\delta p(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{(\delta\eta)(y)}{\|x-y\|} dv_y = 0$. By Poisson's equation, we have $\Delta p = -\eta$ in $\mathbb{R}^3$. Thus, (1) is proved. (2) follows by $\delta\omega = \delta dp = d\delta p - \Delta p = \eta$. $\square$

When ω is defined by η through (3.1) and (3.2), we say that η *induces* ω. By Lemma 3.1, this ω is a C^∞ closed 2-form in $\mathbb{R}^3$ which is harmonic outside the support of η and $\|\omega\|(x) = O(1/\|x\|^2)$ near ∞. Conversely, we have

Lemma 3.2 *Let ω be a C^∞ closed 2-form in $\mathbb{R}^3$ such that*
(1) *ω is harmonic outside a compact set in $\mathbb{R}^3$.*
(2) *$\|\omega\|(x) = O(1/\|x\|)$ near ∞.*
Then there exists a unique C_0^∞ co-closed 1-form η in $\mathbb{R}^3$ which induces ω. Namely, $\eta = \delta\omega$.

Proof. Uniqueness is clear from (2) of Lemma 3.1. Putting $\eta := \delta\omega$, we shall verify that η induces ω. By (1), η is a C_0^∞ co-closed 1-form in $\mathbb{R}^3$. We construct $p(x)$ by (3.1) for this $\eta(x)$. From $d\omega = 0$, we have

$$\begin{aligned} dp(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{(d\delta\omega)(y)}{\|x-y\|}dv_y &= \frac{-1}{4\pi}\int_{\mathbb{R}^3} \frac{(\Delta_x\omega)(y)}{\|x-y\|}dv_y \\ &\equiv \alpha dy\wedge dz + \beta dz\wedge dx + \gamma dx\wedge dy. \end{aligned}$$

We write $\omega = g_1 dy\wedge dz + g_2 dz\wedge dx + g_3 dx\wedge dy$. Since each $g_i(x)$ is a harmonic function outside a compact set, (2) implies (2′) $\|\mathrm{grad}\ g_i(x)\| = O(1/\|x\|^2)$ near ∞. It follows from Stokes' formula that

$$\begin{aligned} \alpha(x) &= \frac{-1}{4\pi}\int_{\mathbb{R}^3} \frac{\Delta_y g_1(x+y)}{\|y\|}dv_y \\ &= g_1(x) + \frac{1}{4\pi}\lim_{r\to\infty} \\ &\quad \left\{\int_{\|y\|=r}\left(g_1(x+y)\frac{\partial}{\partial n_y}\left(\frac{1}{\|y\|}\right) - \frac{1}{\|y\|}\frac{\partial g_1(x+y)}{\partial n_y}\right)dS_y\right\} \\ &= g_1(x) \qquad \text{by (2) and (2}'\text{)}. \end{aligned}$$

Similarly, $\beta = g_2$ and $\gamma = g_3$ in $\mathbb{R}^3$, so that $dp = \omega$. □

Lemma 3.3 *Let ω be a C_0^∞ closed 2-form in $\mathbb{R}^3$ and put $\eta = \delta\omega$ in $\mathbb{R}^3$. If we set, for any $x \in \mathbb{R}^3$,*

$$p(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\eta(y)}{\|x-y\|}dv_y, \qquad \lambda(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\omega(y)}{\|x-y\|}dv_y,$$

then $p = \delta\lambda$ in $\mathbb{R}^3$.

Proof. We note that $\lambda(x)$ is of class C^∞ in $\mathbb{R}^3$. We put $q := p - \delta\lambda \in C_1^\infty(\mathbb{R}^3)$. Since $\delta p = 0$ by $\delta\eta = 0$ in $\mathbb{R}^3$, we have $\delta q = 0$ in $\mathbb{R}^3$. On the other hand, we see that $d\lambda = 0$ by $d\omega = 0$, and $\Delta\lambda = -\omega$ in $\mathbb{R}^3$. Since Lemma 3.2 is applicable for our ω with compact support in $\mathbb{R}^3$, we have $dp = \omega$ in $\mathbb{R}^3$. Thus, $dq = dp - d\delta\lambda = \omega + (\Delta - \delta d)\lambda = 0$ in $\mathbb{R}^3$. Hence, q is a harmonic

1-form in $\mathbb{R}^3$. Since $q(x) = O(1/\|x\|)$ near ∞, it follows that $q = 0$ in $\mathbb{R}^3$. $\square$

If we use a simple notation $\mathcal{N}\sigma(x) = (1/4\pi)\int_{\mathbb{R}^3} \frac{\sigma(y)}{\|x-y\|}\,dv_y$ for $x \in \mathbb{R}^3$, then Poisson's equation says $-\Delta\mathcal{N} =$ identity, and Lemma 3.3 with (2) of Lemma 3.1 gives the following commutative diagram:

$$\begin{tikzcd}
 & \eta \arrow[dl, "\mathcal{N}"'] & \\
p \arrow[rr, "d"] & & \omega \arrow[ul, "\delta"'] \arrow[dl, "\mathcal{N}"] \\
 & \lambda \arrow[ul, "\delta"] &
\end{tikzcd}$$

For $j = 1, 2$, we denote by $C_j^\infty(\mathbb{R}^3)$ or $C_{j0}^\infty(\mathbb{R}^3)$ the set of C^∞ or C_0^∞ j-forms in $\mathbb{R}^3$. For $f_i \in C_0^\infty(\mathbb{R}^3)(i = 1, 2, 3)$, we consider the following injections:

$$\left.\begin{array}{ll}
T_c : Jdv_x = (f_1, f_2, f_3)dv_x & \mapsto \eta = f_1dx + f_2dy + f_3dz \in C_{10}^\infty(\mathbb{R}^3), \\
T_p : A(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{J(y)}{\|x-y\|}dv_y & \mapsto p(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\eta(y)}{\|x-y\|}dv_y \in C_1^\infty(\mathbb{R}^3), \\
T_m : B(x) = \text{rot } A(x) & \mapsto \omega(x) = dp(x) \in C_2^\infty(\mathbb{R}^3).
\end{array}\right\} \tag{3.3}$$

Clearly, Jdv_x is a volume current density in $\mathbb{R}^3$, iff η is a co-closed 1-form in $\mathbb{R}^3$. In that case, B is the magnetic field induced by Jdv_x, and η induces ω.

Let JdS_x be any surface current density on Σ, and denote by A and B its vector potential and its magnetic field. We write $JdS_x = (f_1, f_3, f_3)dS_x$, $A = (a, b, c)$, $B = (\alpha, \beta, \gamma)$ on Σ, $\mathbb{R}^3$, $\mathbb{R}^3 \setminus \Sigma$, respectively. We consider the following injections:

$$\left.\begin{array}{ll}
S_c : JdS_x & \mapsto \eta = f_1dx + f_2dy + f_3dz \quad \text{on } \Sigma, \\
S_p : A & \mapsto p = adx + bdy + cdz \quad \text{in } \mathbb{R}^3, \\
S_m : B & \mapsto \omega = \alpha dy \wedge dz + \beta dz \wedge dx + \gamma dx \wedge dy \quad \text{in } \mathbb{R}^3 \setminus \Sigma.
\end{array}\right\} \tag{3.4}$$

Thus, p is a continuous 1-form in $\mathbb{R}^3$, and $dp = \omega$ in $\mathbb{R}^3 \setminus \Sigma$. The surfaces Σ are regarded as Riemann surfaces with conformal structure induced by the restriction of the Euclidean metric of $\mathbb{R}^3$. Then $b_J(x)$ of (1.7) is equal to the conjugate differential of η on the Riemann surfaces Σ, so that η is a co-closed differential on Σ.

Corollary 3.1 *Let JdS_x be a surface current density on Σ and define the 1-form $b_J(x)$ on Σ by (1.7). Let γ be a 1-cycle in $\mathbb{R}^3 \setminus \Sigma$ bounding a 2-chain Q in $\mathbb{R}^3$ and put $\gamma' = Q \times \Sigma$ (intersection curves $Q \cap \Sigma$). Then we have*

$$J[\gamma] = \int_{\gamma'} b_J(x).$$

Proof. For the JdS_x on Σ we find a sequence of volume current densities $\{J_n dv_x\}_n$ which satisfies Corollary 1.1. We use A_n, A, B_n, B defined in that corollary. By (3.3), we consider η_n, p_n, and ω_n for $J_n dv_x, A_n$, and B_n. By (3.4), we consider η, p, and ω for JdS_x, A, and B. Then η_n is a C_0^∞ co-closed 1-form in $\mathbb{R}^3$ such that $\eta_n = \delta\omega_n$, and ω_n is a C^∞ 2-form in $\mathbb{R}^3$ such that $\omega_n = dp_n$. Moreover, ω_n is harmonic outside the support of η_n. Corollary 1.1 means that $\lim_{n\to\infty} p_n(x) = p(x)$ and $\lim_{n\to\infty} \omega_n(x) = \omega(x)$ uniformly on any compact set in $\mathbb{R}^3 \setminus \Sigma$. Since $\operatorname{Supp} \eta_n \to \Sigma (n \to \infty)$, we thus have

$$d * p = 0, \ \omega = dp, \text{ and } \omega \text{ is harmonic in } \mathbb{R}^3 \setminus \Sigma. \tag{3.5}$$

Let γ and Q be given in Corollary 3.1. Since $\eta_n = \delta\omega_n$ in $\mathbb{R}^3$, it follows that

$$J[\gamma] = \lim_{n\to\infty} J_n[\gamma] = \lim_{n\to\infty} \int_Q *\eta_n = \lim_{n\to\infty} \int_\gamma *\omega_n = \int_\gamma *\omega. \tag{3.6}$$

We simply set $D^+ = D, D^- = D'$, and $\omega(x) = \omega^\pm(x)$ for $x \in D^\pm$. By Proposition 1.1 and (1.7), $\omega^\pm(x)$ are continuous up to Σ and

$$*\omega^+(x) - *\omega^-(x) = b_J(x) \quad \text{on } \Sigma. \tag{3.7}$$

We separate the intersection curves $\gamma' = Q \cap \Sigma$ into the following sets: $\gamma' = \gamma_1' + \ldots + \gamma_N'$ such that

(a) Each γ_k' consists of a finite number of disjoint closed curves,
(b) If we denote by Q_k $(k = 0, 1, \ldots, N-1)$ the subregions of Q bounded by γ_k', then $Q_k \supset Q_{k+1}$, where $\gamma_0' = \gamma'$ and $Q_0 = Q$,
(c) If $Q_k \setminus Q_{k+1} \subset D^\pm$, then $Q_{k+1} - Q_{k+2} \subset D^\mp$.

It follows that $\gamma_k' + \gamma_{k+1}' \sim 0$ $(k = 1, \ldots, N-1)$ and $\gamma_N' \sim 0$ in D^+ or D^-. Consider the case when $\gamma \subset D^+$. Then $\gamma \sim \gamma_1'$ in D^+. Since $d * \omega^+ = 0$ in D^+, we have

$$J[\gamma] = \int_\gamma *\omega^+ = \int_{\gamma_1'} *\omega^+ = \int_{\gamma_1'} b_J + \int_{\gamma_1'} *\omega^-$$

$$= \int_{\gamma_1'} b_J - \int_{\gamma_2'} *\omega^- \qquad \text{by } d*\omega^- = 0 \text{ and } \gamma_1' + \gamma_2' \sim 0 \text{ in } D^-.$$

By repeating this procedure we obtain

$$\begin{aligned} J[\gamma] &= \Big(\sum_{k=1}^{N} \int_{\gamma_k'} b_J \Big) \mp \int_{\gamma_N'} *\omega^{\mp} \\ &= \int_{\gamma'} b_J \qquad \text{by } d*\omega^{\mp} = 0 \ \text{ and } \ \gamma_N' \sim 0 \text{ in } D^{\mp}. \end{aligned}$$

We similarly have the same formula in the case when $\gamma \subset D^-$. □

Corollary 3.2 *Let p and ω be defined in notation* (3.4) *through A and B induced by a surface current density $J dS_x$ on Σ. Then*

(1) *p is a co-closed 1-form in $\mathbb{R}^3$ and satisfies $\int_{\gamma_1} p = \int_Q \omega$ for any 1-cycle γ_1 and 2-chain Q in $\mathbb{R}^3$ such that $\partial Q = \gamma_1$ and $Q \cap \Sigma$ is a 1-chain.*

(2) *ω is a* (*discontinuous*) *closed 2-form in $\mathbb{R}^3$.*

Proof. Since p is continuous in $\mathbb{R}^3$, formula (3.5) implies the integral formula in (1) and $\int_{\gamma_2} *p = 0$ for any 2-cycle γ_2 in $\mathbb{R}^3$. So, $*p$ is closed in $\mathbb{R}^3$ by H. Weyl [Wy]. (1) is proved. Since ω is closed in $\mathbb{R}^3 \setminus \Sigma$ and since the normal component of $\omega^+(x)$ is equal to that of $\omega^-(x)$ on Σ by (3.7), we have $\int_{\gamma_2} \omega = 0$ for any 2-cycle γ_2 in $\mathbb{R}^3$ such that $\gamma_2 \cap \Sigma$ is a 1-cycle. So, ω is closed in $\mathbb{R}^3$ by [Wy]. □

4. Reproducing 1-form $*\Omega_\gamma$

Let $D \subset\subset \mathbb{R}^3, \Sigma, \overline{D}$ and D' be the same as in §2. We usually put $C^\infty(\overline{D})$ = the space of C^∞ functions in a neighborhood of $\overline{D}$ and $C_0^\infty(D)$ = the space of C^∞ functions in D with compact support in D. For $i = 1, 2$, we consider the following spaces:

$$\begin{aligned} C_i^\infty(\overline{D}) &= \text{the space of } i\text{-forms of class } C^\infty \text{ in a neighborhood of } \overline{D}, \\ C_{i0}^\infty(D) &= \text{the space of } i\text{-forms of class } C^\infty \text{ with compact support in } D, \\ C_i^\omega(U) &= \text{the set of real analytic } i\text{-forms in } U \subset \mathbb{R}^3, \\ Z_i^\infty(\overline{D}) &= \text{the space of closed } i\text{-forms of class } C^\infty \text{ in a neighborhood of } \overline{D}, \\ L_i^2(D) &= \text{the Hilbert space of square integrable } i\text{-forms in } D, \\ Z_i(D) &= \mathrm{Cl}_i\, [Z_i^\infty(\overline{D})], \\ B_i(D) &= \mathrm{Cl}_i\, [dC_{i-1,0}^\infty(D)] \ \text{ where } C_{0,0}^\infty(D) = C_0^\infty(D), \end{aligned}$$

$H_i(D)$ = the space of square integrable harmonic i-forms in D,

where $\mathrm{Cl}_i[\quad]$ means the closure of $[\quad]$ in $L_i^2(D)$. Weyl's orthogonal decomposition theorems [Wy] are

$$L_i^2(D) = Z_i(D) \dot{+} * B_{3-i}(D), \qquad Z_i(D) = H_i(D) \dot{+} B_i(D). \qquad (4.1)$$

Let $i = 1, 2$ and $\omega_i \in C_i^\infty(U)$, where $U \supset\supset \Sigma$. If all three coefficients of ω_i vanish identically on Σ, we write $\omega_i = 0$ *on* Σ. If the restriction ω_i' of ω_i to the surfaces Σ is 0 as an i-form on Σ, we write $\omega_i = 0$ *along* Σ. Put $\omega_1 = adx + bdy + cdz$ and $\omega_2 = \alpha dy \wedge dz + \beta dz \wedge dx + \gamma dx \wedge dy$. Then it is clear that $\omega_1 = 0$ *or* $\omega_2 = 0$ along Σ, iff the vector (a, b, c) *or* (α, β, γ) is a normal *or* tangent vector of Σ at each $x \in \Sigma$, respectively.

Proposition 4.1 *Let* $\omega_i \in C_i^\infty(\overline{D})$ $(i = 1, 2)$. *Then*
(1) *If* $\omega_2 \in B_2(D)$, *then* $\omega_2 = 0$ *along* Σ.
(2) *Assume that* $\omega_2 = d\omega_1$ *on* $\overline{D}$. *Then* $\omega_2 = 0$ *along* Σ, *if and only if the restriction* ω_1' *of* ω_1 *to the surfaces* Σ *is a closed* 1*-form on* Σ.

Proof. Let $\omega_2 \in B_2(D) \cap C_2^\infty(\overline{D})$. Then, $(\omega_2, *df)_D = \int_{\partial D} f\omega_2$ for any $f \in C^\infty(\overline{D})$. Thus (4.1) implies (1). Let $\omega_2 = d\omega_1$ on $\overline{D}$. For any 1-cycle γ on Σ which bounds a 2-chain $Q(\subset \Sigma)$, it holds $\int_\gamma \omega_1' = \int_Q \omega_2$. This implies (2). □

Now let γ be a 1-cycle in D. We consider a linear functional L_γ on $Z_1^\infty(\overline{D})$:

$$L_\gamma : \omega \longmapsto \int_\gamma \omega \in \mathbb{R}.$$

By [Wy], we find an $M > 0$ such that $|\int_\gamma \omega| \leq M \|\omega\|_D$ for all $\omega \in Z_1^\infty(\overline{D})$. There thus exists a unique $*\Omega_\gamma \in Z_1(D)$ which satisfies

$$\int_\gamma \omega = (\omega, *\Omega_\gamma)_D \qquad \text{for any } \omega \in Z_1^\infty(\overline{D}). \qquad (4.2)$$

We call $*\Omega_\gamma$ the *reproducing* 1*-form for* (D, γ). We note that $\Omega_\gamma \in H_2(D)$. Indeed, for any $f \in C_0^\infty(D)$, (4.2) implies that $(df, *\Omega_\gamma)_D = \int_\gamma df = 0$, so that $\Omega_\gamma \in Z_2(D)$ from the first formula of (4.1). Hence, $\Omega_\gamma \in H_2(D)$.

We need a rather concrete construction of the 2-form Ω_γ (due to F. Maitani). We consider the u-axially symmetric solid torus $K := I \times A$ with corners in $\mathbb{R}^3$, where

$$I = \{u \in \mathbf{R} \mid -1 < u < 1\},$$
$$A = \{(v, w) \in \mathbb{R}^2 \mid 1/2 < \sqrt{v^2 + w^2} < 2\}.$$

In K, we take the circle $C_0 = \{(0, \cos\theta, \sin\theta) \mid 0 \leq \theta \leq 2\pi\}$ and the rectangle $S_0 = I \times \{(v, 0) \in A \mid 1/2 < v < 2\}$, so that $S_0 \times C_0$ (intersection number) $= 1$. We here construct C^∞ functions $\chi(u)$ on $\bar{I}$ and $\phi(v, w)$ on $\overline{A}$ such that

$$\chi(u) = \begin{cases} 0 & \text{on } [-1, -1/2] \\ 1 & \text{on } [1/2, 1], \end{cases}$$
$$\phi(v, w) = \begin{cases} 0 & \text{on } 1/2 \leq \sqrt{v^2 + w^2} \leq 2/3 \\ 1 & \text{on } 3/2 \leq \sqrt{v^2 + w^2} \leq 2, \end{cases}$$

and put

$$\sigma_0 = d\chi(u) \wedge d\phi(v, w) \in Z_{20}^\infty(K).$$

Proposition 4.2 *This 2-form σ_0 in K has the following properties:*
(1) $(\omega, *\sigma_0)_K = \int_{C_0} \omega$ *for any $\omega \in Z_1^\infty(\overline{K})$.*
(2) $\int_{S_0} \sigma_0 = 1$.

Proof. For any $\omega \in Z_1^\infty(\overline{K})$, we have from Stokes' formula

$$\begin{aligned}(\omega, *\sigma_0)_K &= \int_K d(\chi\, d\phi \wedge \omega) \\ &= \int_{[(\partial I) \times A] \cup [I \times \partial A]} \chi(u) d\phi(v, w) \wedge \omega = \int_{\{1\} \times A} d(\phi(v, w)\omega) \\ &= \int_{\{1\} \times \{\sqrt{v^2+w^2}=2\}} \omega = \int_{C_0} \omega.\end{aligned}$$

Thus, (1) is proved. We similarly have

$$\int_{S_0} \sigma_0 = \int_{S_0} d(\chi\, d\phi) = \int_{\partial S_0} \chi(u) d\phi(v, w) = \int_{1/2}^2 d\phi(v, 0) = 1.$$

□

Now let γ be a smooth 1-cycle in D. We take a tubular neighborhood $\widetilde{K}$ of γ in D which admits a C^∞ (orientation preserving) transformation $T : \widetilde{K} \mapsto K$ with $T(\gamma) = C_0$. We denote by $T\sharp\sigma_0$ the pull back of σ_0 by T, and put

$$\widetilde{\sigma} = \begin{cases} T\sharp\sigma_0 & \text{in } \widetilde{K} \\ 0 & \text{in } D \setminus \widetilde{K}, \end{cases} \tag{4.3}$$

so that $\widetilde{\sigma} \in Z_{20}^{\infty}(D)$. For any $\omega \in Z_1^{\infty}(\overline{D})$, (1) of Proposition 4.2 implies

$$(\omega, *\widetilde{\sigma})_D = (\omega, *T\sharp\sigma_0)_{\widetilde{K}} = (T^{-1}\sharp\omega, *\sigma_0)_K = \int_{C_0} T^{-1}\sharp\omega = \int_{\gamma} \omega.$$

It follows from the first formula of (4.1) that Ω_γ is the orthocomponent of $\widetilde{\sigma}$ to $H_2(D)$ in the second one of (4.1):

$$\widetilde{\sigma} = \Omega_\gamma + \tau \quad \text{where } \Omega_\gamma \in H_2(D) \text{ and } \tau \in B_2(D). \tag{4.4}$$

Note that $\tau \in B_2(D) \cap C_2^{\infty}(D)$ and $\Omega_\gamma + \tau = 0$ in $D \setminus \widetilde{K}$. Further we have

Proposition 4.3 *Ω_γ and τ in (4.4) is extended onto a neighborhood U of Σ in $\mathbb{R}^3$ such that $\Omega_\gamma \in H_2(D) \cap C_2^{\omega}(U)$ and $\tau \in B_2^{\infty}(D) \cap C_2^{\omega}(U)$ with*

$$(a) \quad \Omega_\gamma + \tau = 0 \quad \text{in } U, \qquad (b) \quad \Omega_\gamma = \tau = 0 \quad \text{along } \Sigma.$$

Proof. For a given $x_0 \in \Sigma$, we take a small ball $B \subset \mathbb{R}^3$ centered at x_0 such that $B \cap D \subset D \setminus \widetilde{K}$ and $B \cap D$ is simply connected. Then there exists a single-valued harmonic function u in $B \cap D$ such that $*\Omega_\gamma = du$ in $B \cap D$. Since $\Omega_\gamma + \tau = 0$ in $B \cap D$ and $Z_1(D) \perp *B_2(D)$, it holds, for any $f \in C_0^{\infty}(B)$,

$$0 = (df, *\tau)_D = -(df, du)_{B\cap D} = \lim_{n\to\infty} \int_{L_n} f \frac{\partial u}{\partial n_x} dS_x,$$

where $L_n = \{x \in D \cap B \mid R(x) = -1/n\}$ and $R(x)$ is defined by (1.8). Since $L_n \to \Sigma \cap B$ $(n \to \infty)$ and $\|du\|_{B\cap D} \leq \|\Omega_\gamma\|_D < \infty$, it follows that u is of class C^1 up to $\Sigma \cap B$ and $\partial u/\partial n_x = 0$ on $\Sigma \cap B$. Σ being real analytic in $\mathbb{R}^3$, u is extended harmonic in a neighborhood $U(x_0)$ of $\Sigma \cap B$ in B (see [LM]). Thus, Ω_γ is harmonic in $U(x_0)$, and $\Omega_\gamma = 0$ along $\Sigma \cap B$. If we put $\tau = -\Omega_\gamma$ in $U(x_0)$, then Ω_γ and τ satisfy (a) and (b) in $U(x_0)$. Since $x_0 \in \Sigma$ is arbitrary, Proposition 4.3 follows from the uniqueness theorem for harmonic functions. □

Corollary 4.1 *For any 2-chain Q in $\overline{D}$ such that $\partial Q \subset \partial D$, it holds*

$$\int_Q \Omega_\gamma = Q \times \gamma.$$

Proof. Let Q be given as above. For later use we first show that, for any $\tau \in B_2(D) \cap C_2^\infty(U)$ where $U \supset\supset \Sigma$,

$$\int_Q \tau = 0. \tag{4.5}$$

In fact, take a larger smooth domain $\widetilde{D}$ in $\mathbb{R}^3$ and a 2-chain $\widetilde{Q}$ in $\widetilde{D}$ such that $\widetilde{D} \supset\supset D, D \cap \widetilde{Q} = Q$ and $\partial\widetilde{Q} \subset \partial\widetilde{D}$. We put $\widetilde{Q} = \sum_{i=1}^n \widetilde{Q}_i$ where $\widetilde{Q}_i$ $(1 \le i \le n)$ is a 2-cell in $\mathbb{R}^3$. For any $x \in \mathbb{R}^3$, we consider a solid angle $\mu_i(x)$ of $\widetilde{Q}_i$ seen from the point x such that

$$\mu_i(x) = \frac{1}{4\pi}\int_{\widetilde{Q}_i}\left(\frac{\partial}{\partial n_y}\frac{1}{\|x-y\|}\right)dS_y \qquad \text{(Gauss integral)}.$$

Then $\mu_i(x)$ is a bounded harmonic function in $\mathbb{R}^3 - \widetilde{Q}_i$ and is harmonically extended beyond $\widetilde{Q}_i$ from both sides such that $\mu_i(x^+) - \mu_i(x^-) = 1$ for any $x \in \widetilde{Q}_i$, and $d\mu_i(x)$ is a harmonic 1-form in $\mathbb{R}^3 \setminus \partial\widetilde{Q}_i$. If we put $\mu(x) = \sum_{i=1}^n \mu_i(x)$ for $x \in D$, then $\mu(x)$ is a harmonic function in $D \setminus Q$ such that $d\mu(x) \in H_1(\overline{D})$ and $\mu(x^+) - \mu(x^-) = 1$ for any $x \in Q$. Since $B_2(D) \perp *Z_1(D)$ and $\tau = 0$ along Σ, we have

$$0 = (\tau, *d\mu)_D = \int_{D\setminus Q} d(\mu\tau) = \int_{\Sigma\cup Q^\pm} \mu\tau = \int_Q \tau.$$

Thus (4.5) is proved. We next show Corollary 4.1. By (4.4), (4.5) and (2) of Proposition 4.2, we have

$$\begin{aligned}\int_Q \Omega_\gamma &= \int_Q \widetilde{\sigma} = \int_{Q\cap\widetilde{K}} T\sharp\sigma_0 = \int_{T(Q)\cap K}\sigma_0 \\ &= (T(Q)\cap K)\times C_0 = Q\times\gamma,\end{aligned}$$

which proves Corollary 4.1. $\square$

We here consider the subspaces $H_{20}(D)$ of $H_2(D)$ and $H_{1e}(D)$ of $H_1(D)$:

$$H_{20}(D) = \{\omega \in H_2(D) \mid \omega \text{ is harmonic on } \overline{D} \text{ and } \omega = 0 \text{ along } \Sigma\}$$
$$H_{1e}(D) = \{du \in H_1(D) \mid u \text{ is a harmonic function in } D\}.$$

Proposition 4.4 *Let $\{\gamma_i\}_{i=1,\dots,q}$ be a 1-dimensional homology base of D, and denote by $*\Omega_i$ the reproducing 1-form for (D, γ_i). Then*
(1) *$\{\Omega_i\}_{i=1,\dots,q}$ is a base of $H_{20}(D)$.*

(2) *For each* i $(1 \leq i \leq q)$, *there exists a unique* $\omega_i \in H_{20}(D)$ *such that*

$$\int_{\gamma_j} *\omega_i = \delta_{ij} \ (1 \leq j \leq q).$$

(3) *The orthogonal decomposition* $H_2(D) = H_{20}(D) \dot{+} * H_{1e}(D)$ *holds.*

(4) *If we put* $\omega_i = \sum_{j=1}^{q} c_{ij}\Omega_j$, *then the* (q, q)*-matrix* $(c_{ij})_{i,j}$ *is non-singular.*

Proof. First we show (a) $\{\Omega_i\}_{i=1,\ldots,q}$ are linearly independent in $H_{20}(D)$. Indeed, Proposition 4.3 implies $\Omega_i \in H_{20}(D)$. By de Rahm's theorem there exists a $\sigma_i \in Z_1^{\infty}(\overline{D})$ with $\int_{\gamma_j} \sigma_i = \delta_{ij}$ $(1 \leq j \leq q)$. Thus (a) follows by (4.2). Next we show (b) An element $\omega \in H_{20}(D)$ such that $*\omega$ has no periods in D is 0. In fact, we then find a single-valued harmonic function u on $\overline{D}$ such that $du = *\omega$. From $\omega \in H_{20}(D)$, we have $\partial u/\partial n_x = 0$ on Σ. Hence $u =$ const. in D, which proves (b). By (a) and (b), $(\int_{\gamma_j} *\Omega_i)_{i,j}$ is non-singular. This fact implies (1) $\sim$ (3). From (4.2) we have $c_{ij} = (\omega_i, \omega_j)_D$, by which (4) follows. □

This proposition is analogue to L.V. Ahlfors [Ah]. Precisely, let R be a Riemann surface with border ∂R and let γ be a 1-cycle in R. He studied about the reproducing differential $*\Omega_\gamma$ defined by

$$\int_\gamma \omega = (\omega, *\Omega_\gamma)_R \quad \text{for any } C^\infty \text{ closed differential } \omega \text{ on } \overline{R},$$

and has then proved the corresponding results for R to Proposition 4.4.

Further, A. Accola [Ac] showed a geometrical meaning of the norm $\|\Omega_\gamma\|_R^2$ in terms of the extremal length of the family of curves C such that $C \sim \gamma$ (homologous) in R. Modifying his method, we have the following different kind of geometric result which will be useful in §7:

Corollary 4.2 *Let* D *be any bounded domain of* $\mathbb{R}^3$ *with* C^ω *smooth boundary surfaces* Σ. *Let* $\{\gamma_j\}_{j=1,\ldots,q}$ *be a* 1*-dimensional homology base of* D. *Then we find* C^ω *smooth* 2 *dimensional surfaces* $\{Q_i\}_{i=1,\ldots,q}$ *on* $\overline{D}$ *such that*

$$Q_i \times \gamma_j = \delta_{ij}, \qquad \partial Q_i \subset \Sigma, \qquad Q_i \perp \Sigma. \tag{4.6}$$

Proof. For each i $(1 \leq i \leq q)$, take ω_i in (2) of Proposition 4.4. We consider the Abel integral $u(x)$ of $*\omega_i$. That is, for any $x \in \overline{D}$, $u(x) = \int_l *\omega_i$, where l is an arc on $\overline{D}$ connecting a fixed starting point x_0 and x. Then $u(x)$

is a locally harmonic function on $\overline{D}$ such that $\omega_i = \frac{\partial u}{\partial n_x} dS_x = 0$ at any $x \in \Sigma$. By (2) of Proposition 4.4, the period for any 1-cycle in D of $u(x)$ is always integer. It follows that, for a fixed $c \in \mathbb{R}$ (except for an isolated set), the level surface Q of u in $\overline{D}$ defined by $Q = \{x \in \overline{D} \mid u(x) = c \bmod \mathbb{Z}\}$ consists of a finite number of 2-dimensional C^ω disjoint smooth surfaces $\theta_\nu (\nu = 1, \ldots, p)$ in $\overline{D}$ such that $\partial\theta_\nu \subset \Sigma$ and $\theta_\nu \perp \Sigma$. We divide $D \backslash Q = \sum_{k=1}^m D_k$ (connected components). Then $\partial D_k = \Theta_k + \Sigma_k$ where $\Sigma_k \subset \Sigma$ and Θ_k consists of a finite number of $+\theta_\nu$ or $-\theta_\nu$, and $u(x)$ is a single-valued harmonic function on $\overline{D}_k$ with $\partial u / \partial n_x = 0$ on Σ_k and with boundary values

$$u(x) = \begin{cases} c + 1 + N_k & \text{on } \Theta'_k \\ c + N_k & \text{on } \Theta''_k, \end{cases} \tag{4.7}$$

where N_k ia an integer and $\Theta_k = \Theta'_k - \Theta''_k$. If we set $Q_i = \sum_{k=1}^m \Theta'_k$, then we see from $*\omega_k = du$ in D that, for any $1 \leq j \leq q$,

$$\begin{aligned} \delta_{ij} &= \int_{\gamma_j} *\omega_i = (*\omega_i, *\Omega_j)_D \quad \text{by (4.2)} \\ &= \sum_{k=1}^m \int_{D_k} du \wedge \Omega_j = \sum_{k=1}^m \int_{\Theta'_k} \Omega_j \quad \text{by (4.7) and } \Omega_j = 0 \text{ along } \Sigma \\ &= \int_{Q_i} \Omega_j = Q_i \times \gamma_j \quad \text{by Corollary 4.1.} \end{aligned}$$

Consequently, this Q_i satisfies all three conditions in Corollary 4.2. $\square$

We say that $\{Q_i\}_{i=1,\cdots,q}$ is a *dual base of* $\{\gamma_j\}_{j=1,\cdots,q}$.

5. Vector potential $\mathcal{A}$ with boundary values 0

We shall show a general property for elements in $B_2(D)$:

Lemma 5.1 *Let $\tau \in B_2^\infty(D) \cap C_2^\omega(U)$ where $\Sigma \subset\subset U \subset \mathbb{R}^3$. Then we find an $e_0 \in C_1^\infty(D) \cap C_1^\omega(U')$ where $\Sigma \subset\subset U' \subset U$ such that*

$$(a) \quad \tau = de_0 \quad \text{in } D \cup U', \qquad (b) \quad e_0 = 0 \quad \text{on } \Sigma.$$

Proof. We take a tubular neighborhood $U_1 \subset U$ of Σ in $\mathbb{R}^3$. First we show

(α) There exists an $e \in C_1^\infty(D) \cap C_1^\omega(U_1)$ such that $\tau = de$ in $D \cup U_1$. Indeed, since $\tau \in B_2^\infty(D) \cap C_2^\omega(U_1), \tau$ belongs to $Z_2^\infty(D \cup U_1) \cap Z_2^\omega(U_1)$ and has no periods along any 2-cycle in $D \cup U_1$. De Rahm's theorem implies that there exists an $e_1 \in C_1^\infty(D \cup U_1)$ such that $\tau = de_1$ in $D \cup U_1$. Analogously, Cartan's theorem [C] (de Rahm's theorem for real analytic category) implies that there exists a $\sigma \in C_1^\omega(U_1)$ such that $\tau = d\sigma$ in U_1. Note that $e_1 - \sigma \in Z_1^\infty(U_1)$. We choose a $\mu \in Z_1^\omega(U_1)$ such that μ and $e_1 - \sigma$ have the same period along each 1-cycle in U_1. Thus, we find an $f \in C^\infty(D \cup U_1)$ such that $df = \mu - (e_1 - \sigma)$ in U_1. By putting $e = e_1 + df$ in $D \cup U_1$, we obtain (α). By (2) of Proposition 4.1, the restriction e' of e to the surfaces Σ is a C^ω closed 1-form on Σ. Moreover, e' has the following property (P):

$$\int_{\gamma'} e' = 0 \text{ for any 1-cycle } \ \gamma' \subset \Sigma \text{ such that } \ \gamma' \sim 0 \text{ on } D \cup \Sigma.$$

For, since we can take a 2-chain $Q \subset D$ such that $\partial Q = \gamma'$, it follows from (4.5) that $\int_{\gamma'} e' = \int_Q de = \int_Q \tau = 0$. Next we show

(β) There exists an $\eta \in C_1^\infty(D) \cap C_1^\omega(U_2)$ where $\Sigma \subset\subset U_2 \subset U_1$ such that $\tau = d\eta$ in $D \cup U_2$ and the restriction η' of η to Σ is a C^ω closed 1-form on Σ with no periods.

Indeed, for each 1-cycle $\gamma_i \subset D$ $(1 \le i \le q)$ we find a 1-cycle $\gamma_i' \subset \Sigma$ such that $\gamma_i \sim \gamma_i'$ on $D \cup \Sigma$. We put $a_i = \int_{\gamma_i'} e'$. Using $\omega_i \in H_{20}(D)$ in (2) of Proposition 4.4, we set $\eta = e - \sum_{k=1}^q a_k * \omega_k$ on $D \cup \Sigma$. Then $\eta \in C_1^\infty(D) \cup C_1^\omega(U_2)$ where $\Sigma \subset\subset U_2 \subset U_1$, and $\int_{\gamma_i'} \eta' = 0$ $(1 \le i \le q)$. Since $*\omega_k \in H_1(D \cup U_2)$, it follows that $d\eta = \tau$ in $D \cup U_2$ and η' on Σ has property (P) like e'. Let γ' be any 1-cycle on Σ. We find a cycle $\delta' = \sum_{i=1}^q n_i \gamma_i'$ on Σ such that $\gamma' \sim \delta'$ on $D \cup \Sigma$. Then $\int_{\gamma'} \eta' = \int_{\gamma' - \delta'} \eta' + \sum_{i=1}^q n_i \int_{\gamma_i'} \eta' = 0$. Hence, ($\beta$) holds. Finally, from ($\beta$) we find an $f(x) \in C^\omega(\Sigma)$ such that $df = \eta'$ on Σ. We denote by $h(x)$ the outer normal component of η at $x \in \Sigma$. Since Σ is real analytic, $h(x) \in C^\omega(\Sigma)$. Then we can construct an $F(x) \in C^\omega(U') \cap C^\infty(D)$ where $\Sigma \subset\subset U' \subset U_2$ such that $F = f$ and $\partial F / \partial n_x = h$ on Σ. If we put $e_0 = \eta - dF$ in $D \cup U'$, then e_0 satisfies (a) and (b) in Lemma 5.1. □

Theorem 5.1 *Let* $\omega \in H_{20}(D)$. *Then there exists a unique* $\mathcal{A} \in C_1^\omega(V)$ *where* $\Sigma \subset\subset V \subset \mathbb{R}^3$ *such that*

$$\text{(i)} \quad d\mathcal{A} = \omega \quad \text{in } D \cap V, \qquad \text{(ii)} \quad \delta\mathcal{A} = 0 \quad \text{in } V,$$
$$\text{(iii)} \quad \mathcal{A} = 0 \quad \text{on } \Sigma.$$

Proof. (*Uniqueness*) Assume that there exists another $\widetilde{\mathcal{A}} \in C_1^\omega(V)$ satisfying (i) $\sim$ (iii). Then (i) implies that $\mathcal{A} - \widetilde{\mathcal{A}}$ is closed in $D \cap V$ and hence in V. For any $x_0 \in \Sigma$, we can find a ball $B \subset V$ centered at x_0 and an

$f \in C^\omega(B)$ such that $\mathcal{A} - \widetilde{\mathcal{A}} = df$ in B. By (ii) we have $\Delta f = -\delta df = 0$, so that f is harmonic in U. Since (iii) implies grad $f = 0$ on $\Sigma \cap B$, we see that $f =$ const. in B. Hence $\mathcal{A} = \widetilde{\mathcal{A}}$ in B, or in V.

(*Existence*) By Proposition 4.4 it suffices to prove for $\omega = \Omega_\gamma$ defined in (4.4). By Proposition 4.3, there exists a neighborhood $U: \Sigma \subset\subset U$ such that $\Omega_\gamma \in H_2(D \cup U), \tau \in B_2^\infty(D) \cap C_2^\omega(U), \Omega_\gamma + \tau = 0$ in U and $\Omega_\gamma = \tau = 0$ along Σ. By Lemma 5.1, there exists an $e_0 \in C_1^\infty(D) \cap C_1^\omega(U')$ where $\Sigma \subset\subset U' \subset U$ such that $\tau = de_0$ in $D \cup U'$ and $e_0 = 0$ on Σ. For any $x_0 \in \Sigma$, we take a ball $B \subset U'$ centered at x_0. From $\Omega_\gamma \in H_2(B) \subset Z_2^\omega(B)$, Poisson's equation implies that there exists an $\alpha \in C_1^\omega(B)$ such that $d\alpha = \Omega_\gamma$ and $\delta\alpha = 0$ in B. Since $\Omega_\gamma + \tau = 0$ in B, there exists an $f \in C^\omega(B)$ such that $\alpha + e_0 = df$ in B. Now let $S = B \cap \Sigma$, which is a C^ω smooth surface in B such that $\partial S \subset \partial B$. By solving the Cauchy problem:

$$\begin{cases} \Delta u(x) = 0 & \text{near } S \text{ in } B \\ \text{grad } u(x) = \text{grad } f(x) & \text{on } S, \end{cases} \tag{5.1}$$

we find a harmonic function u in a neighborhood $V(x_0)$ of S in B such that $du = df$ on S. Put $\mathcal{A} = \alpha - du$ in $V(x_0)$. Then, in $V(x_0)$, $d\mathcal{A} = \Omega_\gamma$ and $\delta\mathcal{A} = \delta\alpha - \Delta u = 0$, while, on S, $\mathcal{A} = -e_0 + df - du = 0$. Hence $\mathcal{A}$ satisfies (i) $\sim$ (iii) in $V(x_0)$. Since $x_0 \in \Sigma$ is arbitrary, it follows from the uniqueness that we find a neighborhood V of Σ in $\mathbb{R}^3$ and $\mathcal{A} \in C_1^\omega(V)$ which satisfy (i) $\sim$ (iii). $\square$

$\mathcal{A}$ is called the *vector potential of* ω *with boundary values* 0 *in* V.

Lemma 5.2 *Let* $\omega \in H_{20}(D)$. *Denote by* $\mathcal{A}$ *the vector potential of* ω *with boundary values* 0 *in* V. *Then there exist a triple* $\{W, \sigma_2, e_1\}$, *where* W *with* $\Sigma \subset\subset W \subset V$, $\sigma_2 \in Z_{20}^\infty(D)$ *with* Supp $\sigma_2 \subset D \setminus W$, *and* $e_1 \in C_1^\infty(D) \cap C_1^\omega(W)$, *such that, putting* $\sigma_2 = 0$ *in* $W \setminus D$, *we have*

$$(a) \quad \sigma_2 = \omega + de_1 \quad \text{in } D \cup W, \qquad (b) \quad \mathcal{A} + e_1 = 0 \quad \text{in } W.$$

Proof. It suffices to prove for $\omega = \Omega_\gamma$ of (4.4). Proposition 4.3 and Lemma 5.1 imply that there exist U' with $\Sigma \subset\subset U' \subset V$, $\widetilde{\sigma} \in C_{20}^\infty(D)$ with Supp $\widetilde{\sigma} \subset D \setminus U'$, and $e_0 \in C_1^\infty(D) \cap C_1^\omega(U')$ such that $e_0 = 0$ on Σ and $\widetilde{\sigma} = \Omega_\gamma + de_0$ in $D \cup U'$. Since $\Omega_\gamma = d\mathcal{A}$ in V and $\mathcal{A} = 0$ on Σ, it follows that $\mathcal{A} + e_0$ is a C^ω closed 1-form in U' such that $\mathcal{A} + e_0 = 0$ on Σ. Thus, for a tubular neighborhood W of Σ in U', there exists a function $g \in C^\omega(W)$ such that $\mathcal{A}+e_0 = dg$ in W. We can extend g to a function $\tilde{g} \in C^\infty(D \cup W)$. By putting $\sigma_2 = \tilde{\sigma}$ and $e_1 = e_0 - d\tilde{g}$ in $D \cup W$, we obtain Lemma 5.2. $\square$

In Theorem 5.1 we write $\mathcal{A} = A_1 dx + A_2 dy + A_3 dz$ in V, where A_i ($i = 1, 2, 3$) is necessarily a *harmonic* function in V. We identify the 1-form $\mathcal{A}$ with the vector field (A_1, A_2, A_3) in V. Thus, (ii) and (iii) for $\mathcal{A}$ are of the forms

$$\sum_{i=1}^{3} \frac{\partial A_i}{\partial x_i} = 0 \quad \text{in } V, \qquad A_i = 0 \quad \text{on } \Sigma \ (i = 1, 2, 3). \tag{5.2}$$

For any vector r in $\mathbb{R}^3$ with $\|r\| = 1$, we put $\frac{\partial \mathcal{A}}{\partial r} = (\frac{\partial A_1}{\partial r}, \frac{\partial A_2}{\partial r}, \frac{\partial A_3}{\partial r})$ in V.

Lemma 5.3 *For any* $x \in \Sigma$, $\frac{\partial \mathcal{A}}{\partial r}(x)$ *is a tangent vector of* Σ *at* x, *while* grad $A_i(x)$ ($i = 1, 2, 3$) *is a normal vector of* Σ *at* x.

Proof. We use the function $R(x)$ of (1.8) defined near Σ. By the second formula of (5.2), we have a C^ω function $c_i(x)$ such that $A_i(x) = c_i(x)R(x)$ in a neighborhood W of Σ. Thus, for any direction r, $\frac{\partial A_i}{\partial r} = c_i \frac{\partial R}{\partial r}$ on Σ. By the first one of (5.2), we have $\sum_{i=1}^{3} c_i \frac{\partial R}{\partial x_i} = 0$ on Σ, by which Lemma 5.3 follows. $\square$

6. Key lemma

The following lemma gives the relation between elements of $H_{20}(D)$ and magnetic fields induced by equilibrium current densities on Σ:

Key Lemma *Let* $\omega \in H_{20}(D)$. *We write* $\omega = \alpha dy \wedge dz + \beta dz \wedge dx + \gamma dx \wedge dy$ *in* D. *Define the following vector field* $B(x)$ *in* $\mathbb{R}^3 \setminus \Sigma$:

$$B(x) = \begin{cases} (\alpha, \beta, \gamma) & \text{in } D \\ (0, 0, 0) & \text{in } D'. \end{cases} \tag{6.1}$$

Then there exists a unique surface current density $J dS_x$ *on* Σ *which induces* $B(x)$ *as a magnetic field. Precisely,* $J dS_x = \frac{\partial \mathcal{A}}{\partial n_x} dS_x$ *on* Σ *where* $\mathcal{A}$ *is the vector potential of* ω *with boundary values* 0.

Proof. Uniqueness is clear from Fleming's law. To prove existence, let $\omega \in H_{20}(D)$ and let $B(x)$ be defined by (6.1). By Theorem 5.1, we have the vector potential $\mathcal{A}$ of ω with boundary values 0 in V where $\Sigma \subset\subset V \subset \mathbb{R}^3$. We write $\mathcal{A} = A_1 dx + A_2 dy + A_3 dz$ in V and consider $\frac{\partial \mathcal{A}}{\partial n_x}$ at $x \in \Sigma$. It is enough to prove the following claims:

(c$_1$) $\dfrac{\partial \mathcal{A}}{\partial n_x} dS_x$ is a surface current density on Σ.

(c$_2$) $\dfrac{\partial \mathcal{A}}{\partial n_x} dS_x$ induces the above $B(x)$ as a magnetic field.

Indeed, we apply Lemma 5.2 for the given ω in Key Lemma to obtain a triple $\{W, \sigma_2, e_1\}$ satisfying all conditions in Lemma 5.2. We recall the functions $R(x)$ in $U \supset\supset \Sigma$, $\chi_n(R)$ on $(-\infty, +\infty)$, and $\widetilde{\chi}_n(x)$ in $\mathbb{R}^3$ defined by (1.8), (1.9) and (1.14), respectively. We may assume that $W = U$. We put

$$D_n = \{x \in D \mid \widetilde{\chi}_n(x) > 0\}, \qquad D_{n,1} = \{x \in D \mid \widetilde{\chi}_n(x) = 1\}.$$

We always consider sufficiently large n such that $D_{n,1} \supset D \setminus W$. By (1.10), we have $\Gamma_n \cup D_{n,1} \subset D_n \subset\subset D$ and $D_{n,1} \nearrow D\,(n \to \infty)$. We set

$$\widetilde{\omega}_n := \begin{cases} d(-\widetilde{\chi}_n e_1) & \text{in } D \\ 0 & \text{in } \mathbb{R}^3 \setminus D \end{cases} \quad \text{and} \quad \widetilde{\eta}_n := \delta\, \widetilde{\omega}_n \ \text{ in } \mathbb{R}^3. \tag{6.2}$$

Then $\widetilde{\omega}_n \in Z^{\infty}_{20}(\mathbb{R}^3)$ with $\operatorname{Supp} \widetilde{\omega}_n \subset D_n$, and $\widetilde{\eta}_n \in *Z^{\infty}_{20}(\mathbb{R}^3)$. Lemma 3.2 implies that $\widetilde{\eta}_n$ induces $\widetilde{\omega}_n$. By direct calculation, we have

$$\widetilde{\eta}_n = -\delta[(d\widetilde{\chi}_n) \wedge e_1] - *[(d\widetilde{\chi}_n) \wedge *de_1] - \widetilde{\chi}_n \delta de_1 \quad \text{in } D.$$

Since $\omega \in H_2(D)$ and $D_{n,1} \supset \operatorname{Supp} \sigma_2$, it follows from (a) of Lemma 5.2 that

$$\widetilde{\chi}_n \delta de_1 = \widetilde{\chi}_n \delta(\sigma_2 - \omega) = \delta\sigma_2 \quad \text{in } D,$$

which is independent of n. We set

$$\widehat{\sigma} := \begin{cases} \sigma_2 & \text{in } D \\ 0 & \text{in } \mathbb{R}^3 \setminus D \end{cases} \qquad \text{and} \quad \widehat{\eta} := \delta\widehat{\sigma} \text{ in } \mathbb{R}^3.$$

Then $\widehat{\sigma} \in Z^{\infty}_{20}(\mathbb{R}^3)$ with $\operatorname{Supp} \widehat{\sigma} \subset D \setminus W$, and $\widehat{\eta} \in *Z^{\infty}_{20}(\mathbb{R}^3)$. By Lemma 3.2, $\widehat{\eta}$ induces $\widehat{\sigma}$. We put $\eta_n := \widetilde{\eta}_n + \widehat{\eta}$ in $\mathbb{R}^3$, namely,

$$\eta_n = \begin{cases} -\delta[(d\widetilde{\chi}_n) \wedge e_1] - *[(d\widetilde{\chi}_n) \wedge *de_1] & \text{in } D \\ 0 & \text{in } \mathbb{R}^3 \setminus D. \end{cases}$$

It follows that $\eta_n \in *Z^{\infty}_{20}(\mathbb{R}^3)$ and $\operatorname{Supp} \eta_n \subset \Gamma_n \subset W \cap D$. We denote by ω_n the C^∞ 2-form in $\mathbb{R}^3$ induced by η_n in the sense of Lemma 3.1. Since

$\omega_n = \widetilde{\omega}_n + \widehat{\sigma}$ in $\mathbb{R}^3$, it holds by (a) of Lemma 5.2

$$\omega_n = \begin{cases} d(-e_1) + \sigma_2 = \omega & \text{in } D_{n,1} \\ 0 & \text{in } \mathbb{R}^3 \setminus D. \end{cases} \tag{6.3}$$

In particular, we pointwise have

$$\overline{\omega} := \lim_{n\to\infty} \omega_n(x) = \begin{cases} \omega(x) & \text{for } x \in D \\ 0 & \text{for } x \in \mathbb{R}^3 \setminus D. \end{cases} \tag{6.4}$$

By (a) and (b) of Lemma 5.2, we have

$$\begin{aligned} \eta_n &= \begin{cases} \delta[(d\widetilde{\chi}_n) \wedge \mathcal{A}] + *[(d\widetilde{\chi}_n) \wedge *d\mathcal{A}] & \text{in } D \cap W \\ 0 & \text{in } \mathbb{R}^3 \setminus (D \cap W) \end{cases} \\ &\equiv f_{1n}dx + f_{2n}dy + f_{3n}dz \quad \text{in } \mathbb{R}^3. \end{aligned} \tag{6.5}$$

We put $\omega_n(x) := \alpha_n dy \wedge dz + \beta_n dz \wedge dx + \gamma_n dx \wedge dy$ for $x \in \mathbb{R}^3$, so that

$$\alpha_n(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x_3 - y_3) f_{2n}(y) - (x_2 - y_2) f_{3n}(y)}{\|x - y\|^3} dv_y \quad etc., \tag{6.6}$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. By (3.3), we put

$$J_n dv_x := T_c^{-1}(\eta_n) = (f_{1n}, f_{2n}, f_{3n}) dv_x \quad \text{in } \mathbb{R}^3. \tag{6.7}$$

Since $\eta_n \in *Z^{\infty}_{20}(\mathbb{R}^3)$, $J_n dv_x$ is a volume current density in $\mathbb{R}^3$. It thus suffices for claim (c_1) to prove

$$J_n dv_x \to \frac{\partial \mathcal{A}}{\partial n_x} dS_x \, (n\to\infty) \text{ on } \Sigma \text{ in the sense of distribution.} \tag{6.8}$$

In fact, to get the explicit formula of f_{in} $(i = 1, 2, 3)$, we note that, in W,

$$d\widetilde{\chi}_n(x) = \chi_n'(R(x)) \sum_{i=1}^{3} \frac{\partial R}{\partial x_i} dx_i,$$

$$d\chi_n'(R(x)) = \chi_n''(R(x)) \sum_{i=1}^{3} \frac{\partial R}{\partial x_i} dx_i.$$

Substituting these for (6.5), we have, in $D \cap W$,

$$\eta_n = \chi_n'(R(x)) \sum_{i=1}^{3} g_i dx_i + \chi_n''(R(x)) \sum_{i=1}^{3} G_i dx_i,$$

where

$$\begin{aligned}
g_1(x) &= \frac{\partial}{\partial y}\left(\frac{\partial R}{\partial x}A_2 - \frac{\partial R}{\partial y}A_1\right) - \frac{\partial}{\partial z}\left(\frac{\partial R}{\partial z}A_1 - \frac{\partial R}{\partial x}A_3\right) \\
&\quad + \frac{\partial R}{\partial y}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) - \frac{\partial R}{\partial z}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right), \\
G_1(x) &= \frac{\partial R}{\partial y}\left(\frac{\partial R}{\partial x}A_2 - \frac{\partial R}{\partial y}A_1\right) - \frac{\partial R}{\partial z}\left(\frac{\partial R}{\partial z}A_1 - \frac{\partial R}{\partial x}A_3\right),
\end{aligned}$$

and $g_i(x)$, $G_i(x)\,(i = 2, 3)$ are written cyclically. Hence,

$$f_{in}(x) = \chi_n'(R(x))g_i(x) + \chi_n''(R(x))G_i(x) \quad \text{in } D \cap W. \tag{6.9}$$

From Lemma 5.3 and (5.2), we have, on Σ,

$$g_1(x) = \sum_{i=1}^{3}\left(\frac{\partial R}{\partial x_1}\frac{\partial A_i}{\partial x_i} + \frac{\partial R}{\partial x_i}\frac{\partial A_i}{\partial x_1} - 2\frac{\partial R}{\partial x_i}\frac{\partial A_1}{\partial x_i}\right) = -2\frac{\partial A_1}{\partial n_x}.$$

From (1) of Lemma 1.1, $\chi_n'(R(x))g_1(x)dv_x \to 2\frac{\partial A_1}{\partial n_x}dS_x$ $(n \to \infty)$ on Σ in the sense of distribution. Again using Lemma 5.3 and (5.2), we have, on Σ,

$$\begin{aligned}
G_1(x) &= 0, \\
\frac{\partial G_1}{\partial n_x} &= -\|\operatorname{grad} R\|^2\frac{\partial A_1}{\partial n_x} + \frac{\partial R}{\partial x_1}\sum_{i=1}^{3}\left(\frac{\partial R}{\partial x_i}\frac{\partial A_i}{\partial n_x}\right) = -\frac{\partial A_1}{\partial n_x}.
\end{aligned}$$

It follows from (2) of Lemma 1.1 that $\chi_n''(R(x))G_1(x)dv_x \to -\frac{\partial A_1}{\partial n_x}dS_x$ $(n \to \infty)$ on Σ in the sense of distribution. Therefore, (6.9) implies that $f_{1n}dv_x \to \frac{\partial A_1}{\partial n_x}dS_x$ $(n \to \infty)$ on Σ in the sense of distribution. Cyclically we have similar formulas for $i = 2, 3$. Hence, (c_1) is proved.

Next we shall prove (c_2). We put $J_0dS_x = \frac{\partial A}{\partial n_x}dS_x$, and denote by $B_0(x)$ the magnetic field induced by J_0dS_x. By (3.4) we consider

$$\begin{aligned}
\eta_0(x) &:= S_c(J_0dS_x) = \sum_{i=1}^{3}\frac{\partial A_i}{\partial n_x}dx_i \quad \text{on } \Sigma, \\
\omega_0(x) &:= S_m(B_0(x)) = \alpha_0 dy \wedge dz + \beta_0 dz \wedge dx + \gamma_0 dx \wedge dy \text{ in } \mathbb{R}^3 \backslash \Sigma,
\end{aligned}$$

so that

$$\alpha_0(x) = \frac{1}{4\pi}\int_\Sigma \frac{(x_3 - y_3)\dfrac{\partial A_2}{\partial n_y} - (x_2 - y_2)\dfrac{\partial A_3}{\partial n_y}}{\|x - y\|^3}dS_y \qquad etc.$$

For any fixed $x \in \mathbb{R}^3 \setminus \Sigma$, limiting formula (6.8), together with (6.6), implies that $\omega_n(x) \to \omega_0(x)$ $(n \to \infty)$. By (6.4), we thus have $\omega_0(x) = \overline{\omega}(x)$ in $\mathbb{R}^3 \setminus \Sigma$, or equivalently, $B_0(x) = B(x)$ where $B(x)$ is defined by (6.1). Hence, (c_2) holds. Key lemma is completely proved. $\square$

Converse of Key Lemma *Let B be a magnetic field induced by an equilibrium current density on Σ. We write $B(x) = (\alpha, \beta, \gamma)$ in D. If we put $\omega := \alpha dy \wedge dz + \beta dz \wedge dx + \gamma dx \wedge dy$ in D, then $\omega \in H_{20}(D)$.*

Proof. By (3.5), we proved $\omega \in H_2(D)$. By Proposition 1.1, ω is continuous up to Σ and the normal component of ω with respect to Σ vanishes on Σ. By the same use of [LM] in the proof of Proposition 4.3, we see that ω is harmonic beyond Σ and $\omega = 0$ along Σ, so that $\omega \in H_{20}(D)$. $\square$

We extend any $\omega \in H_{20}(D)$ to $\overline{\omega} \in H_2(\mathbb{R}^3 \setminus \Sigma)$ by putting $\overline{\omega} = 0$ in D', and define the following spaces:

$$\overline{H}_{20}(D) = \{\overline{\omega} \in L_2^2(\mathbb{R}^3) \mid \omega \in H_{20}(D)\},$$
$$\mathcal{B} = \text{the set of all magnetic fields } B_J \text{ induced by equilibrium current densities } J dS_x \text{ on } \Sigma.$$

By (3.4), we considered the injection S_m from the set of magnetic fields induced by all surface current densities on Σ, into $H_2(\mathbb{R}^3 \setminus \Sigma)$. Key Lemma and its converse imply that

$$S_m : \mathcal{B} \longmapsto \overline{H}_{20}(D) \quad \textit{is bijective.} \tag{6.10}$$

We here give the final step of

Proof of the main theorem (*Uniqueness in* (1)) Let JdS_x be an equilibrium current density on Σ and denote by B its magnetic field in $\mathbb{R}^3 \setminus \Sigma$. It is enough to prove that, if the total current $J[\gamma_j] = 0$ for each 1-cycle γ_j $(1 \leq j \leq q)$ in D, then $J(x) = 0$ on Σ. In fact, we put $\overline{\omega} = S_m(B) \in \overline{H}_{20}(D)$. By (3.6), $\int_{\gamma_j} *\omega = J[\gamma_j] = 0$ $(1 \leq j \leq q)$. It follows from (2) of Proposition 4.4 that $\omega = 0$ in D. Thus, $B^+(x) = 0$ for any $x \in \Sigma$. Since $B^-(x) = 0$ on Σ and $J(x)$ is tangential on Σ, Proposition 1.1 implies $J(x) = 0$ on Σ.

(*Existence in* (1), *and* (2)) For any fixed i $(1 \leq i \leq q)$, we obtain, from (2) of Proposition 4.4, $\omega_i \in H_{20}(D)$ with $\int_{\gamma_j} *\omega_i = \delta_{ij}$ $(1 \leq j \leq q)$. By (6.10), we have $B := S_m^{-1}(\overline{\omega}_i) \in \mathcal{B}$. We thus find an equilibrium current density JdS_x on Σ which induces B as a magnetic field. Again using (3.6), we have $J[\gamma_j]=\delta_{ij}$, so that JdS_x on Σ is the desired one for the existence in (1). (2) of the main theorem is clear from (1) and Proposition 4.4. $\square$

This proof implies that

$$S_m(\mathbf{B}_i)=\overline{\omega}_i\ (1\le i\le q), \tag{6.11}$$

where $\mathbf{B}_i$ and ω_i are stated in the main theorem of §2 and in (2) of Proposition 4.4, respectively. For the reproducing 1-form $*\Omega_i$ for (D,γ_i), we have a unique $\widetilde{\mathbf{B}}_i\in\mathcal{B}$ such that $S_m(\widetilde{\mathbf{B}}_i)=\overline{\Omega}_i$. We put

$$\begin{aligned}\widetilde{\mathbf{A}}_i &= \text{ the equilibrium vector potential of } \widetilde{\mathbf{B}}_i,\\ \widetilde{\mathbf{p}}_i &= S_p(\widetilde{\mathbf{A}}_i)\in C_1(\mathbb{R}^3),\end{aligned}$$

so that $d\widetilde{\mathbf{p}}_i=\overline{\Omega}_i$ in $\mathbb{R}^3\setminus\Sigma$. Then we have the following result which is related to how D is embedded in $\mathbb{R}^3$:

Remark 6.1. Let A be an equilibrium vector potential in $\mathbb{R}^3$ and put $\mathbf{p}=S_p(A)\in C_1(\mathbb{R}^3)$. Then the restriction of $\mathbf{p}$ on Σ is a closed 1-form on Σ. In the case when $\mathbf{A}=\widetilde{\mathbf{A}}_i$, namely, $\mathbf{p}=\widetilde{\mathbf{p}}_i$, its period of any 1-cycle δ on Σ is given by

$$\int_\delta \widetilde{\mathbf{p}}_i=Q\times\gamma_i,$$

where Q is a 2-chain in $\mathbb{R}^3$ such that $\partial Q=\delta$.

Proof. Let γ be any small 1-cycle on Σ. Then we can take a 2-chain $S\subset D'$ such that $\partial S=\gamma$. Since $dp=0$ in D' and $p(x)$ is continuous in $\mathbb{R}^3$, we have $\int_\gamma p=\int_S dp=0$. Hence p is closed on Σ. In the case when $\mathbf{p}=\mathbf{p}_i$, let δ and Q be given as above. We divide Q into two 2-chains $\{Q_1,Q_1'\}$: $Q=Q_1+Q_1'$ such that $Q_1\subset D$, $Q_1'\subset D'$, $\partial Q_1\subset\Sigma$ and $\partial Q_1'\subset\Sigma$. Since $\overline{\Omega}_i=0$ in D', it follows from Corollary 4.1 and (1) of Corollary 3.2 that

$$\int_\delta\widetilde{\mathbf{p}}_i=\int_Q\overline{\Omega}_i=\int_{Q_1}\Omega_i=Q_1\times\gamma_i=Q\times\gamma_i.$$

$\square$

Let JdS_x be a surface current density on Σ and denote by A and B its vector potential and its magnetic field. We put $p:=S_p(A)\in C_1^\omega(D\cup D')$ and $\omega:=S_m(B)\in H_2(D\cup D')$. Since Σ is of class C^ω, we have the C^ω

extensions p^+ and p^- of p from D and D' beyond Σ, respectively. Hence, $\widetilde{p} := p^+ - p^-$ is of class C^ω in a neighborhood W of Σ in $\mathbb{R}^3$. By continuity of p in $\mathbb{R}^3$, $\widetilde{p} = 0$ on Σ. Moreover, (3.5) implies $\delta\widetilde{p} = 0$ in W. Therefore, we obtain

Remark 6.2. Under these notaions, if JdS_x is an equilibrium current density on Σ, then the vector potential $\mathcal{A}$ of ω with boundary values 0 is written into

$$\mathcal{A} = p^+ - p^- \quad \text{in } W.$$

7. Extremal properties

We need the following approximation condition for the equilibrium current densities which is compared with Corollary 1.1 for the surface current densities.

Lemma 7.1 *Let JdS_x be an equilibrium current densities on Σ and denote by A and B the equilibrium vector potential and the magnetic field for JdS_x. Then we find a sequence of volume current densities $\{J_n dv_x\}_n$ in $\mathbb{R}^3$ converging to JdS_x on Σ in the sense of distribution such that, denoting by A_n and B_n the vector potential and the magnetic field for $J_n dv_x$, we have*

(1) $\operatorname{Supp} J_n \subset D$ *for* $n \geq 1$ *and* $\operatorname{Supp} J_n \to \Sigma$ $(n \to \infty)$.

(2) $B_n = B = 0$ *in* D' *for* $n \geq 1$. *Given* $K \subset\subset D$, *there exists an* n_1 *such that* $B_n = B$ *in* K *for* $n \geq n_1$.

(3) $A_n(x) \to A(x)$ *and* $B_n \to B(x)$ *uniformly on any compact set in* $\mathbb{R}^3 \backslash \Sigma$.

(4) *There exists an* $M_0 > 0$ *such that* $\|B_n(x)\| \leq M_0$ *for* $n \geq 1$ *and* $x \in \mathbb{R}^3$.

(5) $\lim_{n\to\infty} \int_{\mathbb{R}^3} \|B_n(x)\|^2 dv_x = \int_{\mathbb{R}^3\backslash\Sigma} \|B(x)\|^2 dv_x$.

Lemma 7.1$'$ *All are same as Lemma* 7.1 *except that* (1) *and* (2) *are replace by*

(1$'$) $\operatorname{Supp} J_n \subset D'$ *for* $n \geq 1$ *and* $\operatorname{Supp} J_n \to \Sigma$ $(n \to \infty)$.

(2$'$) $B_n = B$ *in* D *for* $n \geq 1$. *Given* $G \supset\supset D$, *there exists an* n_1 *such that* $B_n = B = 0$ *in* $\mathbb{R}^3 \setminus G$ *for* $n \geq n_1$.

Proof of Lemma 7.1 Let JdS_x, A and B be given as above. By (6.10) we put $\overline{\omega} = S_m(B) \in \overline{H}_{20}(D)$. From the uniqueness, JdS_x is identical with the one obtained from this ω in Key Lemma. In that proof we here use C^∞ functions $\chi_n(R)$ on $(-\infty, +\infty)$ with (1.16). We keep all notations $\eta_n, \omega_n, \widetilde{\eta}_n, \cdots$

in the proof of Key Lemma. By (6.7) we again define a sequence of volume current densities $\{J_n dv_x\}_n$ in $\mathbb{R}^3$, so that it converges to JdS_x on Σ in the sense of distribution. Further, we can show that the vector potential $A_n(x)$ and the magnetic field $B_n(x)$ for such $J_n dv_x$ satisfy (1) $\sim$ (5) in Lemma 7.1. In fact, since $\eta_n = T_c(J_n dv_x)$ and $\omega_n = T_m(B_n)$, (1) and (2) follow from (6.5) and (6.3), respectively. Since Supp $d\widetilde{\chi}_n \subset \Gamma_n \subset W \cap D$, we have from (6.9) and (6.2)

$$\begin{aligned} J_n &= \chi_n'(R(x))(g_1, g_2, g_3) + \chi_n''(R(x))(G_1, G_2, G_3) \quad \text{in } \Gamma_n, \\ \omega_n(x) &= \widetilde{\omega}_n(x) + \widehat{\sigma}(x) \quad \text{in } \mathbb{R}^3, \end{aligned}$$

where

$$\widetilde{\omega}_n(x) = \begin{cases} (d\widetilde{\chi}_n) \wedge \mathcal{A} + \widetilde{\chi}_n \omega & \text{in } W \cap D \\ -de_1 & \text{in } D \setminus W \\ 0 & \text{in } D' \cup \Sigma. \end{cases}$$

Since $\mathcal{A} = G_i = 0$ $(i = 1, 2, 3)$ on Σ, it follows from (1.16) that there exists an $M_1 > 0$ such that $\|J_n(x)\| \leq nM_1$ and $\|\widetilde{\omega}_n\|(x) \leq M_1$ for all $n \geq 1$ and all $x \in \mathbb{R}^3$. The first inequality like Corollary 1.1 implies (3) of Lemma 7.1. The second one with $\widehat{\sigma} \in C_{20}^\infty(\mathbb{R}^3)$ implies (4). (5) follows from (2) and (4). $\square$

Proof of Lemma 7.1′ Instead of $\chi_n(R)$ and $\widetilde{\chi}_n(x)$ in the above proof, we take C^∞ functions $K_n(R)$ on $(-\infty, +\infty)$ and $\widetilde{K}_n(x)$ in $W(\supset\supset \Sigma)$ such that

$$0 \leq K_n(R) \leq 1, \qquad K_n(R) = \begin{cases} 1 & \text{on } \left(-\infty, \frac{1}{2n}\right] \\ 0 & \text{on } \left[\frac{1}{n}, +\infty\right), \end{cases}$$

$$|K_n'(R)| \leq nM, \qquad |K_n''(R)| \leq n^2 M, \qquad \widetilde{K}_n(x) = K_n(R(x)),$$

where $M > 0$ is a constant independent of $n \geq 1$ and $R \in (-\infty, +\infty)$. Then, by the same argument as Lemma 7.1, we have Lemma 7.1′. $\square$

Now let Jdv_x be a volume current density in $\mathbb{R}^3$ and denote by $B_J(x)$ the magnetic field in $\mathbb{R}^3$ induced by Jdv_x. We put

$$\|B_J\|_{\mathbb{R}^3}^2 = \int_{\mathbb{R}^3} \|B_J(x)\|^2 dv_x,$$

which is called the *total energy of the magnetic field* $B_J(x)$. We analogously define the total energy $\|B\|_{\mathbb{R}^3 \setminus \Sigma}^2$ of the magnetic field $B(x)$ induced by a surface current density JdS_x on Σ. We consider

$$\begin{aligned}
\mathcal{V} &= \text{the set of all volume current densities } Jdv_x \text{ in } \mathbb{R}^3,\\
\mathcal{S} &= \text{the set of all surface current densities } JdS_x \text{ on } \Sigma,\\
\mathcal{E} &= \text{the set of all equilibrium current densities } JdS_x \text{ on } \Sigma.
\end{aligned}$$

For any fixed i $(1 \le i \le q)$, we put

$$\begin{aligned}
\mathcal{V}_i &= \{Jdv_x \in \mathcal{V} \mid \mathrm{Supp}J \subset D' \quad \text{and} \quad J[\gamma_j] = \delta_{ij}\ (1 \le j \le q)\},\\
\mathcal{S}_i &= \{JdS_x \in \mathcal{S} \mid J[\gamma_j] = \delta_{ij}\ (1 \le j \le q)\}.
\end{aligned}$$

For $\mathbf{J}_i dS_x \in \mathcal{S}_i$ and $\mathbf{B}_i(x)$ stated in the main theorem of §2 we have

Theorem 7.1 *$\mathbf{J}_i dS_x$ and $\mathbf{B}_i(x)$ have the following extremal properties:*
(1) $\|\mathbf{B}_i\|^2_{\mathbb{R}^3\setminus\Sigma} = \inf\{\|B_J\|^2_{\mathbb{R}^3} \mid Jdv_x \in \mathcal{V}_i\}$.
(2) *$\mathbf{J}_i dS_x$ is a unique element in $\mathcal{S}_i$ minimizing* $\{\|B_J\|^2_{\mathbb{R}^3\setminus\Sigma} \mid JdS_x \in \mathcal{S}_i\}$.

Proof. By (6.11), we put $\overline{\omega}_i = S_m(\mathbf{B}_i) \in \overline{H}_{20}(D)$. To prove (1), we first take any $Jdv_x \in \mathcal{V}_i$. By (3.3), we put $\omega_J = T_m(B_J) \in Z_2^\infty(\mathbb{R}^3)$. From Supp $J \subset D'$, $*\omega_J \in H_1(\overline{D})$ by (3.5). Since $\omega_i = \sum_{j=1}^q c_{ij}\Omega_j$ by (4) of Proposition 4.4, it follows from (4.2) and (3.5) that

$$\begin{aligned}
\|\omega_i\|^2_D &= \sum_{j=1}^q c_{ij}(*\omega_i, *\Omega_j)_D = c_{ii}\\
&= \sum_{j=1}^q c_{ij}(*\omega_J, *\Omega_j)_D = (*\omega_J, *\omega_i)_D,
\end{aligned}$$

so that $\|\mathbf{B}_i\|^2_{\mathbb{R}^3\setminus\Sigma} = \|\omega_i\|^2_D \le \|\omega_J\|^2_D < \|\omega_J\|^2_{\mathbb{R}^3} = \|B_J\|^2_{\mathbb{R}^3}$. We next apply Lemma 7.1$'$ to the case when $JdS_x = \mathbf{J}_i dS_x$ on Σ to find a sequence $\{J_n dv_x\}_n$ in $\mathcal{V}$ converging to $\mathbf{J}_i dS_x$ on Σ in the sense of distribution which satisfies $(1') \sim (5)$ in Lemma 7.1$'$. By $(1')$, we have Supp $J_n \subset D'$. By $(2')$ and (3.5), $J_n[\gamma_j] = \mathbf{J}_i[\gamma_j] = \delta_{ij}$ for all n, so that $J_n dv_x \in \mathcal{V}_i$. Further, (5) of Lemma 7.1$'$ implies $\lim_{n\to\infty} \|B_{J_n}\|^2_{\mathbb{R}^3} = \|\mathbf{B}_i\|^2_{\mathbb{R}^3\setminus\Sigma}$. Thus, (1) of Theorem 7.1 is proved. We similarly have (2) of Theorem 7.1. □

To show the extremal property of equilibrium vector potentials A (see Theorem 7.2), we generalize the notion of volume current density Jdv_x in $\mathbb{R}^3$. Let $J = (f_1, f_2, f_3)$ be a vector field in $\mathbb{R}^3$ with compact support such that each f_i is (not necessarily continuous) *bounded* and *piecewise smooth*

in $\mathbb{R}^3$. Assume that there exists a sequence $\{J_n dv_x\}_n$ in $\mathcal{V}$ such that
(i$''$) $\{\|J_n(x)\|\}_n$ is uniformly bounded in $\mathbb{R}^3$,
(ii$''$) $J_n dv_x \to J dv_x$ $(n \to \infty)$ in the sense of distribution.
Then $J dv_x$ is called a *generalized volume current density* in $\mathbb{R}^3$. We put

$$A_J(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{J(y)}{\|x-y\|} dv_y \qquad \text{for } x \in \mathbb{R}^3,$$

$$B_J(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} J(y) \times \frac{x-y}{\|x-y\|^3} dv_y \quad \text{for } x \in \mathbb{R}^3,$$

which are called the *vector potential for* $J dv_x$ and the *magnetic field induced by* $J dv_x$. It is clear that $A_J \in C^1(\mathbb{R}^3)$ and $B_J \in C(\mathbb{R}^3)$. Moreover, Lebesgue's convergence theorem implies

$$B_J(x) = \text{rot } A_J(x) \quad \text{in } \mathbb{R}^3, \qquad \text{div } A_J(x) = 0 \quad \text{in } \mathbb{R}^3. \tag{7.1}$$

We define

$$\|J dv_x\|^2_{\mathbb{R}^3} = \int_{\mathbb{R}^3} \|J(x)\|^2 dv_x,$$

which is called the *total energy of the generalized current density* $J dv_x$. Let Q be any 2-dimensional smooth surface in $\mathbb{R}^3$ such that Q intersects the set of discontinuous points of J at most along a 1-chain. We then define the *total current* $J[Q]$ *through* Q *of* $J dv_x$:

$$J[Q] = \int_Q J(x) \cdot n_x \, dS_x.$$

Using $\{J_n dv_x\}_n$ with (i$''$) and (ii$''$), we have

$$J[Q] = \lim_{n\to\infty} \int_Q J_n(x) \cdot n_x \, dS_x, \;\; \|J dv_x\|_{\mathbb{R}^3} = \lim_{n\to\infty} \|J_n dv_x\|_{\mathbb{R}^3}. \tag{7.2}$$

Let $\{\gamma_j\}_{j=1,\dots,q}$ be a 1-dimensional homology base of D, and $\{Q_j\}_{j=1,\cdots,q}$ its dual base defined in Corollary 4.2. For each i $(1 \le i \le q)$, we define

$$\begin{aligned} \mathcal{G} &= \text{the set of all generalized volume current densities } J dv_x \text{ in } \mathbb{R}^3, \\ \mathcal{G}_i &= \{J dv_x \in \mathcal{G} \mid \exists \{J_n dv_x\}_n \text{ satisfying (i}''\text{), (ii}''\text{), and} \\ &\qquad\qquad \operatorname{Supp} J_n \subset D \;\text{ and }\; J[Q_j] = \delta_{ij} \;(1 \le j \le q)\,\}. \end{aligned}$$

Then we have the following extremal property for equilibrium vector potentials:

Theorem 7.2 *For each i $(1 \le i \le q)$, we have*

(1) *There exists a unique element $\mathbf{G}_i dv_x$ in $\mathcal{G}_i$ which minimizes $\{\|Jdv_x\|^2_{\mathbb{R}^3} \mid Jdv_x \in \mathcal{G}_i\}$.*

(2) *Let $Jdv_x \in \mathcal{G}_i$. Then $Jdv_x = \mathbf{G}_i dv_x$, if and only if $B_J(x)$ is reduced to an equilibrium vector potential $\widehat{A}(x)$ for a certain $\widehat{J}dS_x \in \mathcal{E}$.*

To prove this theorem we need two lemmas.

Lemma 7.2 (*Recurrence*) *If $JdS_x \in \mathcal{E}$, then $B_J dv_x \in \mathcal{G}$ and $A_J = B_{B_J}$ in $\mathbb{R}^3$.*

Physically speaking, a surface current density JdS_x on Σ always induces the magnetic field B_J and the vector potential A_J. In case when JdS_x is equilibrium, B_J makes a generalized volume current density $B_J dv_x$, whose magnetic field is identical with A_J:

$$A_J(x) = \mathrm{rot}\ \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{B_J(y)}{\|x-y\|} dv_y \qquad \text{for } x \in \mathbb{R}^3. \tag{7.3}$$

Proof. Let $JdS_x \in \mathcal{E}$. We find a sequence $\{J_n dv_x\}_n$ in $\mathcal{V}$ satisfying (1) $\sim$ (5) of Lemma 7.1, and use the same notations A_n, B_n defined in that lemma. Since $\operatorname{div} B_n(x) = 0$ in $\mathbb{R}^3$, it follows from (2) and (4) that $\{B_n dv_x\}_n \subset \mathcal{V}$ such that $\operatorname{Supp} B_n \subset D$ and $B_n dv_x \to B dv_x$ $(n \to \infty)$ in the sense of distribution, so that $B_J dv_x \in \mathcal{G}$. By (3.4), we put $\eta = S_c(JdS_x)$ on Σ, $p = S_p(A_J) \in C_1(\mathbb{R}^3)$ and $\overline{\omega} = S_m(B_J) \in \overline{H}_{20}(D)$. If we define

$$\lambda(x) = \mathcal{N}\omega(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\overline{\omega}(y)}{\|x-y\|}\, dv_y \qquad \text{for } x \in \mathbb{R}^3,$$

then $\lambda(x)$ is a C^1 2-form in $\mathbb{R}^3$. It suffices for (7.3) to verify

$$p(x) = \delta\lambda(x) \qquad \text{for } x \in \mathbb{R}^3. \tag{7.4}$$

For each $n = 1, 2, \ldots$, we put, by (3.3), $\eta_n = T_c(J_n dv_x)$, $p_n = T_p(A_n)$ and $\omega_n = T_m(B_n)$, so that $\omega_n = dp_n$ and $p_n(x) = \mathcal{N}\eta_n(x)$ for $x \in \mathbb{R}^3$. If we put $\lambda_n(x) = \mathcal{N}\omega_n(x)$ for $x \in \mathbb{R}^3$, then Lemma 3.3 implies $p_n = \delta\lambda_n$ in $\mathbb{R}^3$. On the other hand, (2), (3), and (4) of Lemma 7.1 imply that $p_n(x) \to p(x)$, $\lambda_n(x) \to \lambda(x)$, and $\delta\lambda_n(x) \to \delta\lambda(x)$ uniformly on any compact set in $\mathbb{R}^3 \setminus \Sigma$. Therefore, (7.4) is true for $x \in \mathbb{R}^3 \setminus \Sigma$. Since both $p(x)$ and $\delta\lambda(x)$ are continuous in $\mathbb{R}^3$, (7.4) holds for all $x \in \mathbb{R}^3$. $\square$

For a given $JdS_x \in \mathcal{E}$, we consider the vector potential Λ_J for $B_J dv_x$:

$$\Lambda_J(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{B_J(y)}{\|x-y\|} dv_y \qquad \text{for } x \in \mathbb{R}^3, \tag{7.5}$$

so that $A_J = \text{rot}\ \Lambda_J$ by Lemma 7.2. For each i $(1 \le i \le q)$, we define the following subspaces of $Z_2(D)$:

$$\begin{aligned}\varXi &= \{\sigma \in Z_2^\infty(\overline{D}) \mid \sigma = 0 \text{ along } \Sigma\} \supset H_{20}(D),\\ \varXi_i &= \{\sigma \in \varXi \mid \int_{Q_j} \sigma = \delta_{ij}\ (1 \le j \le q)\}.\end{aligned}$$

Lemma 7.3 *The 2-form $\omega_i \in H_{20}(D)$ defined in (2) of Proposition 4.4 satisfies*
(1) $(\omega_i, \sigma)_D = \int_{Q_i} \sigma$ *for any* $\sigma \in \varXi$.
(2) *The set $H_{20}(D) \cap \varXi_i$ consists of a unique element, which we denote by σ_i. This σ_i satisfies $\|\sigma - \sigma_i\|_D^2 = \|\sigma\|_D^2 - \|\sigma_i\|_D^2$ for any $\sigma \in \varXi_i$.*

Proof. Since $Q_i \times \gamma_j = \delta_{ij}$ and $\int_{\gamma_j} *\omega_i = \delta_{ij}$, the 1-form $*\omega_i$ in D is exact in $D \setminus Q_i$. We thus find a harmonic function u_i in $D \setminus Q_i$ such that $*\omega_i = du_i$ and $u_i(x^+) - u_i(x^-) = 1$ for any $x \in Q_i$. Then, for any $\sigma \in \varXi$,

$$(\omega_i, \sigma)_D = \int_{D-Q_i} du_i \wedge \sigma = \int_{\Sigma + Q_i^+ - Q_i^-} u_i \sigma = \int_{Q_i} \sigma.$$

(1) is proved. To prove (2), we put $\sigma = \omega_j \in H_{20}(D)$ in (1). Then, (4) of Proposition 4.4 implies that $\int_{Q_i} \omega_j = c_{ij}$. Since $(c_{ij})_{i,j}$ is non-singular and $\{\omega_j\}_{j=1,\dots,q}$ is a base of $H_{20}(D)$, $H_{20} \cap \varXi_i$ consists of a unique element, say σ_i, of the form $\sigma_i = \sum_{k=1}^q x_k \omega_k$. Using assertion (1) twice, we have, for any $\sigma \in \varXi_i$,

$$(\sigma_i, \sigma)_D = \sum_{k=1}^q x_k (\omega_k, \sigma)_D = x_i = \sum_{k=1}^q x_k(\omega_k, \sigma_i)_D = \|\sigma_i\|_D^2.$$

□

Proof of Theorem 7.2 For each i $(1 \le i \le q)$, we take $\sigma_i \in H_{20}(D)$ in (2) of Lemma 7.3 and define $\mathbf{G}_i := S_m^{-1}(\overline{\sigma}_i) \in \mathcal{B}$. By Lemma 7.2, $\mathbf{G}_i dv_x \in \mathcal{G}$. We also have $\mathbf{G}_i[Q_j] = \int_{Q_j} \sigma_i = \delta_{ij}\,(1 \le j \le q)$, so that $\mathbf{G}_i dv_x \in \mathcal{G}_i$. For any $Jdv_x \in \mathcal{V} \cap \mathcal{G}_i$ with Supp $J \subset D$, we put $\eta = T_c(Jdv_x) \in *Z_{20}^\infty(\mathbb{R}^3)$. Since $\delta_{ij} = J[Q_j] = \int_{Q_j} *\eta$, we have $*\eta \in \varXi_i$. It follows from (2) of Lemma

7.3 that

$$\|Jdv_x - \mathbf{G}_i dv_x\|^2_{\mathbb{R}^3} = \| * \eta - \sigma_i\|^2_D = \|Jdv_x\|^2_{\mathbb{R}^3} - \|\mathbf{G}_i dv_x\|^2_{\mathbb{R}^3} \ (\geq 0).$$

This equality, together with (7.2), implies (1) of Theorem 7.2.

To prove (2) of Theorem 7.2, let $Jdv_x \in \mathcal{G}_i$. First, assume that $Jdv_x = \mathbf{G}_i dv_x$. By (6.10), $\mathbf{G}_i$ is a magnetic field induced by an equilibrium current density $\widehat{J}_i dS_x$ on Σ. If we denote by $\widehat{A_i}$ the equilibrium vector potential for $\widehat{J}_i dS_x$, then we have $B_J = B_{\mathbf{G}_i} = \widehat{A_i}$ by Lemma 7.2, so that "only if" part is proved. Next, assume that the magnetic field B_J induced by Jdv_x is identical with an equilibrium vector potential $\widehat{A}(x)$ for some equilibrium current density $\widehat{J}dS_x$ on Σ : $B_J = \widehat{A}$ in $\mathbb{R}^3$. If we denote by $\widehat{B}$ the magnetic field induced by $\widehat{J}dS_x$, then we have from Lemma 7.2,

$$\operatorname{rot} \int_{\mathbb{R}^3} \frac{J(y)}{\|x-y\|} dv_y = \operatorname{rot} \int_{\mathbb{R}^3} \frac{\widehat{B}(y)}{\|x-y\|} dv_y \quad \text{for } x \in \mathbb{R}^3. \tag{7.6}$$

Like (3.3) we let correspond the integral in the left or right hand side to a 1-form $p(x)$ or $\widehat{p}(x)$ in $\mathbb{R}^3$, respectively. Then both $p(x)$ and $\widehat{p}(x)$ belong to $C^1_1(\mathbb{R}^3)$. By (7.6), we have $dp = d\widehat{p}$ in $\mathbb{R}^3$. On the other hand, the second formula of (7.1) implies $\delta p = \delta \widehat{p} = 0$ in $\mathbb{R}^3$. Therefore, $p - \widehat{p}$ is a harmonic 1-form in $\mathbb{R}^3$. Since $p(x)$, $\widehat{p}(x) = O(1/\|x\|)$ near ∞, we have $p = \widehat{p}$ in $\mathbb{R}^3$. Thus, Poisson's equation implies that $Jdv_x = \widehat{B}dv_x$, so $\widehat{B}[Q_j] = J[Q_j] = \delta_{ij}$ $(1 \leq j \leq q)$. Hence, $S_m(\widehat{B}) \in \overline{H}_{20}(D) \cap \Xi_i$. It follows from (2) of Lemma 7.3 that $S_m(\widehat{B}) = \overline{\sigma}_i = S_m(\mathbf{G}_i)$, or equivalently, $\widehat{B} = \mathbf{G}_i$ by (6.10). We thus have $Jdv_x = \mathbf{G}_i dv_x$. $\square$

By Lemma 7.2, any equilibrium vector potential $A(x)$ satisfies

$$\|A(x)\| \leq O(1/\|x\|^2) \qquad \text{near } x = \infty. \tag{7.7}$$

We finally make the following two remarks in this section:

(I) Professor L. Hörmander gave a comment that the main theorem is so stable that it would permit passage to the limit, and the same result would then hold also in the non-analytic case. This is true. In fact, let D be a bounded domain in $\mathbb{R}^3$ with C^∞ smooth boundary surfaces Σ, and let $\{\gamma_j\}_{j=1,\ldots,q}$ be a 1-dimensional homology base of D. We then find a sequence of bounded domains $\{D_n\}_n$ with C^ω smooth boundary surfaces Σ_n such that, as $n \to \infty$, $D_n \nearrow D$ and $\Sigma_n \to \Sigma$ with their first and second derivatives. We may assume that $\{\gamma_j\}_{j=1,\ldots,q}$ is a 1-dimensional homology base of each D_n. Fix $1 \leq i \leq q$. The same reasoning as Proposition

4.4 implies that there exists a unique $\omega \in H_2(D)$ such that $\int_{\gamma_j} *\omega = \delta_{ij}$ $(1 \leq j \leq q)$, ω is of class C^∞ up to Σ and the normal component of ω vanishes on Σ. For each $n = 1, 2, \ldots$, we consider $\omega_n \in H_{20}(D_n)$ such that $\int_{\gamma_j} *\omega_n = \delta_{ij}$. Then

$$\|\omega_m - \omega_n\|^2_{D_n} \leq \|\omega_m\|^2_{D_m} - \|\omega_n\|^2_{D_n} \qquad (n < m \leq \infty),$$

where $\omega_\infty = \omega$ and $D_\infty = D$. It follows that $\omega(x) = \lim_{n\to\infty} \omega_n(x)$ uniformly on any compact set in D. We put $\omega_n(x) = \alpha_n dy \wedge dz + \beta_n dz \wedge dx + \gamma_n dx \wedge dy$ on $D_n \cup \Sigma_n$ and $\omega(x) = \alpha dy \wedge dz + \beta dz \wedge dx + \gamma dx \wedge dy$ on D. By the normal extension, we can continuously extend ω_n and ω to $\widetilde{\omega_n}$ and $\widetilde{\omega}$ in a domain $G \supset\supset \overline{D}$, respectively, such that $\lim_{n\to\infty} \|\widetilde{\omega_n} - \widetilde{\omega}\|^2_G = 0$. It follows that the vector field $(\alpha_n, \beta_n, \gamma_n)$ converges (α, β, γ) uniformly on $\overline{D}$. Thus we find a sequence of volume current densities $\{I_n dv_x\}_{n=1,2,\ldots}$ in $\mathbb{R}^3$ such that $I_n dv_x \to J dS_x$ $(n \to \infty)$ on Σ in the sense of distribution. Since formulas (6.11) and (3.6) for each ω_n hold, we conclude that JdS_x on Σ is an equilibrium current density on Σ such that $J[\gamma_j] = \delta_{ij}$. Therefore, the existence of (1) in the main theorem for C^∞ category is proved. The uniqueness of (1) and (2) are similarly proved.

(II) We have treated the *bounded* domain D with C^∞ boundary surfaces Σ so far. We consider the *unbounded* domain $D'(= \mathbb{R}^3 \setminus \overline{D})$ with the same boundary Σ and with $\{\infty\}$. Since any $\omega \in H_2(D')$ always satisfies $\|\omega\|(x) = O(1/\|x\|^2)$ near ∞ (which is proved by Kelvin's transformation), all arguments for D are available for D'. So, the results for D similarly hold for D'. For example, the main result for D' is stated as follows:

Theorem *Let $\{\gamma_j\}_{j=1,\ldots,q'}$ be a 1-dimensional homology base of D'.* (1) *For any fixed i $(1 \leq i \leq q')$, there exists a unique surface current density $\mathbf{J}_i dS_x$ on Σ such that $\mathbf{J}_i[\gamma_j] = \delta_{ij}$ $(1 \leq j \leq q')$ and the magnetic field $\mathbf{B}_i(x)$ in $\mathbb{R}^3 \setminus \Sigma$ induced by $\mathbf{J}_i dS_x$ is identically 0 in D.* (2) *Any surface current density JdS_x on Σ which induces a magnetic field $B(x)$ identically 0 in D is written by a linear combination of $\{\mathbf{J}_i dS_x\}_{i=1,\ldots,q'}$.*

8. Examples

In the textbooks of electromagnetism, we see the description about a solenoid: For $b > a > 0$, consider a symmetric torus Σ_0 in $\mathbb{R}^3$ given by $(r - b)^2 + z^2 = a^2$, where $[r, \theta, z]$ is the cylindrical coordinates of $\mathbb{R}^3$. We denote by D_0 the solid torus bounded by Σ_0, and put $D_0' = \mathbb{R}^3 \setminus (D_0 \cup \Sigma_0)$.

We positively and symmetrically wind a coil L with electric current I around Σ_0, n times. Then the solenoid ($= \Sigma_0$ equipped with I) induces the static magnetic field:

$$B_0(x) = \begin{cases} c\left(-\frac{\sin\theta}{r}, \frac{\cos\theta}{r}, 0\right) & \text{in } D_0 \\ (0,0,0) & \text{in } D_0', \end{cases}$$

where $c = nI/2\pi$. This formula holds approximately but not rigorously. For, $B_0(x)$ must be singular only on L, but not on Σ_0. Let us show that $B_0(x)$ is the magnetic field induced by an equilibrium current density $J_0 dS_x$ on Σ. We use the torus coordinates $\{R, \phi, \theta\}$ as well as cylindrical coordinates $[r, \theta, z]$:

$$\begin{aligned} x &= (x, y, z) \\ &= \{R, \phi, \theta\} = ((b - R\cos\phi)\cos\theta, (b - R\cos\phi)\sin\theta, R\sin\phi) \\ &= [r, \theta, z] = (r\cos\theta, r\sin\theta, z), \end{aligned}$$

where $0 \le R < b$ and $0 \le \phi, \theta \le 2\pi$. Hence, $\Sigma_0 = \{R = a\}, D_0 = \{R < a\}$, $dS_x = ar\, d\phi d\theta$ on Σ_0 and $dv_x = R(b - R\cos\phi)\, dR d\phi d\theta$. We consider

$$\Omega_0 = c\left(\frac{-\sin\theta}{r} dy \wedge dz + \frac{\cos\theta}{r} dz \wedge dx\right).$$

It is clear that $\Omega_0 \in H_{20}(D)$. So, Key Lemma implies that $B_0(x)$ is a magnetic field induced by the following equilibrium current density $J_0 dS_x$ on Σ:

$$J_0 dS_x = (B_0(x) \times n_x) dS_x = c\left(\frac{\cos\theta\sin\phi}{r}, \frac{\sin\theta\sin\phi}{r}, \frac{\cos\phi}{r}\right) dS_x.$$

The equilibrium vector potential $A_0(x)$ for $J_0(x)$ is then

$$A_0(x) = C\int_0^{2\pi}\int_0^{2\pi} \frac{(\cos\Theta\sin\Phi, \sin\Theta\sin\Phi, \cos\Phi)}{\|x - y\|} d\Phi d\Theta \quad \text{for } x \in \mathbb{R}^3,$$

where $C = nIa/(8\pi^2)$ and $y = \{a, \Phi, \Theta\}$. To give examples more, we introduce some notations:

1. We consider the half-plane Π defined by

$$\Pi = \{(r, z) \mid 0 < r < +\infty,\ -\infty < z < +\infty\},$$

and put $\partial\Pi = \{(0,z) \mid -\infty < z < +\infty\}$ and $\overline{\Pi} = \Pi \cup \partial\Pi$. It is occasionally identified Π with the half (x,z)-plane π_+ in $\mathbb{R}^3$ with $x > 0$ by $(r,z) = (x,z)$. Given a set $K \subset \pi_+ (= \Pi)$, we consider the z-axially symmetric set $\ll K \gg$ in $\mathbb{R}^3$ which is obtained by rotating K around the z-axis, namely, $\ll K \gg$ $= \{\, [r,\theta,z] \mid (r,z) \in K,\ 0 \le \theta \le 2\pi \,\}$.

2. Let $\mathbf{v} = (a,b,c)$ be a vector field in a domain G of $\mathbb{R}^3 \setminus \{\text{the } z\text{-axis}\}$. At any fixed point $x = [r,\theta,z] \in G$, we choose the orthogonal basis of $\mathbb{R}^3$: $\{e_r, e_\theta, e_z\}$ (depending on x) such that $e_z = (0,0,1)$, $e_r = (\cos\theta, \sin\theta, 0)$, $e_\theta = e_z \times e_r = (-\sin\theta, \cos\theta, 0)$, so that

$$\begin{aligned}\mathbf{v} &= (a\cos\theta + b\sin\theta)e_r + (-a\sin\theta + b\cos\theta)e_\theta + ce_z \\ &\equiv \widetilde{a}e_r + \widetilde{b}e_\theta + \widetilde{c}e_z.\end{aligned}$$

We then use the abbreviation: $\mathbf{v} = \langle\, \widetilde{a}, \widetilde{b}, \widetilde{c}\,\rangle$ in G. By simple calculation we have

$$\operatorname{div}\langle \widetilde{a}, \widetilde{b}, \widetilde{c}\rangle = \frac{1}{r}\left(\frac{\partial(r\widetilde{a})}{\partial r} + \frac{\partial \widetilde{b}}{\partial\theta} + \frac{\partial(r\widetilde{c})}{\partial z}\right), \tag{8.1}$$

$$\operatorname{rot}\langle \widetilde{a}, \widetilde{b}, \widetilde{c}\rangle = \left\langle \frac{1}{r}\frac{\partial\widetilde{c}}{\partial\theta} - \frac{\partial\widetilde{b}}{\partial z}, \frac{\partial\widetilde{a}}{\partial z} - \frac{\partial\widetilde{c}}{\partial r}, \frac{1}{r}\left(\frac{\partial(r\widetilde{b})}{\partial r} - \frac{\partial\widetilde{a}}{\partial\theta}\right)\right\rangle, \tag{8.2}$$

$$\langle\, \widetilde{a}, \widetilde{b}, \widetilde{c}\,\rangle \times \langle \widetilde{a}_1, \widetilde{b}_1, \widetilde{c}_1\,\rangle = \langle\, \widetilde{b}\,\widetilde{c}_1 - \widetilde{b}_1\widetilde{c},\ \widetilde{c}\,\widetilde{a}_1 - \widetilde{c}_1\widetilde{a},\ \widetilde{a}\widetilde{b}_1 - \widetilde{a}_1\widetilde{b}\,\rangle. \tag{8.3}$$

By means of (8.1), the vector field $\mathbf{v}$ in G of the form $\mathbf{v} = \langle 0,\ f(r,z)\,, 0\rangle$ or $= \frac{1}{r}\,\langle g_1(\theta),\ 0, g_2(\theta)\,\rangle$ is of divergent 0. Therefore, if $f(r,z) \in C_0^\infty(\Pi)$, then

$$\langle 0, f(r,z), 0\rangle\, dv_x \ \text{ is a volume current density in } \mathbb{R}^3. \tag{8.4}$$

Given a vector potential $A = (a,b,c)$ and a magnetic field $B = (\alpha,\beta,\gamma)$, we considered the injections in (3.3): $T_p(A) = p \in C_1^\infty(\mathbb{R}^3)$ and $T_m(B) = \omega \in C_2(\mathbb{R}^3)$. When we denote by $A = \langle\, \widetilde{a}, \widetilde{b}, \widetilde{c}\,\rangle$ and $B = \langle\, \widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}\,\rangle$, such definitions are equivalent to

$$p = \widetilde{a}dr + r\widetilde{b}d\theta + \widetilde{c}dz, \qquad *\omega = \widetilde{\alpha}dr + r\widetilde{\beta}d\theta + \widetilde{\gamma}dz. \tag{8.5}$$

For example, the above Ω_0, B_0, J_0 and A_0 are written into the forms:

$$\begin{aligned}&*\Omega_0 = c\,d\theta \text{ in } D_0,\ \ B_0(x) = c\,\langle 0,1,0\rangle \text{ in } D_0, \\ &J_0 = \frac{c}{r}\langle \sin\phi, 0, \cos\phi\rangle \ \text{ on } \Sigma,\end{aligned}$$

$$A_0(x) = C \int_0^{2\pi} \int_0^{2\pi} \frac{\langle \cos\Theta \sin\Phi, 0, \cos\Phi \rangle}{\|(r,0,z) - \{a, \Phi, \Theta\}\|} \, d\Phi d\Theta \text{ in } \mathbb{R}^3.$$

3. We consider the following Stokes-Beltrami operators $\Delta^{\pm}$ in Π:

$$\Delta^{\pm} = \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2} \pm \frac{1}{r}\frac{\partial}{\partial r}.$$

Lemma 8.1 *Let X be a domain in Π and consider the z-axially symmetric domain $G = \ll X \gg$ in $\mathbb{R}^3$. Then, we have*

(1) (Beltrami) *Let $f(r,z)$ be a C^∞ function in X. Then, $f(r,z)$ is harmonic as a function in G, if and only if $\Delta^+ f = 0$ in X.*

(2) *Assume that X is simply connected. Let ω be a C^∞ 1-form in X of the form $\omega = f(r,z)dr + g(r,z)dz$. Then ω is harmonic as a 1-form in G, if and only if there exists a C^∞ function $v(r,z)$ in X such that*

$$\text{(i)} \quad \Delta^- v(r,z) = 0, \qquad \text{(ii)} \quad (f,g) = \frac{1}{r}\left(-\frac{\partial v}{\partial z}, \frac{\partial v}{\partial r}\right).$$

Proof. (1) is clear from the identity $\Delta f(\sqrt{x^2+y^2}, z) = \Delta^+ f(r,z)$. To prove (2), assume that $\omega = f(r,z)dr + g(r,z)dz$ is harmonic in G. By $d\omega = 0$, we have $\frac{\partial f}{\partial z} = \frac{\partial g}{\partial r}$ in G, or equivalently, in X. Since

$$*dr = r d\theta \wedge dz, \quad *d\theta = \frac{1}{r} dz \wedge dr, \quad *dz = r dr \wedge d\theta, \tag{8.6}$$

we have $*\omega = rf d\theta \wedge dz + rg dr \wedge d\theta$. By $d * \omega = 0$ in G, it follows that $\frac{\partial(rf)}{\partial r} + \frac{\partial(rg)}{\partial z} = 0$ in X. We thus find C^∞ functions $u(r,z)$ and $v(r,z)$ in X such that

$$\begin{cases} \dfrac{\partial u}{\partial r} = f \\[2ex] \dfrac{\partial u}{\partial z} = g, \end{cases} \qquad \begin{cases} \dfrac{\partial v}{\partial r} = rg \\[2ex] \dfrac{\partial v}{\partial z} = -rf. \end{cases}$$

If we eliminate u by differentiations, v satisfies (i) and (ii) in Lemma 8.1.

Conversely, let $\omega = f(r,z)dr + g(r,z)dz \in C_1^\infty(X)$ satisfy (i) and (ii) of Lemma 8.1. Then (i) and (ii) imply $d\omega = 0$ and $d*\omega = 0$ in G, respectively. Hence, ω is harmonic in G. $\square$

Similarly, $\omega = f(r,z)dr + g(r,z)dz \in C_1^\infty(G)$ is harmonic in G, if and only if there exists a C^∞ function $u(r,z)$ in X such that

$$\text{(i}'\text{)} \quad \Delta^+ u(r,z) = 0, \qquad\qquad \text{(ii}'\text{)} \quad (f,g) = \left(\frac{\partial u}{\partial r}, \frac{\partial u}{\partial z}\right).$$

These functions u and v satisfy a kind of Cauchy-Riemann equations:

$$\frac{\partial u}{\partial r} = -\frac{1}{r}\frac{\partial v}{\partial z}, \qquad \frac{\partial u}{\partial z} = \frac{1}{r}\frac{\partial v}{\partial r}. \tag{8.7}$$

In [Wi], u satisfying (i$'$) is called a z-axially symmetric potential and v related with u in (8.7) its associated stream function. We shall find that v satisfying (i) is concerned with a vector potential for the z-axially symmetric domain.

Lemma 8.2 *Let $h(r,z)$ be a bounded continuous function on $\overline{\Pi}$ and of class C^2 in Π. If $h(r,z)$ satisfies $\Delta^- h(r,z) = 0$ in Π and $h(r,z) = 0$ on $\partial\Pi$, then $h(r,z) = 0$ identically on $\overline{\Pi}$.*

Proof. Given $\varepsilon > 0$, we consider the following two functions $\phi_\varepsilon^\pm(r,z)$ on $\overline{\Pi}$:

$$\phi_\varepsilon^\pm(r,z) = \pm h(r,z) + \varepsilon \log\sqrt{(r+1)^2 + z^2}.$$

It is clear that

$$\Delta^- \phi_\varepsilon^\pm(r,z) \leq 0 \quad \text{on } \overline{\Pi}, \quad \phi_\varepsilon^\pm(r,z) \geq 0 \quad \text{on } \partial\Pi,$$
$$\lim_{(r,z)\to\infty} \phi_\varepsilon^\pm(r,z) \geq 0.$$

It follows from the maximum principle that $\phi_\varepsilon^\pm(r,z) \geq 0$ in Π. Hence, $|h(r,z)| \leq \varepsilon \log\sqrt{(r+1)^2+z^2}$. Letting $\varepsilon \to 0$, we have $h(r,z) = 0$ in Π. □

Let $K \subset\subset \Pi$ be a doubly connected domain bounded by two C^ω smooth closed curves C_1 and C_2 such that $\partial K = C_2 - C_1$. We set $K' = \Pi \setminus \overline{K}$ where $\overline{K} = K \cup \partial K$, which consists of the bounded component K_1' such that $\partial K_1' = C_1$ and the unbounded one K_2' such that $\partial K_2' = -C_2$ in Π. For $i = 1, 2$, we define the z-axially symmetric sets in $\mathbb{R}^3$:

$$D = \ll K \gg, \qquad \Sigma_i = \ll C_i \gg, \qquad \Sigma = \partial D = \Sigma_2 - \Sigma_1,$$

so that D' consists of a bounded solid torus $D_1' = \ll K_1' \gg$ with $\partial D_1' = \Sigma_1$ and an unbounded domain $D_2' = \ll K_2' \gg$ with $\partial D_2' = -\Sigma_2$. We draw a closed cycle $\gamma_1 \subset K$ such that $\gamma_1 \sim C_2$ on $\overline{K}$, and a z-axially symmetric circular ring $\gamma_2 = \ll p_0 \gg \subset D$ where p_0 is a fixed point of K. Then, $\{\gamma_1, \gamma_2\}$ forms a 1-dimensional homology base of D. It follows from the main theorem that, for each $i = 1, 2$, there exists a unique equilibrium current density $\mathbf{J}_i dS_x$ on Σ with $\mathbf{J}_i[\gamma_j] = \delta_{ij}$ $(j = 1, 2)$. We denote by $\mathbf{A}_i$ and $\mathbf{B}_i$ the equilibrium vector potential and the magnetic field for $\mathbf{J}_i dS_x$. By (7.5), we have the vector field $\Lambda_{\mathbf{J}_i}$ in $\mathbb{R}^3$ for $\mathbf{J}_i dS_x$, which is simply denoted by Λ_i.

Let us give explicit formulas of $\mathbf{J}_i dS_x, \mathbf{A}_i, \mathbf{B}_i$, and Λ_i. For this purpose we construct the following functions V, W, and U in Π:

(**1°**) We solve the following boundary value problem on $\overline{K}$:

$$\Delta^- v(r,z) = 0 \quad \text{in } K, \qquad v(r,z) = \begin{cases} 1 & \text{on } C_1 \\ 0 & \text{on } C_2. \end{cases}$$

Such function $v(r, z)$ uniquely exists. We put

$$k = \int_{C_1} \frac{1}{r}\frac{\partial v}{\partial n_p} ds_p \ (< 0), \qquad V(r,z) = \begin{cases} 1/k & \text{in } K_1' \\ v(r,z)/k & \text{on } \overline{K} \\ 0 & \text{in } K_2', \end{cases}$$

where $p = (r, z)$. Thus, $V(r, z)$ is a piecewise smooth continuous function with compact support in Π. We also find a unique C^2 function $W(r, z)$ on $\overline{\Pi}$ which satisfies

$$\Delta^- \ W(r,z) = -V(r,z) \quad \text{in } \Pi \tag{8.8}$$

and assumes boundary values 0, namely,

$$W(r,z) = 0 \text{ on } \partial\Pi \qquad \text{and} \qquad \lim_{(r,z)\to\infty} W(r,z) = 0. \tag{8.9}$$

(**2°**) We have a unique function $U(r,z) \in C^1(\overline{\Pi}) \cap C^2(\Pi \setminus \partial K)$ which satisfies

$$\Delta^- U(r,z) = \begin{cases} \dfrac{-1}{2\pi} & \text{on } K \\ 0 & \text{on } K' \end{cases} \tag{8.10}$$

and assumes boundary values 0.

The integral representations of the functions W and U will be given in (A.7) and (A.8). Using these V, W, and U, we obtain

Expression 1° *We have, for any $x = [r, \theta, z]$,*

$$\mathbf{A}_1(x) = \frac{V(r,z)}{r}\,\langle 0,1,0\,\rangle \ \ in\ \mathbb{R}^3,$$

$$\mathbf{B}_1(x) = \begin{cases} \dfrac{1}{r}\left\langle -\dfrac{\partial V}{\partial z}, 0, \dfrac{\partial V}{\partial r}\right\rangle & in\ D \\ 0 & in\ D', \end{cases}$$

$$\mathbf{J}_1(x) = \frac{1}{r}\,\frac{\partial V(r,z)}{\partial n_p}\,\langle 0,1,0\rangle \ \ on\ \Sigma,$$

$$\Lambda_1(x) = \frac{1}{r}\left\langle -\frac{\partial W}{\partial z}, 0, \frac{\partial W}{\partial r}\right\rangle \ \ in\ \mathbb{R}^3,$$

where $\partial/\partial n_p$ is the outer normal derivative at $p = (r,z)$ of ∂K in Π.

Expression 2° *We have, for any $x = [r, \theta, z]$,*

$$\mathbf{A}_2(x) = \frac{1}{r}\left\langle -\frac{\partial U}{\partial z}, 0, \frac{\partial U}{\partial r}\right\rangle \ \ in\ \mathbb{R}^3,$$

$$\mathbf{B}_2(x) = \begin{cases} \dfrac{1}{2\pi r}\,\langle 0,1,0\rangle & in\ D \\ 0 & in\ D', \end{cases}$$

$$\mathbf{J}_2(x) = \frac{1}{2\pi r}\,\langle r', 0, z'\rangle \ \ on\ \Sigma,$$

$$\Lambda_2(x) = \frac{U(r,z)}{r}\,\langle 0,1,0\,\rangle \ \ in\ \mathbb{R}^3,$$

where (r', z') is the unit tangent vector of ∂K at (r,z) in Π.

Proof. For $x = [r,\theta,z] \in D$, we consider $*\Omega(x) = -\frac{1}{r}\frac{\partial V}{\partial z}dr + \frac{1}{r}\frac{\partial V}{\partial r}dz$. Since $V(r,z) =$ const. on C_i $(i = 1,2)$, it follows from (2) of Lemma 8.1 that $\Omega(x) \in H_{20}(D)$. Moreover, $\int_{\gamma_1} *\Omega = \int_{C_2} \frac{1}{r}\frac{\partial V}{\partial n_p} ds_p = 1$ and $\int_{\gamma_2} *\Omega = 0$, because γ_2 is independent of (r,z). Hence, $\Omega = \omega_1$, which is defined in (2) of Proposition 4.4. By (6.11), we get $\mathbf{B}_1(x) = S_m^{-1}(\overline{\Omega})$. By the second formula of (8.5), $S_m^{-1}(\overline{\Omega})$ is equal to the right-hand side of $\mathbf{B}_1$ in Expression 1°. From Fleming's law: $\mathbf{J}_1(x) = \mathbf{B}_1^+(x) \times n_x$ for $x \in \Sigma_2 - \Sigma_1$, we have by (8.3)

$$\begin{aligned}\mathbf{J}_1(x) &= \frac{1}{r}\left\langle -\frac{\partial V}{\partial z}, 0, \frac{\partial V}{\partial r}\right\rangle \times \left\langle \frac{\partial V}{\partial r}, 0, \frac{\partial V}{\partial z}\right\rangle \Big/ \|\mathrm{grad}\, V(r,z)\| \\ &= \frac{1}{r}\frac{\partial V(r,z)}{\partial n_p}\,\langle 0,1,0\rangle.\end{aligned}$$

From the symmetry of Σ and $\mathbf{J}_1 dS_x$, we see from (1.5) that $\mathbf{A}_1(x)$ is of the form $\mathbf{A}_1(x) = \langle 0, F(r,z), 0\rangle$ for $x = [r,\theta,z] \in \mathbb{R}^3$, where $F(r,z) \in C(\Pi) \cap C^\omega(\Pi \setminus \partial K)$. Since $\operatorname{rot} \mathbf{A}_1(x) = \mathbf{B}_1(x)$ in $\mathbb{R}^3 \setminus \Sigma$, we have from (8.2),

$$\left\langle -\frac{\partial F}{\partial z}, 0, \frac{1}{r}\frac{\partial(rF)}{\partial r}\right\rangle = \begin{cases} \dfrac{1}{r}\left\langle -\dfrac{\partial V}{\partial z}, 0, \dfrac{\partial V}{\partial r}\right\rangle & \text{in } D \\ 0 & \text{in } D'. \end{cases}$$

It follows that, for $i = 1, 2$,

$$rF(r,z) = \begin{cases} V(r,z) + \text{const. } c_0 & \text{in } K \\ \text{const. } c_i & \text{in } K_i'. \end{cases}$$

We take a point $q_0 \in C_2$ and consider the 1-cycle $\delta = \ll q_0 \gg$ on Σ_2. Then we can draw a 2-chain Q in D_2' such that $\partial Q = \delta$. From (3.4) and (8.5), we have $p_1 := S_p(\mathbf{A}_1) = rF(r,\theta)d\theta \in C_1(\mathbb{R}^3)$. Since $V(r,z) = 0$ on C_2, we have $p_1(x) = c_0\, d\theta$ on Σ_2. It follows from $dp_1 = 0$ identically in D' that $2\pi c_0 = \int_\delta p_1 = \int_Q dp_1 = 0$, so that $c_0 = 0$. Because $\mathbf{A}_1(x)$ is continuous in $\mathbb{R}^3$, we obtain $c_1 = 1/k$ on C_1 and $c_2 = 0$ on C_2. Thus, $rF = V$ in all $\mathbb{R}^3$, namely, $\mathbf{A}_1(x) = \langle 0, V(r,z)/r, 0\rangle$ in $\mathbb{R}^3$. Expression $\mathbf{1}^\circ$ except for $\Lambda_1(x)$ is proved.

In order to prove Expression $\mathbf{2}^\circ$, we define $*\Omega(x) = \frac{d\theta}{2\pi}$ for $x = [r,\theta,z] \in D$. Then it is clear that $\Omega(x) \in H_{20}(D)$ with $\int_{\gamma_2} *\Omega = 1$ and $\int_{\gamma_1} *\Omega = 0$. Hence, $\Omega = \omega_2$, which is defined in (2) of Proposition 4.4. It follows from (6.11) that $\mathbf{B}_2(x) = S_m^{-1}(\overline{\Omega})$, which is equal to the right-hand side of $\mathbf{B}_2(x)$ in Expression 2° by the second formula of (8.5). From Fleming's law, we have $\mathbf{J}_2(x) = \mathbf{B}_2^+(x) \times n_p = \frac{1}{2\pi r}\langle r', 0, z'\rangle$ for $x \in \Sigma$. By use of the symmetry of Σ and $\mathbf{J}_2 dS_x$ with respect to the z-axis, we see from (1.5) that $\mathbf{A}_2(x)$ is of the form

$$(1) \quad \mathbf{A}_2(x) = \langle f(r,z), 0, g(r,z)\rangle \quad \text{for any } \; x = [r,\theta,z] \in \mathbb{R}^3,$$

where f and g belong to $C(\overline{\Pi}) \cap C^\omega(\Pi \setminus \partial K)$. By (8.1) and (8.2), we have

$$\operatorname{rot} \mathbf{A}_2(x) = \left\langle 0, \frac{\partial f}{\partial z} - \frac{\partial g}{\partial r}, 0\right\rangle,$$
$$\operatorname{div} \mathbf{A}_2(x) = \frac{1}{r}\left(\frac{\partial(fr)}{\partial r} + \frac{\partial(gr)}{\partial z}\right).$$

Since rot $\mathbf{A}_2 = \mathbf{B}_2$ and div $\mathbf{A}_2 = 0$ in $\mathbb{R}^3 \setminus \Sigma$, f and g satisfy

$$(2) \quad \frac{\partial f}{\partial z} - \frac{\partial g}{\partial r} = \begin{cases} 1/(2\pi r) & \text{in } K \\ 0 & \text{in } K', \end{cases}$$

$$(3) \quad \frac{\partial (fr)}{\partial r} + \frac{\partial (gr)}{\partial z} = 0 \quad \text{in } \Pi \setminus \partial K.$$

From (7.7) there exists an $M > 0$ such that

$$(4) \quad \sqrt{f(r,z)^2 + g(r,z)^2} \leq \frac{M}{r^2 + z^2} \qquad \text{on } \overline{\Pi}.$$

For a given $(r, z) \in \overline{\Pi}$, we connects the origin $(0,0)$ and the point (r, z) by an arc ℓ in $\overline{\Pi}$, and consider the line integral:

$$u(r,z) = \int_\ell -f(r,z) r\, dz + g(r,z) r\, dr.$$

By (3), $u(r, z)$ is independent of the choice of the arc ℓ connecting $(0,0)$ and (r, z) on $\overline{\Pi}$. Therefore, $u(r,z) \in C^1(\overline{\Pi}) \cap C^\omega(\Pi \setminus \partial K)$ and satisfies

$$(5) \quad \begin{cases} \dfrac{\partial u}{\partial z} = -fr \\ \dfrac{\partial u}{\partial r} = gr \end{cases} \quad \text{in } \Pi \setminus \partial K, \qquad (6) \quad u = 0 \quad \text{on } \partial \Pi.$$

Further, u is bounded in $\overline{\Pi}$. Indeed, let (r_0, z_0) be any point in $\overline{\Pi}$ and let $\sqrt{r_0^2 + z_0^2} = R_0$. We take, as an arc ℓ connecting $(0,0)$ and (r_0, z_0), $\ell = \ell_1 + \ell_2$ such that $\ell_1 = \{(0, z) \mid 0 \leq z \leq R_0\}$ and $\ell_2 = \{(r, z) \mid r = R_0 \sin\varphi,\ z = R_0 \cos\varphi,\ 0 \leq \varphi \leq \varphi_0\}$, where $\tan\varphi_0 = r_0/z_0$ $(0 \leq \varphi_0 \leq \pi)$. Since the integrand of u vanishes identically on l_1 and $|f| + |g| \leq 2M/R_0^2$ on ℓ_2 by (4), we have $|u(r_0, z_0)| = |\int_{\ell_2} -f(r,z) r dz + g(r,z) r dr| \leq 2M\pi$. Hence, u is bounded in $\overline{\Pi}$. By (1) and (5), it is enough for the expression for $\mathbf{A}_2(x)$ to prove that $u = U$ in Π. To show this, we put $h(r,z) := u(r,z) - U(r,z)$ on $\overline{\Pi}$. Hence, $h \in C(\overline{\Pi}) \cap C^1(\Pi)$. Eliminating f, g from (2) and (5) by differentiations, we see that u as well as U satisfies (8.10). Thus, $h \in C^1(\overline{\Pi})$ and $\Delta^- h = 0$ in $\Pi \setminus \partial K$. It easily follows that h is of class C^2 in all Π and satisfies $\Delta^- h = 0$ there. By (6) we have $h = 0$ on $\partial\Pi$. Since h is bounded in Π, it follows from Lemma 8.2 that $h = 0$, or equivalently, $u = U$ on $\overline{\Pi}$. Expression 2° except for $\Lambda_2(x)$ is proved.

By the same methods as for $\mathbf{A}_1(x)$ and $\mathbf{A}_2(x)$ we have the expressions for $\Lambda_2(x)$ and $\Lambda_1(x)$, respectively. $\square$

The above proofs for $\mathbf{A}_i(x)$ and $\Lambda_i(x)$ $(i = 1, 2)$ are due to Professor T. Ueda. Ours will be given in Appendix by use of fundamental solution for Δ^-.

We remark that the constant k defined in (1°) has the following meaning:

$$k = -2\pi \text{ (the total energy of } \mathbf{B}_1(x))^{-1}.$$

Proof. For any f, $g \in C^2(G)$ where $G \subset \Pi$, it holds the following Stokes-Beltrami formula:

$$d\left(\frac{1}{r}f * dg\right) = \left\{\frac{1}{r}f(\Delta^- g) + \frac{1}{r}\left(\frac{\partial f}{\partial r}\frac{\partial g}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial z}\right)\right\} dz \wedge dr, \quad (8.11)$$

where $*dr = -dz$ and $*dz = dr$ by the orientation of Π. From the form of $\mathbf{B}_1(x)$ in Expression 1°, we have

$$\|\mathbf{B}_1(x)\|^2_{\mathbb{R}^3\setminus\Sigma} = 2\pi \int_K \frac{1}{r}\left(\left(\frac{\partial V}{\partial r}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2\right) drdz.$$

Since $\Delta^- V = 0$ in K and $V = 1/k$ (resp. 0) on C_1 (resp. C_2), it follows from (8.11) that

$$\|\mathbf{B}_1(x)\|^2_{\mathbb{R}^3\setminus\Sigma} = 2\pi \int_{C_2 - C_1} \frac{1}{r} V \frac{\partial V}{\partial n_p} ds_p = -\frac{2\pi}{k}\int_{C_1} \frac{1}{r}\frac{\partial V}{\partial n_p} ds_p = -\frac{2\pi}{k},$$

which is the desired formula. $\square$

Appendix

A. Fundamental solutions for $\Delta^{\pm} u = 0$

We briefly recall the works of E. Beltrami [B] and A. Weinstein [Wi] about fundamental solutions for $\Delta^{\pm} u = 0$. In this section we use the simple notations:

$$p = (r, z) \in \Pi, \qquad x = [r, \theta, z] = [p, \theta] \in \mathbb{R}^3,$$
$$q = (\rho, \zeta) \in \Pi, \qquad y = [\rho, \Theta, \zeta] = [q, \Theta] \in \mathbb{R}^3.$$

Considering the uniform charge distribution with magnitude $1/\rho$ on the z-axially symmetric circular ring $\gamma(q)$ passing through the point $(\rho, 0, \zeta)$ in $\mathbb{R}^3$, E. Beltrami made the fundamental solution $\mathcal{K}$ for $\Delta^{+} u = 0$:

$$\begin{aligned} \mathcal{K}(p,q) &= \frac{1}{4\pi}\int_{\gamma(q)} \frac{1/\rho}{\|(r,0,z) - y\|}\, ds_y \\ &= \frac{1}{4\pi}\int_0^{2\pi} \frac{1}{\sqrt{r^2 + \rho^2 + (z-\zeta)^2 - 2r\rho\cos\Theta}}\, d\Theta \quad \text{in } \Pi \times \Pi, \end{aligned}$$

where ds_y is the arc length of $\gamma(q)$. A. Weinstein generalized the equations $\Delta^{\pm} u = 0$ to the following ones for any real number α:

$$\Delta^{\pm}_{\alpha} u = \frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial z^2} \pm \frac{\alpha}{r}\frac{\partial u}{\partial r} = 0$$

and established the generalized axially symmetric potential theory. It was not a formal generalization, but some interesting problems were solved by this theory. By the similar method to E. Beltrami, he found the fundamental solution $\mathcal{K}^{+}_{\alpha}$ for $\Delta^{+}_{\alpha} u = 0$:

$$\mathcal{K}^{+}_{\alpha}(p,q) = S_{\alpha-1}\int_0^{\pi} \frac{\sin^{\alpha-1}\Theta}{[r^2+\rho^2+(z-\rho)^2-2r\rho\cos\Theta]^{\alpha/2}}\, d\Theta \quad \text{in } \Pi \times \Pi,$$

where $1/S_{\alpha-1} = \int_0^{\pi} \sin^{\alpha-1}\Theta\, d\Theta$, and showed that $\mathcal{K}^{+}_{\alpha}(p,q)$ has the following development at pole $p = q$:

$$\mathcal{K}^{+}_{\alpha}(p,q) = u_{\alpha}(p,q)\log\frac{1}{\sqrt{(r-\rho)^2 + (z-\zeta)^2}} + v_{\alpha}(p,q),$$

where u_{α}, v_{α} are regular analytic at $p = q$ and $u_{\alpha}(q,q) = S_{\alpha-1}/\rho^{\alpha}$. By use of the remarkable identity: $r^{\alpha+1}\Delta^{+}_{\alpha+2} f(p) = \Delta^{-}_{\alpha}[r^{\alpha+1} f(p)]$ for any $f(p) \in$

$C^2(\Pi)$, he constructed the fundamental solution $\mathcal{K}_\alpha^-$ for $\Delta_\alpha^- u = 0$:

$$\mathcal{K}_\alpha^-(p,q) = \frac{S_{\alpha-1}}{S_{\alpha+1}}\, r^{\alpha+1}\mathcal{K}_{\alpha+2}^+(p,q) \quad \text{in } \Pi \times \Pi.$$

Now we consider the case $\alpha = 1$ and put

$$\begin{aligned}\mathcal{L}(p,q) :&= \frac{\rho}{2\pi}\frac{S_0}{S_2}\mathcal{K}_1^-(p,q)\\ &= \frac{\rho r^2}{2\pi}\int_0^\pi \frac{\sin^2\Theta}{\left[r^2+\rho^2+(z-\rho)^2-2r\rho\cos\Theta\right]^{3/2}}d\Theta \quad \text{in } \Pi\times\Pi.\end{aligned}$$

Then $\mathcal{L}(p,q)$ is a C^ω function in $\Pi \times \Pi$ except for the diagonal set and has the following properties:

Proposition (*A. Weinstein* [Wi])
(i) *For any fixed* $q \in \Pi$, $\Delta^-\mathcal{L}(p,q) = 0$ *for* $p \in \Pi \setminus \{q\}$.
(ii) $\mathcal{L}(p,q)$ *has the following development at pole* $p = q$*:*

$$\mathcal{L}(p,q) = u(p,q)\log\frac{1}{\sqrt{(r-\rho)^2+(z-\zeta)^2}} + v(p,q),$$

where u, v *are regular analytic at* $p = q$, *and* $u(q,q) = 1/(2\pi)$.
(iii) (*Boundary values*) *For any fixed* $q \in \Pi$, $\mathcal{L}(p,q) = \frac{\partial\mathcal{L}}{\partial n_p}(p,q) = 0$ *for* $p \in \partial\Pi$ *and* $\lim_{p\to\infty}\mathcal{L}(p,q) = 0$.
(iv) (*Symmetry*) $\frac{\mathcal{L}(p,q)}{r} = \frac{\mathcal{L}(q,p)}{\rho}$ *in* $\Pi \times \Pi$.
(v) *For any* $f \in C_0^\infty(\Pi)$, $\Delta^-\{\int_\Pi f(q)\mathcal{L}(p,q)\,d\rho d\zeta\} = -f(p)$ *for* $p \in \Pi$.

Let us show the electomagnetic meaning of $\mathcal{L}(p,q)$. We first note that the integration by parts implies the following expression of $\mathcal{L}$:

$$\mathcal{L}(p,q) = \frac{r}{4\pi}\int_0^{2\pi}\frac{\cos\Theta}{\sqrt{r^2+\rho^2+(z-\zeta)^2-2r\rho\cos\Theta}}\,d\Theta \quad \text{in } \Pi\times\Pi. \tag{A.1}$$

We fix $q \in \Pi$ and move $y = [q,\Theta]$ $(0 \le \Theta \le 2\pi)$ in $\mathbb{R}^3$. We then have, for any $x = [p,\theta] \in \mathbb{R}^3$ and any $f,\ g \in C(\Pi)$,

$$\frac{1}{4\pi}\int_0^{2\pi}\frac{\langle 0,1,0\rangle_y}{\|x-y\|}\,d\Theta = \frac{\mathcal{L}(p,q)}{r}\,\langle 0,1,0\rangle_x, \tag{A.2}$$

$$\frac{1}{4\pi}\int_0^{2\pi}\frac{\langle f(q),0,g(q)\rangle_y}{\|x-y\|}d\Theta = \left\langle f(q)\frac{\mathcal{L}(p,q)}{r}, 0,\ g(q)\,\mathcal{K}(p,q)\right\rangle_x. \tag{A.3}$$

Further, the integration by parts implies the identities:

$$\frac{\partial \mathcal{L}(p,q)}{\partial z} = -\frac{\partial \mathcal{L}(p,q)}{\partial \zeta}, \qquad \frac{\partial \mathcal{L}(p,q)}{\partial r} = -r\frac{\partial \mathcal{K}(p,q)}{\partial \rho} \quad \text{in } \Pi \times \Pi.$$

Given any piecewise smooth continuous function f in Π with compact support in Π (like the function V defined in (1°) of §8), we set

$$\mathcal{L}_f(p) = \frac{1}{4\pi}\int_\Pi f(q)\mathcal{L}(p,q)\,d\rho d\zeta \qquad \text{for } p \in \Pi.$$

By the above identities and the integration by parts under property (ii),

$$\begin{aligned} \frac{\partial \mathcal{L}_f(p)}{\partial z} &= \frac{1}{4\pi}\int_\Pi \frac{\partial f(q)}{\partial \zeta}\mathcal{L}(p,q)\,d\rho d\zeta \qquad \text{for } p \in \Pi, \\ \frac{\partial \mathcal{L}_f(p)}{\partial r} &= \frac{r}{4\pi}\int_\Pi \frac{\partial f(q)}{\partial \rho}\mathcal{K}(p,q)\,d\rho d\zeta \qquad \text{for } p \in \Pi. \end{aligned} \tag{A.4}$$

We next recall the magnetic field $B_\gamma(x)$ induced by the usual electric current $J_\gamma ds_y$ along the ring coil γ. Precisely, let $q = (\rho, \zeta)$ be any fixed point in Π and draw the z-axially symmetric circular ring $\gamma(q)$ passing through the point $(\rho, 0, \zeta)$ in $\mathbb{R}^3$. We consider the electric current $J_{\gamma(q)}ds_y$ of magnitude $1/\rho$ along the ring $\gamma(q)$ such that, for $y = [q, \Theta] \in \gamma(q)$,

$$J_{\gamma(q)}ds_y = \frac{1}{\rho}(-\sin\Theta, \cos\Theta, 0)ds_y = \langle 0,1,0\rangle_y\, d\Theta,$$

where ds_y is the arc length of $\gamma(q)$ at y. The current $J_{\gamma(q)}ds_x$ induces the vector potential $A_{\gamma(q)}(x)$ and the magnetic field $B_{\gamma(q)}(x)$ for $x \in \mathbb{R}^3 \setminus \gamma(q)$. From (A.2) and (8.2), we have

$$A_{\gamma(q)}(x) = \frac{1}{4\pi}\int_{\gamma(q)} \frac{J_{\gamma(q)}(y)}{\|x-y\|}ds_y = \frac{\mathcal{L}(p,q)}{r}\langle 0,1,0\rangle_x, \tag{A.5}$$

$$B_{\gamma(q)}(x) = \operatorname{rot} A_{\gamma(q)}(x) = \frac{1}{r}\left\langle -\frac{\partial \mathcal{L}(p,q)}{\partial z}, 0, \frac{\partial \mathcal{L}(p,q)}{\partial r}\right\rangle_x. \tag{A.6}$$

It follows that

$$\frac{\mathcal{L}(p,q)}{r} \textit{ means the magnitude of the vector potential } A_{\gamma(q)}(x).$$

Our consideration naturally leads us to property (i) for $\mathcal{L}(p,q)$ in the above proposition as follows: We construct a sequence of volume current densities $\{J_n dv_x\}_n$ such that $J_n dv_x \to J_{\gamma(q)}ds_y$ in the sense of distribution and such that, if we denote by $B_n(x)$ the magnetic field induced by $J_n dv_x$, then $B_n(x) \to B_{\gamma(q)}(x)$ uniformly on any compact set in $\mathbb{R}^3 \setminus \gamma(q)$. (For example, put $J_n(x)dv_x = \langle 0, f_n(R), 0\rangle\, dv_x$ by (8.4), where $R = \sqrt{(r-\rho)^2 + (z-\zeta)^2}$ for $x = [r, \theta, z]$, and $f_n(R)$ (≥ 0) is a C^∞ function on $[0, +\infty)$ with support $[0, 1/n)$ and with $\int_0^{1/n} f_n(R) R\, dR = 1/\rho$.) Since $\operatorname{rot} B_n(x) = 0$ outside $\operatorname{Supp} J_n$, it follows from (A.6) and (8.2) that $\langle 0,0,0\rangle = \operatorname{rot} B_{\gamma(q)}(x) = \frac{-1}{r}\langle 0, \Delta^- \mathcal{L}(p,q), 0\rangle$ in $\mathbb{R}^3 \setminus \gamma(q)$. Hence, $\mathcal{L}(p,q)$ satisfies (i).

Let us give our proofs of the expressions for $\mathbf{A}_i$ and Λ_i ($i = 1, 2$) by use of those for $\mathbf{B}_i$ and $\mathbf{J}_i$ in Expression 1° and 2° of §8: By property (v) for $\mathcal{L}$, the functions W and U of (1°) and (2°) of §8, are written into the following forms:

$$W(p) = \mathcal{L}_V(p) = \int_\Pi V(q)\mathcal{L}(p,q)\, d\rho d\zeta \qquad \text{for } p \in \Pi, \tag{A.7}$$

$$U(p) = \tfrac{1}{2\pi}\int_K \mathcal{L}(p,q)\, d\rho d\zeta \qquad \text{for } p \in \Pi. \tag{A.8}$$

Since $dv_y = \rho\, d\rho d\zeta d\Theta$ at $y \in \mathbb{R}^3$, we have

$$\mathbf{B}_1(y)dv_y = \begin{cases} \left\langle -\frac{\partial V}{\partial \zeta}, 0, \frac{\partial V}{\partial \rho}\right\rangle d\rho d\zeta d\Theta & \text{for } y \in D \\ 0 & \text{for } y \in D'. \end{cases}$$

We thus have from definition (7.5)

$$\begin{aligned} \Lambda_1(x) &= \int_\Pi \left\{ \frac{1}{4\pi}\int_0^{2\pi} \frac{\left\langle -\frac{\partial V}{\partial \zeta}, 0, \frac{\partial V}{\partial \rho}\right\rangle_y}{\|x-y\|}\, d\Theta \right\} d\rho d\zeta \\ &= \int_\Pi \left\langle -\frac{1}{r}\frac{\partial V}{\partial \zeta}\mathcal{L}(p,q), 0, \frac{\partial V}{\partial \rho}\mathcal{K}(p,q)\right\rangle_x d\rho d\zeta \quad \text{by (A.3)} \\ &= \frac{1}{r}\left\langle -\frac{\partial W}{\partial z}, 0, \frac{\partial W}{\partial r}\right\rangle_x \qquad \text{by (A.4) and (A.7).} \end{aligned}$$

Hence, the expression for $\Lambda_1(x)$ is proved. By applying Lemma 7.2 to $\mathbf{J}_1(x)$, we have from (7.3) and (8.2)

$$\begin{aligned} \mathbf{A}_1(x) &= \operatorname{rot} \Lambda_1(x) = \operatorname{rot}\left\{ \frac{1}{r}\left\langle -\frac{\partial W}{\partial z}, 0, \frac{\partial W}{\partial r}\right\rangle\right\} \\ &= -\frac{\Delta^- W(p)}{r}\langle 0,1,0\rangle = \frac{V(p)}{r}\langle 0,1,0\rangle \qquad \text{by (8.8).} \end{aligned}$$

Thus, the expression for $\mathbf{A}_1(x)$ is proved.

Since $\mathbf{B}_2(x)$ in Expression 2° is written into the following form:

$$\mathbf{B}_2(y)\,dv_y = \begin{cases} \dfrac{1}{2\pi}(J_{\gamma(q)}\,ds_y)\,d\rho d\zeta & \text{for } y \in D \\ 0 & \text{for } y \in D', \end{cases}$$

it follows from definition (7.5) that

$$\begin{aligned} \Lambda_2(x) &= \frac{1}{2\pi}\int_K \left\{ \frac{1}{4\pi}\int_{\gamma(q)} \frac{J_{\gamma(q)}(y)}{\|x-y\|}\,ds_y \right\} d\rho d\zeta \\ &= \frac{1}{r}\langle 0,1,0\rangle_x \cdot \left(\frac{1}{2\pi}\int_K \mathcal{L}(p,q)\,d\rho d\zeta \right) \qquad \text{by (A.5)} \\ &= \frac{U(p)}{r}\langle 0,1,0\rangle_x \qquad \text{by (A.8).} \end{aligned}$$

Thus the formula for $\Lambda_2(x)$ is proved. For the expression of $\mathbf{A}_2(x)$, we apply Lemma 7.2 to $\mathbf{J}_2(x)$ and obtain

$$\mathbf{A}_2(x) = \text{rot } \Lambda_2(x) = \text{rot } \left\{ \frac{U(p)}{r}\langle 0,1,0\rangle \right\} = \frac{1}{r}\left\langle -\frac{\partial U}{\partial z}, 0, \frac{\partial U}{\partial r} \right\rangle.$$

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