Hokkaido Mathematical Journal Vol. 26 (1997) p. 699–711

Hyperbolic Besov functions and Bloch-to-Besov composition operators

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(Received October 21, 1996)

Abstract. Compactness of composition operators from the Bloch space $\mathcal{B}$ into the analytic Besov spaces B_p is characterized by the behavior of the hyperbolic derivative of self-maps of the unit disk D.

Key words: Besov space, Bloch space, composition operator, hyperbolic derivative.

1. Introduction

Let $H(D)$ be the space of analytic functions on the unit disk $D = \{z : |z| < 1\}$. The function $f \in H(D)$ is called a Bloch function if

$$\|f\|_{\mathcal{B}} = \sup_{z \in D}(1-|z|^2)|f'(z)| < \infty.$$

This defines a semi-norm. The Bloch functions form a Banach space $\mathcal{B}$ with the norm $\|f\| = |f(0)| + \|f\|_{\mathcal{B}}$.

The function f is called a little Bloch function, i.e. $f \in \mathcal{B}_0$, if

$$\lim_{|z|\to 1}(1-|z|^2)|f'(z)| = 0.$$

The analytic Besov functions are defined as follows

$$B_p = \left\{ f \in H(D) : \right.$$
$$\left. \|f\|_{B_p} = \left(\iint_D ((1-|z|^2)|f'(z)|)^p \, d\lambda(z) \right)^{\frac{1}{p}} < \infty \right\},$$

where $d\lambda(z) = \frac{dA(z)}{(1-|z|^2)^2}$ is the hyperbolic area measure on D and $dA(z) = \frac{1}{\pi}\, dx\, dy$. The analytic Besov functions form a Banach space B_p, $1 < p < \infty$, with the norm $\|f\| = |f(0)| + \|f\|_{B_p}$. (See e.g. [Z1].)

Let B be the set of holomorphic self-maps $\varphi : D \to D$ and $\varphi^*(z) =$

1991 Mathematics Subject Classification : 30D50, 47B38.

$\frac{|\varphi'(z)|}{1-|\varphi(z)|^2}$ is the hyperbolic derivative of φ. By the Schwarz-Pick lemma $\sup_{z\in D}(1-|z|^2)\varphi^*(z) \le 1$.

We say that $\varphi \in B_0$ if $\lim_{|z|\to 1}(1-|z|^2)\varphi^*(z) = 0$.

Every $\varphi \in B$ induces a linear composition operator C_φ from $H(D)$ into itself as follows: $C_\varphi f = f\circ\varphi$, whenever $f \in H(D)$. Let X be a Banach space of holomorphic functions in D. By the definition, composition operator $C_\varphi : \mathcal{B} \to X$ is *compact* on $\mathcal{B}$ if it takes the unit ball $b(\mathcal{B})$ of $\mathcal{B}$ into a set whose closure is compact.

Composition operators on the Bloch space $\mathcal{B}$ were studied from the general positions by K. Madigan and A. Matheson [MMa]. They proved that a holomorphic mapping φ of the unit disk D into itself induces a compact composition operator $C_\varphi : \mathcal{B} \to \mathcal{B}$ if and only if for every $\varepsilon > 0$ there exists r, $0 < r < 1$, such that $(1-|z|^2)\varphi^*(z) < \varepsilon$ whenever $|\varphi(z)| > r$. They also proved that C_φ is compact on the little Bloch space $\mathcal{B}_0$ if and only if φ belongs to the class B_0.

J. Arazy, S.D. Fisher and J. Peetre [AFP] considered composition operators which map B_p into B_p for $p \ge 2$. They proved that holomorphic mapping φ of D into D induces a bounded composition operator $C_\varphi : B_p \to B_p$ if there is an integer number N such that the cordinality of $\varphi^{-1}(w)$ is N or less for all $w \in D$.

In section 2 we consider the composition operators from the Bloch space $\mathcal{B}$ to the spaces of analytic Besov functions B_p, $1 < p < \infty$. We prove that C_φ maps the Bloch space $\mathcal{B}$ in the Besov space B_p if and only if φ belongs to the hyperbolic Besov class B_p^h and that every such C_φ is always a compact composition operator. The class B_p^h is defined as follows:

Definition The hyperbolic analytic Besov class B_p^h, $1 < p < \infty$, contains such functions $\varphi \in B$ that

$$\|\varphi\|_{B_p^h} = \left(\iint_D (1-|z|^2)^p \varphi^*(z)^p \, d\lambda(z)\right)^{\frac{1}{p}} < \infty.$$

For $p = \infty$ we set $B_\infty^h = B$.

In section 3 we give a necessary and sufficient condition for hyperbolic Besov functions and prove that these classes satisfy the nesting property. Examples of hyperbolic Besov functions are given in the end of this section.

In section 4 we consider the meromorphic (spherical) Besov classes $B_p^\#$.

These classes were defined by R. Aulaskari and G. Csordas [AuCs]. We prove that $\varphi \in B_p^h$ also induce composition operators from the class of normal functions into the meromorphic Besov class $B_p^{\#}$.

Now we consider examples of functions which are in B_p^h or are not in B_p^h.

1. Let $S_\alpha = \{z = x + iy : |x|^\alpha + |y|^\alpha < 1\}$, $0 < \alpha \leq 1$, and φ_α be a conformal mapping of D into S_α, then $\varphi_1 \notin B_2^h$. If $\alpha < 1$ then $\varphi_\alpha(z) \in B_2^h$.

2. Let φ be bounded holomorphic function in D with $\|\varphi\|_\infty \leq k < 1$. If φ is continuous in $\overline{D}$ and $\varphi(e^{i\theta}) \in \Lambda_\alpha$, $0 < \alpha \leq 1$, then by the Hardy-Littlewood Theorem ([D], Theorem 5.1)

$$(1 - |z|^2)|\varphi'(z)| = O((1 - |z|^2)^\alpha) \quad \text{as} \quad |z| \to 1,$$

and also

$$(1 - |z|^2)\varphi^*(z) = O((1 - |z|^2)^\alpha) \quad \text{as} \quad |z| \to 1.$$

Thus $\varphi \in B_p^h$ for $p > \frac{1}{\alpha}$. See also [Y].

2. Composition operators on the Bloch space

Our main result is the following:

Theorem 1 *Let $1 < p < \infty$. The following conditions are equivalent*

(1) $\varphi \in B_p^h$;

(2) C_φ *takes the Bloch space $\mathcal{B}$ into the analytic Besov space B_p;*

(3) $C_\varphi : \mathcal{B} \to B_p$ *is compact.*

Proof. (1) $\Rightarrow$ (2). Let φ be arbitrary function of B_p^h, $1 < p < \infty$, and f be arbitrary Bloch function. Estimate the p-Besov norm of $f \circ \varphi$.

$$\begin{aligned}
\|f \circ \varphi\|_{B_p}^p &= \iint\limits_D (1 - |z|^2)^p |f' \circ \varphi(z)|^p |\varphi'(z)|^p \, d\lambda(z) \\
&= \iint\limits_D (1 - |z|^2)^p (\varphi^*(z))^p (1 - |\varphi(z)|^2)^p |f' \circ \varphi(z)|^p \, d\lambda(z) \\
&\leq \|f\|_{\mathcal{B}}^p \, \|\varphi\|_{B_p^h}^p < \infty.
\end{aligned}$$

To prove (2) $\Rightarrow$ (1) we use a trick in [CRU]. Pick such Bloch functions

f and g that

$$|f'(z)| + |g'(z)| \geq \frac{1}{1-|z|^2}$$

(existence of such functions was proved in [RU]). Then for every $p > 1$

$$|f'(z)|^p + |g'(z)|^p \geq \frac{2^{1-p}}{(1-|z|^2)^p}$$

and hence

$$2^{1-p}\|\varphi\|^p_{B^h_p} \leq \|f \circ \varphi\|^p_{B_p} + \|g \circ \varphi\|^p_{B_p} < \infty.$$

(1) $\Rightarrow$ (3). Let $b(\mathcal{B})$ be the unit ball in $\mathcal{B}$ and $\{f_n\} \subset b(\mathcal{B})$. Sequence $\{f_n\}$ is a normal family in D and therefore there is such subsequence $\{f_{n_k}\}$ that it converges uniformly on every compact subset of D to $f \in b(\mathcal{B})$. Then the sequence $\{g_k\}$, $g_k(z) = f_{n_k}(z) - f(z)$, converges uniformly to 0 on every compact subset of D. Thus for compactness of operator $C_\varphi : \mathcal{B} \to B_p$ it is enough to prove that if $\{g_k\} \in b(\mathcal{B})$ and $\{g_k\}$ converges to 0 uniformly on every compact subset of D then $\lim_{k\to\infty} \|g_k \circ \varphi\|_{B_p} = 0$.

Let $\{g_k\} \in b(\mathcal{B})$ and $\{g_k\}$ converges to 0 uniformly on every compact subset of D. Since $\varphi \in B^h_p$, for every $\varepsilon > 0$ there exists such a compact $K \subset D$ that

$$\iint\limits_{D\setminus K} (1-|z|^2)^p(\varphi^*(z))^p \, d\lambda(z) < \varepsilon$$

and there exists a number N such that

$$\sup_{w\in\varphi(K)} (1-|w|^2)|g'_k(w)| < \varepsilon^{\frac{1}{p}}$$

for any $k \geq N$. Then

$$\begin{aligned}
&\|g_k \circ \varphi\|^p_{B_p} \\
&\quad = \iint\limits_D (1-|z|^2)^p|(g_k \circ \varphi)'(z)|^p \, d\lambda(z) \\
&\quad = \iint\limits_K (1-|z|^2)^p(\varphi^*(z))^p(1-|\varphi(z)|^2)^p|g'_k \circ \varphi(z)|^p \, d\lambda(z) \\
&\qquad\quad + \iint\limits_{D\setminus K} (1-|z|^2)^p(\varphi^*(z))^p(1-|\varphi(z)|^2)^p|g'_k \circ \varphi(z)|^p \, d\lambda(z)
\end{aligned}$$

$$\leq \varepsilon \iint\limits_{K} (1-|z|^2)^p (\varphi^*(z))^p \, d\lambda(z)$$
$$+ 1 \cdot \iint\limits_{D\setminus K} (1-|z|^2)^p (\varphi^*(z))^p \, d\lambda(z)$$
$$\leq \varepsilon \|\varphi\|^p_{B^h_p} + \varepsilon = \varepsilon \cdot const.$$

The implication (3) $\Rightarrow$ (2) is obvious. □

3. Hyperbolic Besov functions

In this section we obtain some properties of hyperbolic Besov functions. Let $T_a(z) = \frac{a-z}{1-\bar{a}z}$, $a \in D$, and $\varphi_a(z) = \varphi(T_a(z))$.

For every $\varphi \in B^h_p$ and every $a \in D$ functions $T_a \circ \varphi(z)$ and $\varphi \circ T_a(z)$ belong to B^h_p.

Denote by $\rho(a,b)$ the pseudohyperbolic distance on D

$$\rho(a,b) = \left| \frac{a-b}{1-a\bar{b}} \right|$$

and by $\sigma(a,b)$ the hyperbolic distance on D

$$\sigma(a,b) = \frac{1}{2} \ln \frac{|1-a\bar{b}| + |a-b|}{|1-a\bar{b}| - |a-b|}, \quad a, b \in D.$$

Denote by $K(z,w)$ the Bergman kernal of D

$$K(z,w) = \frac{1}{(1-z\bar{w})^2}.$$

Then the Jacobian of T_w transformation at z is

$$J_{T_w}(z) = \frac{(1-|w|^2)^2}{|1-z\bar{w}|^4} = \frac{|K(z,w)|^2}{K(w,w)}. \tag{3.1}$$

The following theorem is an analogy to the corresponding result for the Besov space B_p [Z2]. For the proof of necessity we need a different idea to estimate the hyperbolic distance by the hyperbolic Besov norm because of their non-linearity.

Theorem 2 *Let φ be holomorphic self-map of the unit disk D. Then*

$\varphi \in B_p^h$, $1 < p < \infty$, if and only if

$$\iint_D \iint_D \frac{\sigma(\varphi(z), \varphi(w))^p}{|1 - z\bar{w}|^4} \, dA(z) \, dA(w) < \infty.$$

Proof. Necessity. At first we estimate the hyperbolic distance between $\varphi(z)$ and $\varphi(0)$.

$$\begin{aligned}
&\sigma(\varphi(z), \varphi(0)) \\
&\quad \le \int_0^1 \varphi^*(tz)|z| \, dt = \int_0^1 \frac{|z|}{(1 - t|z|)^{\frac{1}{2}}} (1 - t|z|)\varphi^*(tz) \frac{dt}{(1 - t|z|)^{\frac{1}{2}}} \\
&\quad \le \left(\int_0^1 \frac{|z|^q}{(1 - t|z|)^{\frac{1+q}{2}}} \, dt \right)^{\frac{1}{q}} \left(\int_0^1 \frac{(1 - t|z|)^p \varphi^*(tz)^p}{(1 - t|z|)^{\frac{1}{2}}} \, dt \right)^{\frac{1}{p}} \\
&\quad = \left(\frac{2|z|^{q-1}}{q - 1} \left(\frac{1}{(1 - |z|)^{\frac{q-1}{2}}} - 1 \right) \right)^{\frac{1}{q}} \left(\int_0^1 \frac{(1 - t|z|)^p \varphi^*(tz)^p}{(1 - t|z|)^{\frac{1}{2}}} \, dt \right)^{\frac{1}{p}} \\
&\quad \le C \left(\frac{|z|}{\sqrt{1 - |z|}} \int_0^1 \frac{(1 - t|z|)^p \varphi^*(tz)^p}{\sqrt{1 - t|z|}} \, dt \right)^{\frac{1}{p}},
\end{aligned}$$

where $C = (\frac{2}{q-1})^{1/q} = (2p - 2)^{p/(p-1)}$. Thus, setting $C_1 = (2p - 2)^{p^2/(p-1)}$ we have

$$\sigma(\varphi(z), \varphi(0))^p \le \frac{C_1|z|}{\sqrt{1 - |z|}} \int_0^1 \frac{(1 - t|z|)^p \varphi^*(tz)^p}{\sqrt{1 - t|z|}} \, dt, \quad p > 1.$$

Then

$$\begin{aligned}
&\iint_D \sigma(\varphi(z), \varphi(0))^p \, dA(z) \\
&\quad \le C_1 \iint_D dA(z) \int_0^1 \frac{|z|(1 - t|z|)^p \varphi^*(tz)^p}{\sqrt{1 - |z|}\sqrt{1 - t|z|}} \, dt \\
&\quad = C_1 \int_0^1 \iint_{tD} \frac{|z|(1 - |z|)^p \varphi^*(z)^p}{\sqrt{1 - |z|}} \cdot \frac{dA(z) \, \frac{dt}{t^3}}{\sqrt{1 - \frac{|z|}{t}}} \\
&\quad = C_1 \iint_D \frac{|z|(1 - |z|)^p \varphi^*(z)^p}{\sqrt{1 - |z|}} \, dA(z) \int_{|z|}^1 \frac{dt}{t^2\sqrt{t^2 - |z|t}}.
\end{aligned}$$

Since

$$\int_{|z|}^{1} \frac{dt}{t^2\sqrt{t^2-|z|t}} = \frac{2}{3}\left(\frac{1}{|z|t^2}+\frac{2}{|z|^2 t}\right)\sqrt{t^2-t|z|}\,\Bigg|_{|z|}^{1} \leq \frac{2\sqrt{1-|z|}}{|z|^2}$$

we conclude that

$$\iint\limits_D \sigma(\varphi(z),\varphi(0))^p\,dA(z) \leq 2C_1 \iint\limits_D \frac{(1-|z|^2)^p\varphi^*(z)^p}{|z|}\,dA(z). \quad (3.2)$$

Now we apply (3.2) to function $\varphi\circ T_w(z)$. Then

$$\begin{aligned}
&\iint\limits_D\iint\limits_D \frac{\sigma(\varphi(w),\varphi(z))^p}{|1-w\bar{z}|^4}\,dA(w)\,dA(z)\\
&\quad= \iint\limits_D\iint\limits_D \sigma(\varphi\circ T_w(z),\varphi(w))^p\,dA(z)\,d\lambda(w)\\
&\quad\leq 2C_1 \iint\limits_D d\lambda(w)\iint\limits_D \frac{(1-|z|^2)^p((\varphi\circ T_w)^*(z))^p}{|z|}\,dA(z)\\
&\quad= 2C_1 \iint\limits_D d\lambda(w)\iint\limits_D \frac{(1-|T_w(z)|^2)^p\varphi^*(T_w(z))^p}{|z|}\,dA(z)
\end{aligned}$$

by (3.1) and noting $T_w^{-1}=T_z$

$$\begin{aligned}
&\quad= 2C_1 \iint\limits_D d\lambda(w)\iint\limits_D \frac{(1-|z|^2)^p\varphi^*(z)^p}{|T_z(w)|}\frac{|K(w,z)|^2}{K(w,w)}\,dA(z)\\
&\quad= 2C_1 \iint\limits_D (1-|z|^2)^p\varphi^*(z)^p\,dA(z)\iint\limits_D \frac{|K(w,z)|^2}{|T_w(z)|}\,dA(w).
\end{aligned}$$

Note that

$$|K(z,T_z(w))| = \frac{K(z,z)}{|K(z,w)|}$$

for any $z,w\in D$. Since $|K(z,w)|=|K(w,z)|$ and $|T_z'(w)|^2=\frac{|K(w,z)|}{K(z,z)}$, we have

$$\iint_D \frac{|K(w,z)|^2}{|T_z(w)|}\,dA(w) = \iint_D \frac{|K(z,w)|^2}{|T_z(w)|}\,dA(w)$$

$$\text{(change of variable } w := T_z(w))$$

$$= \iint_D \frac{|K(z,T_z(w))|^2}{|w|}\frac{|K(z,w)|^2}{K(z,z)}\,dA(w)$$

$$= K(z,z)\iint_D \frac{dA(w)}{|w|}$$

$$= \frac{2}{(1-|z|^2)^2}.$$

Thus we obtain

$$\iint_D \iint_D \frac{\sigma(\varphi(z),\varphi(w))^p}{|1-z\bar{w}|^4}\,dA(z)\,dA(w)$$

$$\leq 4C_1 \iint_D (1-|z|^2)^p \varphi^*(z)^p d\lambda(z).$$

Sufficiency. Let $g(z)$ be holomorphic in D and $g(0)=0$, then

$$|g'(0)| = 2\left|\iint_D \bar{z} g(z)\,dA(z)\right|. \tag{3.3}$$

We apply (3.3) to function $T_{\varphi(w)} \circ \varphi \circ T_w(z) = g(z)$. Then for any $z, w \in D$

$$|g(z)| = \rho(\varphi \circ T_w(z), \varphi(w))$$

and we have

$$(1-|w|^2)^p \varphi^*(w)^p \leq 2^p \iint_D \rho(\varphi \circ T_w(z), \varphi(w))^p\,dA(z).$$

Thus

$$\iint_D (1-|w|^2)^p \varphi^*(w)^p\,d\lambda(w)$$

$$\leq 2^p \iint_D d\lambda(w) \iint_D \rho(\varphi \circ T_w(z), \varphi(w))^p\,dA(z)$$

$$= 2^p \iint\limits_D \iint\limits_D \frac{\rho(\varphi(z),\varphi(w))^p}{|1 - z\bar{w}|^4}\, dA(z)\, dA(w)$$

$$< 2^p \iint\limits_D \iint\limits_D \frac{\sigma(\varphi(z),\varphi(w))^p}{|1 - z\bar{w}|^4}\, dA(z)\, dA(w) < \infty.$$

□

Denote by $D(a,r)$ the pseudohyperbolic disk

$$D(a,r) = \{z \in D : \rho(a,z) \leq r\}$$

with center at $a \in D$ and radius r, $0 < r < 1$.

Lemma *Let $0 < p < \infty$, $0 < r < 1$ and $\varphi \in B$. Then $\varphi \in B_0$ if and only if*

$$\lim_{|a|\to 1} \iint\limits_{D(a,r)} (1 - |z|^2)^p \varphi^*(z)^p\, d\lambda(z) = 0.$$

Proof. The necessity is obvious.

Conversely, suppose that $\varphi \notin B_0$. Then there exists such a sequence $\{a_n\}$, $\lim\limits_{n\to\infty} |a_n| = 1$, that

$$\lim_{n\to\infty} (1 - |a_n|^2)\varphi^*(a_n) > 0.$$

Let $g_n(z) = T_{\varphi(a_n)} \circ \varphi \circ T_{a_n}(z)$. Then $\{g_n(z)\}$ is a normal family in D. Choosing a subsequence if necessary we may suppose that $\lim_{n\to\infty} g_n(z) = g(z)$.

Since

$$|g'(0)| = \lim_{n\to\infty} |g_n'(0)| = \lim_{n\to\infty} (1 - |a_n|^2)\varphi^*(a_n) > 0$$

$g(z)$ is a non-constant function. Noting

$$\iint\limits_{D(a_n,r)} (1 - |z|^2)^p \varphi^*(z)^p\, d\lambda(z) = \iint\limits_{D(0,r)} (1 - |z|^2)^{p-2} g_n^*(z)^p\, dA(z)$$

$$\geq (1 - r^2)^{p-2} \iint\limits_{D(0,r)} g_n^*(z)^p\, dA(z).$$

The last integral converges to a positive number and it contradicts our hypothesis. □

Theorem 3 *$B_p^h \subset B_q^h \subset B_0$ for any $1 < p < q < \infty$.*

Proof. If $\varphi \in B_p^h$ then for every $\varepsilon > 0$ there exists R, $0 < R < 1$, such that

$$\iint\limits_{R \le |z| < 1} (1-|z|^2)^p \varphi^*(z)^p \, d\lambda(z) < \varepsilon.$$

Hence for every $r > 0$

$$\lim_{|a|\to 1} \iint\limits_{D(a,r)} (1-|z|^2)^p \varphi^*(z)^p \, d\lambda(z) = 0.$$

and by the Lemma $\varphi \in B_0$.

The classes B_p^h satisfy the nesting property $B_p^h \subset B_q^h$ for $1 < p < q$. This follows from the Schwarz-Pick lemma and the inequality

$$\begin{aligned}
&\iint\limits_{D} (1-|z|^2)^{q-2} (\varphi^*(z))^q \, dA(z) \\
&\quad = \iint\limits_{D} ((1-|z|^2)\varphi^*(z))^{q-p} (1-|z|^2)^{p-2} (\varphi^*(z))^p \, dA(z) \\
&\quad \le \iint\limits_{D} (1-|z|^2)^{p-2} (\varphi^*(z))^p \, dA(z).
\end{aligned}$$

□

4. Normal functions

Let f be a meromorphic function in the unit disk D and $f^{\#}(z)$ be the spherical derivative

$$f^{\#}(z) = \frac{|f'(z)|}{1+|f(z)|^2}.$$

R. Aulaskari and G. Csordas [AuCs] defined the meromorphic (spherical) Besov classes $B_p^{\#}$, $1 < p < \infty$. By the definition

$$B_p^{\#} = \Big\{ f - \text{ meromorphic in } D : \iint\limits_{D} (1-|z|^2)^p f^{\#}(z)^p \, d\lambda(z) < \infty \Big\}.$$

We can assume that $B^{\#}_{\infty} = \mathcal{N}$, where

$$\mathcal{N} = \Big\{ f - \text{ meromorphic in } D : \sup_{z\in D}(1-|z|^2)f^{\#}(z) < \infty \Big\}.$$

The class $\mathcal{N}$ is the family of normal functions in the unit disk D. It was defined by K.Noshiro [N]. R. Aulaskari and G. Csordas proved [AuCs] that $B^{\#}_p \subset \mathcal{N}_0$, $2 \le p < \infty$, where $\mathcal{N}_0$ is the class of little normal functions and contains meromorphic functions which satisfy the condition

$$\lim_{|z|\to 1}(1-|z|^2)f^{\#}(z) = 0.$$

Theorem 4 *If $\varphi \in B^h_p$, $1 < p < \infty$, then the composition $f \circ \varphi$ belongs to the class $B^{\#}_p$ for every $f \in \mathcal{N}$.*

Proof. Let f be normal function and $\sup_{z\in D}(1-|z|^2)f^{\#}(z) = K_f$. Let $\|\varphi\|_{B^h_p} = M_\varphi$. Then

$$\begin{aligned}
&\iint_D (1-|z|^2)^p (f\circ\varphi)^{\#}(z)^p \, d\lambda(z) \\
&\quad = \iint_D (1-|z|^2)^p (f^{\#}\circ\varphi)(z)^p |\varphi'(z)|^p \, d\lambda(z) \\
&\quad = \iint_D (1-|z|^2)^p (\varphi^*(z))^p (1-|\varphi(z)|^2)^p (f^{\#}\circ\varphi)(z)^p \, d\lambda(z) \\
&\quad \le K_f^p \cdot M_\varphi^p < \infty.
\end{aligned}$$

□

Let $\chi(A,B)$ be the chordal distance between points A and B of the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

F.Colonna [Col] proved

Theorem A *Let f be meromorphic function in D. Then f is a normal function in D if and only if there exists such a constant C that*

$$\chi(f(z), f(w)) \le C\sigma(z,w)$$

whenever $z, w \in D$.

Basing on Theorem 2, Theorem 4 and Theorem A we have

Corollary *Let $g \in B_p^{\#}$, $1 < p < \infty$, and $g = f \circ \varphi$, where f is a normal function, and $\varphi \in B_p^h$. Then*

$$\iint_D \iint_D \frac{(\chi(g(z), g(w)))^p}{|1 - z\bar{w}|^4}\, dA(z)\, dA(w) < \infty.$$

Addendum Professor T. Gamelin informed the author that Maria Tjani [T] independently obtained similar results to the Theorem 1.

Professor R. Aulaskari informed the author that his student Ruhan Zhao [Zh] also obtained similar results to the Theorem 1. All these proofs are different.

Acknowledgments The author would like to thank Professor M. Hayashi for helpful suggestions and valuable discussions during the preparation of the paper.

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