Hokkaido Mathematical Journal Vol. 26 (1997) p. 631–667

Asymptotic characterization of stationary interfacial patterns for reaction diffusion systems

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(Received July 19, 1996)

Abstract. We discuss asymptotic characterization of stationary interfacial patterns for reaction diffusion systems in higher dimensional spaces when the thickness of interface (denoted by ε) tends to zero. The fact due to [14] that any stationary interfacial pattern, which has a smooth limiting configuration up to $\varepsilon=0$, must become unstable for small ε implies that stable solutions must become finer and finer as $\varepsilon\downarrow 0$. This leads to the necessity of rescaling in order to capture the limiting configuration of the stable stationary interfacial patterns. An appropriate scaling turns out to be order $\varepsilon^{1/3}$. The rescaled reduced equation, which determines the asymptotic profile of the interface, as well as the spectral behavior of the associated linearized eigenvalue problems are studied by using the matched asymptotic expansion method.

Key words: interfacial pattern, reaction diffusion system, matched asymptotic expansion.

1. Introduction

One of the pioneering works in pattern formation problem can be traced back to Turing [17] who found that spatially inhomogeneous patterns can be formed by diffusion-driven instability if the inhibitor diffuses faster than the activator. A typical model system is of the form

$$\begin{cases} u_t=\varepsilon^2\Delta u+f(u,v), & \\ & (x,t)\in\Omega\times(0,\infty),\\ \delta v_t=D\Delta v+g(u,v), & \\ \dfrac{\partial u}{\partial n}=0=\dfrac{\partial v}{\partial n}, & (x,t)\in\partial\Omega\times(0,\infty). \end{cases} \tag{1.1}$$

where u is the activator, v is the inhibitor, $\delta>0$, and ε is a small positive parameter, Ω is a bounded domain in $\mathbf{R}^N$ $(N\geq 2)$, and its boundary $\partial\Omega$ is a connected C^∞ hypersurface. The precise assumptions for (f,g) are displayed at the end of this section. It is widely accepted that (1.1) capture the essence of pattern formation driven by Turing's instability observed typically in chemical reactions (see [1]). Although (1.1) exhibits a variety of

1991 Mathematics Subject Classification : 35B25, 35C20, 35K57.

patterns depending on diffusion and\or reaction rates, we focus our study on the stationary patterns in higher space dimensions. The basic issue is that "*Does* (1.1) *have stable stationary solutions for small* ε *? And, if it does, what are the asymptotic configurations of them as* $\varepsilon \downarrow 0$ *?*" As we shall see, this is closely related to knowing the location of free boundaries called an *interface* separating two different states. Numerically a variety of stationary patterns have been observed such as hexagons, stripes, and snaky patterns for (1.1) (see for instance [1]). Hence one can naively expect that (1.1) has a lot of stationary solutions for small ε and there seems to be no essential difficulty in the construction of stationary patterns in a way parallel to one dimensional cases, once we know the asymptotic forms as $\varepsilon \downarrow 0$. In this paper and the accompanied one [14], we shall show that this view point is too optimistic and the situation in higher dimensional spaces is completely different from the one dimensional case.

For one dimensional cases, many important works have been done. Especially, singular perturbation methods are quite useful for the construction of stationary solutions, e.g., Fife [2], Mimura, Tabata, and Hosono [8], Ito [7]. Concerning the stability of such solutions, the SLEP method is a powerful tool originally developed by Nishiura and Fujii [12] and Nishiura [10]. Fujii, Mimura, and Nishiura [4] and Nishiura [9] study the global bifurcation structure of the stationary solution of (1.1), and Hale and Sakamoto [5] and Sakamoto [15] give an bifurcation theoretical approach to construction and stability analysis of the stationary solutions.

On the other hand, we know very little about the existence and stability of stationary solutions of (1.1) in higher space dimensions. The main difficulty lies in the fact that we have to control the objects which have infinite degrees of freedom, such as curves and surfaces, in order to match quantities at the interface.

Here we restate the first part of our basic question:

Does (1.1) *have an* ε*-family of stable stationary layered solutions with smooth interface* Γ^{ε} *up to* $\varepsilon = 0$ *?*

Unfortunately the answer turns out to be negative.

Theorem A (Nishiura and Suzuki [13], [14]). *Suppose that* (1.1) *has an* ε*-family of stationary matched asymptotic solutions whose interface is smooth up to* $\varepsilon = 0$*. Then, it must be unstable for small* ε*.*

Theorem A suggests that the configuration of a stable interface must become *fine* and/or *complicated* as $\varepsilon \downarrow 0$. In order to answer the second part of our question, namely, "*What are the asymptotic configuration of stable solutions?*", it seems necessary to apply an appropriate rescaling to blow up the degenerate situation and consider again the whole issue for the resulting new system, since there are no well-defined asymptotic limit of interfaces in the original coordinate. In fact, under nice rescaling (see §2), we have the *rescaled problem* for (1.1), that is,

$$\begin{cases} \tilde{u}_t = \tilde{\varepsilon}^2 \Delta \tilde{u} + f(\tilde{u}, \tilde{v}), & \\ & (x,t) \in \Omega^* \times (0,\infty), \\ \delta\tilde{\varepsilon}\tilde{v}_t = D\Delta\tilde{v} + \tilde{\varepsilon} g(\tilde{u}, \tilde{v}), & \\ \dfrac{\partial \tilde{u}}{\partial n} = 0 = \dfrac{\partial \tilde{v}}{\partial n}, & (x,t) \in \partial\Omega^* \times (0,\infty), \end{cases} \tag{1.2}$$

where Ω^* is a rescaled domain and $\tilde{\varepsilon} \equiv \varepsilon^{2/3}$. Our main result is twofold: firstly we present a new reduced problem derived from (1.2); secondly we show the singular limit linearized eigenvalue problem on the interface at a stationary pattern of (1.2) when $\varepsilon \downarrow 0$. These two results give the basic mathematical ingredient to the second problem.

Theorem B (i) *The rescaled reduced equation which determines the limiting profiles of the interface Γ^0 is given by*

$$\begin{aligned} &D\Delta v_1^{\pm} = -g(h_{\pm}(v^*), v^*) \qquad \text{in } \Omega^{*\pm}, \\ &v_1^{\pm} = \beta^*(N-1)H_0, \quad \frac{\partial v_1^+}{\partial \nu} = \frac{\partial v_1^-}{\partial \nu} \quad \text{on } \Gamma^0, \\ &\frac{\partial v_1^-}{\partial n} = 0 \quad \text{on } \partial\Omega^*. \end{aligned}$$

(ii) *The principal part of critical eigenvalues λ^ε and eigenfunctions $(w^{(n)}(x,\varepsilon), z^{(n)}(x,\varepsilon))$ of the linearized eigenvalue problem of (1.2), associated with a stationary solution which has the limiting interface Γ^0, are given by the following forms*:

$$\lambda^\varepsilon \approx \varepsilon^2 \lambda_2^{(n)},$$

$$w^{(n)}(x,\varepsilon) \approx \begin{cases} \omega(\frac{Y(x)}{d})\frac{\dot{\phi}_0(Y(x)/\varepsilon)}{\dot{\phi}_0(0)}\Theta_0^{(n)}(S(x)), & x \in U_d(\Gamma^0), \\ 0, & x \in \Omega^*\backslash U_d(\Gamma^0), \end{cases}$$

$$z^{(n)}(x,\varepsilon) \approx -\varepsilon^2 \frac{1}{D}\frac{[g]}{\dot{\phi}_0(0)} K_N^*(\delta_{\Gamma^0} \otimes \Theta_0^{(n)}).$$

Here, $(\lambda_2^{(n)}, \Theta_0^{(n)})$ *are determined by the following eigenvalue problem on* Γ^0 :

$$L^*\Theta_0^{(n)} + \frac{1}{D}c_2[g]J'(v^*)\left\langle K_N^*(\delta_{\Gamma^0} \otimes \Theta_0^{(n)}), \delta_{\Gamma^0}\right\rangle = \lambda_2^{(n)}\Theta_0^{(n)}.$$

with

$$\int_{\Gamma^0} \Theta_0^{(n)} dS = 0,$$

where

$$L^* \equiv \Delta^{\Gamma^0} + \frac{1}{2}H_1(s) - c_1 P_3(s) + \hat{\Lambda}(s).$$

(*For the notation and the proof of this theorem, see* §3).

The outline of this paper is as follows. In §2, we make several key observations obtained so far, then expand an ε-family of matched asymptotic stationary solution to rescaled system of (1.1), and finally derive the rescaled reduced equation. We study the associated limiting eigenvalue problem on the interface in §3.

Finally we state the assumptions for f and g (Figure 1.1), and notations.

(A.0) f and g are smooth functions of u and v defined on some open set $\mathcal{O}$ in $\mathbf{R}^2$.

(A.1) (a) The nullcline of f is sigmoidal and consists of three smooth curves $u = h_-(v)$, $h_0(v)$ and $h_+(v)$ defined on the intervals I_-, I_0, and I_+, respectively. Let $\min I_- = \underline{v}$ and $\max I_+ = \overline{v}$, then the inequality $h_-(v) < h_0(v) < h_+(v)$ holds for $v \in I^* \equiv (\underline{v}, \overline{v})$ and $h_+(v)$ ($h_-(v)$) coincides with $h_0(v)$ at only one point $v = \overline{v}$ ($\underline{v}$), respectively.

(b) The nullcline of g intersects with that of f at one or three points transversally as in Fig. 1.1.

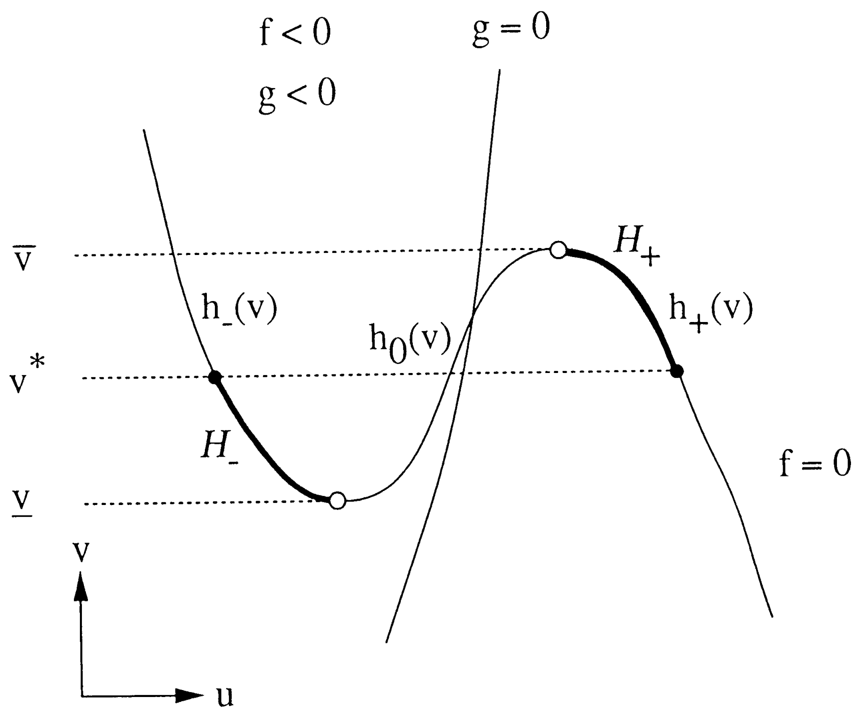

Fig. 1.1. Functional forms of $f = 0$ and $g = 0$

(A.2) $J(v)$ has an isolated zero at $v = v^* \in I^*$ such that $dJ/dv < 0$ at $v = v^*$, where $J(v) = \int_{h_-(v)}^{h_+(v)} f(s,v)ds$. Moreover we assume that $v_- < v^* < v_+$.

(A.3) $f_u < 0$ on $H_+ \cup H_-$, where H_- (H_+) denotes the part of the curve $u = h_-(v)$ ($h_+(v)$) defined by H_- (H_+) $= \{(u,v) \mid u = h_-(v)(h_+(v))$ for $\underline{v} \leq v < v^*(v^* < v \leq \overline{v})\}$, respectively.

(A.4) (a)

$$g|_{H_-} < 0 < g|_{H_+}$$

(b)

$$\det \left(\frac{\partial(f,g)}{\partial(u,v)} \right) \Bigg|_{H_+ \cup H_-} > 0.$$

(A.5) $\quad g_v|_{H_+ \cup H_-} \leq 0.$

Remark 1.1. Let $G_\pm(v) \equiv g(h_\pm(v), v)$ for $v \in I_\pm$. Then, the assumption (A.4) (b) is equivalent to

$$\frac{d}{dv} G_\pm(v) \Bigg|_{H_\pm} < 0, \text{ respectively,}$$

since it follows from $f(h_\pm(v),v)=0$ and (A.3) that

$$\frac{d}{dv}G_\pm(v)\Big|_{H_\pm} = \frac{f_u g_v - f_v g_u}{f_u}\Big|_{H_\pm}.$$

Remark 1.2. It holds that $f_u = 0$ at $(h_+(\overline{v}),\overline{v})$ and $(h_-(\underline{v}),\underline{v})$.

We use the following notation throughout the paper: Let $\sigma = (\sigma_1,\sigma_2,\dots,\sigma_N)$ denote the usual multi-index of order $|\sigma| = \sigma_1+\sigma_2+\cdots+\sigma_N$ with nonnegative integers σ_i, and write $\partial_i = \partial/\partial x_i$ $(1 \le i \le N)$.

(i) Let k be a nonnegative integer and $\alpha \in (0,1)$. By $C^{k+\alpha}(\overline{\Omega})$ we mean the Banach space of all real-valued functions $u \in C^k(\overline{\Omega})$ for which the derivatives $\partial^\sigma u$ with $|\sigma| = k$ are Hölder continuous on $\overline{\Omega}$ with exponent α. The norm is

$$\| u \|_{C^{k+\alpha}(\overline{\Omega})} = \sum_{j=0}^{k} |u|_{j,\overline{\Omega}} + |u|_{k+\alpha,\overline{\Omega}},$$

where

$$|u|_{j,\overline{\Omega}} = \max_{|\sigma|=j}\sup_{x\in\overline{\Omega}} |\partial^\sigma u(x)|,$$

and

$$|u|_{k+\alpha,\overline{\Omega}} = \max_{|\sigma|=k}\sup_{x,y\in\overline{\Omega}} \frac{|\partial^\sigma u(x) - \partial^\sigma u(y)|}{|x-y|^\alpha} \quad (x \ne y)$$

(ii) $C_0^{k+\alpha}(\overline{\Omega})$ is the subspace of $C^{k+\alpha}(\overline{\Omega})$ whose elements are functions vanishing on $\partial\Omega$.

(iii) $C_\varepsilon^{k+\alpha}(\overline{\Omega})$ is the Banach space of all real-valued functions in $C^{k+\alpha}(\overline{\Omega})$, but with the special norm depending on ε:

$$\| u \|_{C_\varepsilon^{k+\alpha}(\overline{\Omega})} = \sum_{j=0}^{k} \varepsilon^j |u|_{j,\overline{\Omega}} + \varepsilon^{k+\alpha} |u|_{k+\alpha,\overline{\Omega}}.$$

(iv) $C_{\varepsilon,0}^{k+\alpha}(\overline{\Omega})$ is the subspace of $C_\varepsilon^{k+\alpha}(\overline{\Omega})$ whose elements are functions vanishing on $\partial\Omega$.

2. Matched asymptotic expansion of the rescaled stationary interfacial pattern and the rescaled reduced problem

Theorem A in §1 suggests us that any stationary interfacial pattern, which is smooth up to $\varepsilon = 0$, becomes unstable for sufficiently small $\varepsilon > 0$. On the other hand, for any fixed small ε, we know the existence of observable steady states by numerical simulation. How do we make these two observations consistent with each other? The following two results give an insight into the issue.

Theorem 2.1 (Taniguchi and Nishiura [16]). *Let Ω be a rectangular domain. Then the planar front solution loses its stability when $\varepsilon \downarrow 0$, and the fastest growing wavelength is exactly of order $\varepsilon^{1/3}$.*

Theorem 2.2 (Nishiura [11]). *The stripe pattern with width of $O(\varepsilon^{1/3})$ is stable for small ε in a rectangular domain.*

These two results strongly suggest that the stable stationary patterns do exist for small ε, however they become finer and finer in the limit of ε with the characteristic domain size of order $\varepsilon^{1/3}$. Hence a unit pattern itself shrinks to zero and does not have a well-defined limit. To capture the limiting configuration of the interface, we have to magnify the shrinking pattern by introducing the following rescaled variable

$$X \equiv \frac{x - x^*}{\varepsilon^{1/3}} \tag{2.1}$$

for an appropriate $x^* \in \mathbf{R}^N$. The aim of this section is to derive a *rescaled reduced problem*, which determines the limiting configuration of the interface.

We derive a new rescaled system by an appropriate magnification, and obtain the rescaled reduced problem by using matched asymptotic expansion. Our working hypotheses throughout this section are the following:

Hypotheses

1. An ε-family of stationary patterns of (1.1) has a periodic structure, namely, it has a unit periodic cell Ω^ε. Moreover the domain size of Ω^ε is of order $\varepsilon^{1/3}$ and

$$\lim_{\varepsilon \downarrow 0} \frac{\partial \Omega^\varepsilon - x^*}{\varepsilon^{1/3}} = \partial \Omega^*$$

exists for an appropriate $x^* \in \Omega^\varepsilon$, where

$$\partial\Omega^\varepsilon - x^* = \{x - x^* \mid x \in \partial\Omega^\varepsilon\}.$$

The rescaled limiting domain Ω^* as well as Ω^ε is at least piecewise smooth (hence the polygonal shape is allowable).

2. The Neumann boundary condition is satisfied on $\partial\Omega^\varepsilon$ and $\partial\Omega^*$.

Using (2.1), the resulting rescaled system becomes

$$\begin{cases} u_t = \tilde{\varepsilon}^2 \Delta_X u + f(u,v), \\ \delta v_t = D\tilde{\varepsilon}^{-1}\Delta_X v + g(u,v), \end{cases} \quad (X,t) \in \Omega^* \times (0,\infty), \qquad (2.2)_a$$

$$\frac{\partial u}{\partial n} = 0 = \frac{\partial v}{\partial n} \qquad (X,t) \in \partial\Omega^* \times (0,\infty), \qquad (2.2)_b$$

where Δ_X is the Laplacian with respect to $X = (X_1, \ldots, X_N)$, n outward unit normal vector on $\partial\Omega^*$ (we use the same notation as before), and $\tilde{\varepsilon} \equiv \varepsilon^{2/3}$. Note that in this scale, the thickness of interior transition layer is $O(\tilde{\varepsilon})$. Roughly speaking, the difference between (1.1) and (2.2) is the coefficient ε of $g(u,v)$ in (2.2).

Then the stationary problem becomes

$$\begin{cases} 0 = \varepsilon^2 \Delta u + f(u,v), \\ 0 = D\Delta v + \varepsilon g(u,v), \end{cases} \quad \text{in } \Omega^* \qquad (2.3)$$

with Neumann boundary condition. Here we omit tilde ˜ of ε and subscript X of Δ and use the notation x in place of X.

In the following of this subsection, we display how the formal asymptotic expansion can be carried out. We assume that there exists an ε-family of smooth solutions $(U^\varepsilon(x), V^\varepsilon(x))$ of (2.3) with interior transition layers such that the interface

$$\Gamma^\varepsilon \equiv \left\{x \in \Omega^* \mid U^\varepsilon(x) = \alpha^* \equiv \frac{1}{2}(h_+(v^*) + h_-(v^*))\right\}$$

is a compact smooth manifold of dimension $N-1$ embedded in $\mathbf{R}^N$ and have a definite limit Γ^0 as $\varepsilon \downarrow 0$. Γ^0, also, is a compact smooth manifold of dimension $N-1$ embedded in $\mathbf{R}^N$. For simplicity we assume that the region surrounded by Γ^0 as well as Γ^ε is simply connected. Let (X_ϕ, ϕ) be a local chart on Γ^0, with $\phi(X_\phi)$ an open subset of $\mathbf{R}^{N-1}$. For $x_0 \in X_\phi$,

$\phi(x_0) = s = (s^1, \ldots, s^{N-1})$ and we denote the inverse of ϕ by

$$x_0 = (x_0^1(s), \ldots, x_0^N(s)).$$

In some tubular neighborhood $U_d(\Gamma^0) = \{x \in \mathbf{R}^N \mid |y(x)| \leq d\}$ of Γ^0, local coordinate system $(s, y) = (s^1, \ldots, s^{N-1}, y)$ is defined and for $x \in U_d(\Gamma^0)$,

$$x = X(s, y) \equiv x_0(s^1, \ldots, s^{N-1}) + y\nu(s^1, \ldots, s^{N-1}) \tag{2.4}$$

holds, where $\nu(s^1, \ldots, s^{N-1})$ is unit outward normal vector at $s = (s^1, \ldots, s^{N-1})$ to Γ^0. Then, X is a diffeomorphism from $[-d, d] \times \Gamma^0$ to $U_d(\Gamma^0)$ if d is strictly smaller than the infimum of the radii of curvature of Γ^0. Its inverse is denoted by $(S(x), Y(x))$. Then Γ^ε can be represented as

$$\Gamma^\varepsilon = \{x_0(s) + \gamma(s, \varepsilon)\nu(s) \mid s \in \Gamma^0\}$$

where

$$\gamma(s, \varepsilon) = \sum_{k=1}^{m} \varepsilon^k \gamma_k(s) + \varepsilon^m \hat{\gamma}_{m+1}(s, \varepsilon).$$

Here we introduce local shift variable τ by the following relation:

$$y = \tau + \omega\left(\frac{\tau}{d}\right)\gamma(s, \varepsilon), \tag{2.5}$$

where $\omega(\tau) \in C^\infty(\mathbf{R})$ is a cut off function such that

$$\omega(\tau) = 1 \text{ for } |\tau| \leq \frac{1}{2}, \quad \omega(\tau) = 0 \text{ for } |\tau| \geq 1,$$
$$0 \leq \omega(\tau) \leq 1, \quad |\omega'| \leq 3.$$

Then, by the implicit function theorem, $\tau = \tau(s, y, \varepsilon)$ satisfying (2.5) is defined for sufficiently small ε. In place of x, we use a new independent variable $\hat{x}$, defined by

$$\hat{x} = \hat{X}(x, \varepsilon) = \begin{cases} x, & x \in \Omega^* \backslash U_d(\Gamma^0), \\ X(S(x), \tau(S(x), Y(x), \varepsilon)), & x \in U_d(\Gamma^0). \end{cases}$$

Let $\Omega^{\varepsilon+}$ (resp. Ω^{*+}) be the region surrounded by Γ^ε (resp. Γ^0) and $\Omega^{\varepsilon-} \equiv \Omega^* \backslash \overline{\Omega}^{\varepsilon+}$ (resp. $\Omega^{*-} \equiv \Omega^* \backslash \overline{\Omega}^{*+}$). Then, note that $\hat{x} = \hat{X}(x, \varepsilon)$ maps Γ^ε to Γ^0, and $\Omega^{\varepsilon\pm}$ to $\Omega^{*\pm}$, respectively. Throughout this paper, we shall use the following notation

$$u(x) = u(s, y), \quad \hat{u}(\hat{x}) = \hat{u}(s, \tau).$$

After the above transformation, stationary problem (2.3) are equivalent to the following system:

$$\begin{cases} 0 = \varepsilon^2 M^\varepsilon \hat{u} + f(\hat{u},\hat{v}), \\ 0 = DM^\varepsilon \hat{v} + \varepsilon g(\hat{u},\hat{v}). \end{cases} \quad \text{in } \Omega^*, \tag{2.6}$$

$$\frac{\partial \hat{u}}{\partial n} = 0 = \frac{\partial \hat{v}}{\partial n} \quad \text{on } \partial\Omega^*, \tag{2.7}$$

where $\hat{u} = \hat{u}(\hat{x})$, $\hat{v} = \hat{v}(\hat{x})$ and M^ε is the representation of Laplacian Δ_x in $\hat{x}$. In $\Omega^* \backslash U_d(\Gamma^0)$, M^ε is equal to $\Delta_{\hat{x}}$. On the other hand, in the neighborhood $U_d(\Gamma^0)$, M^ε is defined as in the following way: For the local coordinate system (s, y) defined by (2.4) in $\mathbf{R}^N$, let g^{ij} be the contravariant metric tensor and $g = \det(g^{ij})$. Here we regard y as s^N. Then for $u(x) = u(s, y)$, Laplacian Δ_x is represented as

$$\begin{aligned} (\Delta_x u)(x) &= (\Delta_{(s,y)} u)(s, y) \\ &\equiv \frac{\partial^2}{\partial y^2} u(s, y) + (N-1)H(s, y)\frac{\partial}{\partial y} u(s, y) \\ &\qquad + \frac{1}{\sqrt{g}} \sum_{i=1}^{N-1} \frac{\partial}{\partial s^i}\left(\sqrt{g}\sum_{j=1}^{N-1} g^{ij}\frac{\partial}{\partial s^j} u(s, y)\right), \end{aligned} \tag{2.8}$$

where $H = H(s, y)$ is the mean curvature of the hypersurface $\Gamma(y) = \{x_0(s) + y\nu(s) \mid s \in \Gamma^0\}$ at (s, y). Using this representation, for $\hat{u}(\hat{x}) = \hat{u}(s, \tau)$, M^ε is defined by

$$(M^\varepsilon \hat{u})(\hat{x}) \equiv \Delta_{(s,y)} \hat{u}(s, \tau(s, y, \varepsilon)).$$

Noting the above definition, M^ε is expanded as $M^\varepsilon = \sum_{k\geq 0} \varepsilon^k M_k$, where

$$M_0 \equiv \Delta_{\hat{x}}, \qquad \hat{x} \in \Omega^*,$$

and for $k \geq 1$,

$$M_k = \begin{cases} 0, & \hat{x} \in \Omega^* \backslash U_d(\Gamma^0), \\ \text{at most second order differential operator} & \\ \text{in } s^i \ (i = 1, \ldots, N-1) \text{ and } \tau, & \hat{x} \in U_d(\Gamma^0). \end{cases}$$

From this point on, we consider (2.6), so we omit the symbol hat ˆ.

In this situation, we shall find formal matched asymptotic solutions $(U^\varepsilon(x), V^\varepsilon(x))$ to (2.6) that have the following expansions:

$$
\begin{aligned}
U^\varepsilon(x) &\approx \begin{cases} U_+^\varepsilon(x) \equiv U_m^+(x,\varepsilon) + \Phi_m^+(x,\varepsilon), & x \in \Omega^{*+}, \\ U_-^\varepsilon(x) \equiv U_m^-(x,\varepsilon) + \Phi_m^-(x,\varepsilon), & x \in \Omega^{*-}, \end{cases} \\
V^\varepsilon(x) &\approx \begin{cases} V_+^\varepsilon(x) \equiv V_m^+(x,\varepsilon) + \Psi_m^+(x,\varepsilon), & x \in \Omega^{*+}, \\ V_-^\varepsilon(x) \equiv V_m^-(x,\varepsilon) + \Psi_m^-(x,\varepsilon), & x \in \Omega^{*-}, \end{cases}
\end{aligned}
\tag{2.9}
$$

where

$$
U_m^\pm(x,\varepsilon) = \sum_{k=0}^m u_k^\pm(x)\varepsilon^k, \quad V_m^\pm(x,\varepsilon) = \sum_{k=0}^m v_k^\pm(x)\varepsilon^k,
$$

$$
\Phi_m^\pm(x,\varepsilon) = \begin{cases} \omega\left(\dfrac{Y(x)}{d}\right) \displaystyle\sum_{k=0}^m \phi_k^\pm\left(S(x), \frac{Y(x)}{\varepsilon}\right)\varepsilon^k, & x \in U_d(\Gamma^0) \cap \Omega^{*\pm}, \\ 0, & x \in \Omega^{*\pm}\backslash U_d(\Gamma^0), \end{cases}
$$

$$
\Psi_m^\pm(x,\varepsilon) = \begin{cases} \omega\left(\dfrac{Y(x)}{d}\right) \displaystyle\sum_{k=0}^m \psi_k^\pm\left(S(x), \frac{Y(x)}{\varepsilon}\right)\varepsilon^k, & x \in U_d(\Gamma^0) \cap \Omega^{*\pm}, \\ 0, & x \in \Omega^{*\pm}\backslash U_d(\Gamma^0), \end{cases}
$$

ξ is the stretched variable $\xi \equiv \tau/\varepsilon$, $\phi_k^\pm$ and $\psi_k^\pm$ functions of s, Ω^{*+} the region surrounded by Γ^0, and $\Omega^{*-} \equiv \Omega^*\backslash\overline{\Omega}^{*+}$. The coefficients $u_k^\pm$, $v_k^\pm$, $\phi_k^\pm$, and $\psi_k^\pm$ satisfy some equations and relations. We can obtain these equations by making outer and inner expansions and equating like powers of ε^k. Let $\beta^\varepsilon(s) = v^* + \sum_{k=1}^m \beta_k(s)\varepsilon^k + \varepsilon^m\beta_{m+1}(s,\varepsilon)$ be the value of $V^\varepsilon(x)$ on Γ^0.

First, we divide (2.6) into two problems as follows:

$$
\begin{cases} \varepsilon^2 M^\varepsilon u^+ + f(u^+, v^+) = 0, & \\ & \text{in } \Omega^{*+}, \\ DM^\varepsilon v^+ + \varepsilon g(u^+, v^+) = 0, & \\ u^+ = \alpha^*, \quad v^+ = \beta^\varepsilon \text{ on } \Gamma^0, & \end{cases}
\tag{2.10$_+$}
$$

$$\begin{cases} \varepsilon^2 M^\varepsilon u^- + f(u^-, v^-) = 0, & \\ & \text{in } \Omega^{*-}, \\ DM^\varepsilon v^- + \varepsilon g(u^-, v^-) = 0, & \\ u^- = \alpha^*, \quad v^- = \beta^\varepsilon \ \text{ on } \Gamma^0, & \\ \dfrac{\partial u^-}{\partial n} = 0 = \dfrac{\partial v^-}{\partial n} \ \text{ on } \partial\Omega^*. & \end{cases} \tag{2.10$_-$}$$

Then interface is regarded as the boundary layer at Γ^0.

Outer expansion

Let

$$u^\pm = \sum_{k=0}^{m} u_k^\pm(x)\varepsilon^k, \quad v^\pm = \sum_{k=0}^{m} v_k^\pm(x)\varepsilon^k \tag{2.11}$$

and substitute (2.11) into $(2.10)_\pm$. Equating like powers of ε^k, we have the following problem for $(u_k^\pm(x), v_k^\pm(x))$:

$k = 0$

$$\begin{cases} f(u_0^\pm, v_0^\pm) = 0, & \\ & \text{in } \Omega^{*\pm}, \\ M_0 v_0^\pm = 0, & \\ \dfrac{\partial v_0^-}{\partial n} = 0 \ \text{ on } \partial\Omega^*, & \end{cases} \tag{2.12}$$

$k = 1$

$$\begin{cases} f_u^{0\pm} u_1^\pm + f_v^{0\pm} v_1^\pm = 0, & \\ & \text{in } \Omega^{*\pm}, \\ DM_0 v_1^\pm = -g(h_\pm(v_0^\pm), v_0^\pm), & \\ \dfrac{\partial v_1^-}{\partial n} = 0 \ \text{ on } \partial\Omega^*, & \end{cases}$$

$k \geq 2$

$$\begin{cases} f_u^{0\pm} u_k^\pm + f_v^{0\pm} v_k^\pm = \displaystyle\sum_{i+j=k-2} M_i u_j^\pm + P_{k-1}^\pm, & \\ & \text{in } \Omega^{*\pm}, \\ DM_0 v_k^\pm = -D \displaystyle\sum_{i+j=k, i\geq 1} M_i v_j^\pm + Q_{k-1}^\pm, & \\ \dfrac{\partial v_k^-}{\partial n} = 0 \text{ on } \partial\Omega^*, & \end{cases}$$

where $P^{\pm}_{k-1}$ and $Q^{\pm}_{k-1}$ are functions determined only by $u^{\pm}_0, v^{\pm}_0, \dots, u^{\pm}_{k-1}, v^{\pm}_{k-1}$. This expansion is insufficient because the layer part is not represented. For example, u^+_0 and u^-_0 are discontinuous on Γ^0. So we need a new variable $\xi = \tau/\varepsilon$ that stretches a neighborhood of the interface. Also we note that the boundary conditions for $v^{\pm}_k$ are determined by matching conditions.

Inner expansion

We introduce the stretched variable $\xi = \tau/\varepsilon$ and let

$$\begin{aligned} u^{\pm} &= U^{\pm}_m(x,\varepsilon) + \sum_{k=0}^{m} \phi^{\pm}_k(S(x), \frac{Y(x)}{\varepsilon})\varepsilon^k, \\ v^{\pm} &= V^{\pm}_m(x,\varepsilon) + \varepsilon^2 \sum_{k=0}^{m} \psi^{\pm}_k(S(x), \frac{Y(x)}{\varepsilon})\varepsilon^k, \end{aligned} \tag{2.13}$$

where $\phi^{\pm}_k = \phi^{\pm}_k(s,\xi)$ and $\psi^{\pm}_k = \psi^{\pm}_k(s,\xi)$. Substituting (2.13) into $(2.10)_{\pm}$ and equating like powers of ε^k, we obtain the following equations:

$k = 0$

$$\begin{cases} \ddot{\phi}^{\pm}_0 + f(h_{\pm}(v^*) + \phi^{\pm}_0, v^*) = 0, \\ D\ddot{\psi}^{\pm}_0 = 0, \end{cases} \qquad \xi \in I^{\mp}, s \in \Gamma^0,$$

$$\phi^{\pm}_0(s, \mp\infty) = 0, \quad \psi^{\pm}_0(s,\mp\infty) = 0 = \dot{\psi}^{\pm}_0(s,\mp\infty),$$

$k = 1$

$$\begin{cases} \ddot{\phi}^{\pm}_1 + \tilde{f}^{0\pm}_u \phi^{\pm}_1 = -(N-1)H_0(s)\dot{\phi}^{\pm}_0 - \tilde{f}^{0\pm}_u u^{\pm}_1(s,0) - \tilde{f}^{0\pm}_v v^{\pm}_1(s,0), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \xi \in I^{\mp}, s \in \Gamma^0, \\ D\ddot{\psi}^{\pm}_1 = g(h_{\pm}(v^*), v^*) - g(h_{\pm}(v^*) + \phi^{\pm}_0, v^*), \end{cases}$$

$$\phi^{\pm}_1(s,\mp\infty) = 0, \quad \psi^{\pm}_1(s,\mp\infty) = 0 = \dot{\psi}^{\pm}_1(s,\mp\infty),$$

$k \geq 2$

$$\begin{cases} \ddot{\phi}^{\pm}_k + \tilde{f}^{0\pm}_u \phi^{\pm}_k = -\displaystyle\sum_{i+j=k, i\geq 1} \tilde{M}_i \phi^{\pm}_j + \tilde{P}^{\pm}_{k-1}, \\ D\ddot{\psi}^{\pm}_k = -D \displaystyle\sum_{i+j=k, i\geq 1} \tilde{M}_i \psi^{\pm}_j + \tilde{Q}^{\pm}_{k-1}, \end{cases} \qquad \xi \in I^{\mp}, s \in \Gamma^0,$$

$$\phi_k^{\pm}(s, \mp\infty) = 0, \quad \psi_k^{\pm}(s, \mp\infty) = 0 = \dot{\psi}_k^{\pm}(s, \mp\infty),$$

where $\tilde{P}_{k-1}^{\pm}$ depends on $u_0^{\pm}, v_0^{\pm}, \ldots, u_k^{\pm}, v_k^{\pm}, \phi_0^{\pm}, \psi_0^{\pm}, \ldots, \phi_{k-1}^{\pm}, \psi_{k-2}^{\pm}$ and $\tilde{Q}_{k-1}^{\pm}$ does moreover on $\psi_{k-1}^{\pm}$, $I^- \equiv (-\infty, 0)$ and $I^+ \equiv (0, \infty)$. We define $\psi_{-1} \equiv 0$. $\tilde{M}^{\varepsilon}$ is the representation of M^{ε} in variables s and ξ, and expanded as

$$\tilde{M}^{\varepsilon} \equiv \frac{1}{\varepsilon^2} \sum_{k \geq 0} \varepsilon^k \tilde{M}_k.$$

Here $\tilde{M}_k$ $(k \geq 0)$ are at most second order differential operators in s and ξ. The precise forms of $\tilde{M}_k$ are displayed in the following lemma.

Lemma 2.3 *$\tilde{M}_0$, $\tilde{M}_1$, and $\tilde{M}_2$ have the following form:*

$$\begin{aligned} \tilde{M}_0 &\equiv \frac{\partial^2}{\partial \xi^2}, \quad \tilde{M}_1 \equiv (N-1) H_0(s) \frac{\partial}{\partial \xi}, \\ \tilde{M}_2 &\equiv \Delta^{\Gamma^0} - (P_1(s) + P_2(s)) \frac{\partial}{\partial \xi} + P_3(s) \frac{\partial^2}{\partial \xi^2} \\ &\quad - D_s \frac{\partial}{\partial \xi} - H_1(s)(\xi + \gamma_1(s)) \frac{\partial}{\partial \xi}, \end{aligned} \tag{2.14}$$

where

$$\begin{aligned} P_1(s) &= \frac{1}{2G} \sum_{i=1}^{N-1} G_{s^i} \sum_{j=1}^{N-1} G^{ij} \partial_{s^j} \gamma_1, \\ P_2(s) &= \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} [G^{ij}_{s^i} \partial_{s^j} \gamma_1 + G^{ij} \partial_{s^i s^j} \gamma_1], \\ P_3(s) &= \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} G^{ij} \partial_{s^i} \gamma_1 \partial_{s^j} \gamma_1 > 0, \\ D_s &= \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} G^{ij} \left(\partial_{s^i} \gamma_1 \frac{\partial}{\partial s^j} + \partial_{s^j} \gamma_1 \frac{\partial}{\partial s^i} \right), \\ H_1(s) &\equiv \sum_{i=1}^{N-1} \kappa_i(s)^2. \end{aligned}$$

$H_0(s)$ (resp. $\kappa_i(s)$) stands for the mean (resp. principal) curvature of Γ^0 at $s \in \Gamma^0$, G^{ij} is the contravariant metric tensor for the manifold Γ^0 of dimension $N-1$, $G = \det(G^{ij})$, and Δ^{Γ^0} Laplace-Beltrami's operator defined on Γ^0. The coefficients of $\frac{\partial}{\partial s^j}$ in D_s are independent of ξ.

Proof. See Appendix. □

The boundary and C^1-matching conditions

Now we describe the boundary conditions for $v_k^{\pm}$ and $\phi_k^{\pm}$ on Γ^0. Then $u_k^{\pm}$, $v_k^{\pm}$, $\phi_k^{\pm}$, and $\psi_k^{\pm}$ are determined recursively. These conditions are given by

$$\alpha^* = \sum_{k=0}^{m} u_k^{\pm}(s,0)\varepsilon^k + \sum_{k=0}^{m} \phi_k^{\pm}(s,0)\varepsilon^k,$$

$$v^* + \sum_{k=0}^{m} \beta_k(s)\varepsilon^k = \sum_{k=1}^{m} v_k^{\pm}(s,0)\varepsilon^k + \varepsilon^2 \sum_{k=0}^{m-2} \psi_k^{\pm}(s,0)\varepsilon^k.$$

Equating like powers of ε^k, we have the following boundary conditions.

$k = 0$

$$\phi_0^{\pm}(s,0) = \alpha^* - u_0^{\pm}(s,0), \quad v_0^{\pm} = v^* \text{ on } \Gamma^0, \tag{2.15}$$

$k \geq 1$

$$\phi_k^{\pm}(s,0) = -u_k^{\pm}(s,0), \quad v_k^{\pm} = \beta_k(s) - \psi_{k-2}^{\pm}(s,0) \text{ on } \Gamma^0.$$

In this way, we obtain the formal asymptotic solution of $(2.10)_{\pm}$. In order that $(U^\varepsilon, V^\varepsilon)$ become the formal stationary solution of (2.6), $(U_{\pm}^\varepsilon, V_{\pm}^\varepsilon)$ must satisfy the C^1-matching conditions, that is,

$$\varepsilon\frac{\partial U_+^\varepsilon}{\partial \nu} - \varepsilon\frac{\partial U_-^\varepsilon}{\partial \nu} = 0, \quad \frac{\partial V_+^\varepsilon}{\partial \nu} - \frac{\partial V_-^\varepsilon}{\partial \nu} = 0 \quad \text{on } \Gamma^0.$$

After some computation, we have

$k = 0$

$$\frac{\partial v_0^+}{\partial \nu}(s,0) = \frac{\partial v_0^-}{\partial \nu}(s,0), \quad \dot{\phi}_0^+(s,0) = \dot{\phi}_0^-(s,0) \quad \text{on } \Gamma^0. \tag{2.16}$$

$k \geq 1$

$$\left.\begin{aligned} &\frac{\partial v_k^+}{\partial \nu}(s,0) + \dot{\psi}_{k-1}^+(s,0) = \frac{\partial v_k^-}{\partial \nu}(s,0) + \dot{\psi}_{k-1}^-(s,0), \\ &\dot{\phi}_k^+(s,0) + \frac{\partial u_{k-1}^+}{\partial \nu}(s,0) = \dot{\phi}_k^-(s,0) + \frac{\partial u_{k-1}^-}{\partial \nu}(s,0). \end{aligned}\right\} \text{ on } \Gamma^0 \tag{2.17$_k$}$$

Using (2.12), (2.15) and the first relation of (2.16), we can see $v_0^+ \equiv v^* \equiv v_0^-$. Therefore no information can be extracted from the equation of $v_0^\pm$ about the configuration of Γ^0. It turns out that the equation for $v_1^\pm$ determines Γ^0. Here we have a condition for $\beta_1(s)$.

Lemma 2.4 *$\beta_1(s)$ must satisfy the following relation*:

$$\beta_1(s) = \beta^*(N-1)H_0(s), \tag{2.18}$$

where

$$\beta^* \equiv \int_{-\infty}^{\infty} \dot{\phi}_0^2 d\xi \Big/ J'(v^*) < 0.$$

Proof. We consider the C^1-matching condition $(2.17)_1$. By using the representation of $\phi_1^\pm$, we have

$$\begin{aligned}\dot{\phi}_1^\pm(s,0) &= -\frac{1}{\dot{\phi}_0(0)}(N-1)H_0(s)\int_{\mp\infty}^{0} \dot{\phi}_0^2 d\xi \\ &\quad - \frac{1}{\dot{\phi}_0(0)}\beta_1(s)\int_{h_\pm(v^*)}^{0} f_v(u,v^*)du.\end{aligned}$$

Then $(2.17)_1$ becomes

$$\begin{aligned}0 &= \dot{\phi}_1^+(s,0) - \dot{\phi}_1^-(s,0) \\ &= -\frac{1}{\dot{\phi}_0(0)}(N-1)H_0(s)\int_{-\infty}^{\infty} \dot{\phi}_0^2 d\xi \\ &\quad - \frac{1}{\dot{\phi}_0(0)}\beta_1(s)\int_{h_+(v^*)}^{h_-(v^*)} f_v(u,v^*)du,\end{aligned}$$

which implies (2.18). □

Noting Lemma 2.4, the reduced problem can be written as

$$\begin{cases} D\Delta v_1^\pm = -g(h_\pm(v^*),v^*), & \text{in } \Omega^{*\pm}, \\ v_1^\pm(s,0) = \beta^*(N-1)H_0(s), \quad \dfrac{\partial v_1^+}{\partial \nu} = \dfrac{\partial v_1^-}{\partial \nu}, & \text{on } \Gamma^0, \\ \dfrac{\partial v_1^-}{\partial n} = 0 \ \text{ on } \partial\Omega^*. \end{cases} \tag{2.19}$$

Here we used the fact that $v_0^+ \equiv v^* \equiv v_0^-$. The unknowns in (2.19) are $V^\pm$ and Γ^0, and (2.19) is like a free boundary value problem. It is a fundamental problem to solve (2.19) for constructing a solution of (2.6) by using matched asymptotic expansion method.

Let us rewrite (2.19) as

$$\begin{cases} D\Delta v_1^+ = -g(h_+(v^*), v^*) & \text{in } \Omega^{*+}, \\ v_1^+ = \beta^*(N-1)H_0 & \text{on } \Gamma^0, \end{cases} \tag{2.20}$$

$$\begin{cases} D\Delta v_1^- = -g(h_-(v^*), v^*) & \text{in } \Omega^{*-}, \\ v_1^- = \beta^*(N-1)H_0 & \text{on } \Gamma^0, \\ \dfrac{\partial v_1^-}{\partial n} = 0 \quad \text{on } \partial\Omega^*, \end{cases} \tag{2.21}$$

and

$$\frac{\partial v_1^+}{\partial \nu} = \frac{\partial v_1^-}{\partial \nu} \quad \text{on } \Gamma^0. \tag{2.22}$$

For the fixed Γ^0 (then H_0 is determined), (2.20) and (2.21) have unique solution, respectively. In order to obtain the solution of (2.19), we must find a Γ^0 satisfying C^1-matching condition (2.22). Unfortunately, it is not easy to find such Γ^0 since the value of $V^\pm$ on Γ^0 depends on the mean curvature H_0 of Γ^0 itself.

However, when Ω^* is ball, we can easily obtain the spherical symmetric solution of (2.19). Let $\Omega^* = \{x \in \mathbf{R}^N | \ |x| \le R\}$ and $r = |x|$. Then, (2.20), (2.21), and (2.22) are rewritten as

$$\begin{cases} V_{rr}^+ + \dfrac{N-1}{r} V_r^+ = A^+, & r \in (0, r_0), \\ V_r^+(0) = 0, \quad V^+(r_0) = \dfrac{\beta^*(N-1)}{r_0}, \end{cases} \tag{2.23}$$

$$\begin{cases} V_{rr}^- + \dfrac{N-1}{r} V_r^- = A^-, & r \in (r_0, R), \\ V^-(r_0) = \dfrac{\beta^*(N-1)}{r_0}, & V_r^-(R) = 0, \end{cases} \tag{2.24}$$

and

$$V_r^+(r_0) = V_r^-(r_0), \tag{2.25}$$

respectively, where $V^\pm = V^\pm(r)$, $A^\pm = -\frac{1}{D}g(h_\pm(v^*), v^*)$. Here V^+, V^-, and r_0 are unknown functions and parameter. Noting that the first equations of (2.23) and (2.24) are equivalent to

$$(r^{N-1}V_r^\pm)_r = -\frac{1}{D}g(h_\pm(v^*), v^*)r^{N-1},$$

we see that the solutions of (2.23) and (2.24) have the following expressions:

$$\begin{aligned} V^+(r) &= \frac{\beta^*(N-1)}{r_0} - \frac{A^+}{2N}(r_0^2 - r^2), \\ V^-(r) &= \frac{\beta^*(N-1)}{r_0} - \frac{A^-}{N}\int_{r_0}^{r}(R^N t^{1-N} - t)dt. \end{aligned}$$

Then r_0 is uniquely determined by (2.25) and given by

$$r_0 = R\left(\frac{A^-}{A^- - A^+}\right)^{1/N}.$$

For the case that Ω^* is generic domain, the existence of the solution of (2.20), (2.21), and (2.22) are not trivial.

We close this section by presenting the following proposition.

Proposition 2.5 *The solutions of* (2.6) *with interior transition layers are formally expansible and have the asymptotic forms* (2.9). *Then, the rescaled reduced problem is given by* (2.19).

3. Linearized eigenvalue problem for the rescaled system

We proceed to the stability problem of the rescaled stationary patterns. Employing the appropriate scaling obtained in §2, the rescaled stationary pattern can be stable. In this section, we study the linearized eigenvalue problem around a stationary solution $(U^\varepsilon(x), V^\varepsilon(x))$ of (2.6) given by

$$\left\{ \begin{array}{l} \varepsilon^2 M^\varepsilon w + f_u^\varepsilon w + f_v^\varepsilon z = \lambda^\varepsilon w, \\ DM^\varepsilon z + \varepsilon g_u^\varepsilon w + \varepsilon g_v^\varepsilon z = \delta\varepsilon\lambda^\varepsilon z, \end{array} \right. \quad \text{in } \Omega^*, \tag{3.1}$$

$$\frac{\partial w}{\partial n} = 0 = \frac{\partial z}{\partial n} \quad \text{on } \partial\Omega^*. \tag{3.2}$$

Since we employ the method of matched asymptotic expansion, we first divide (3.1) and (3.2) into the following two problems:

$$\begin{cases} \varepsilon^2 M^\varepsilon w^- + f_u^\varepsilon w^- + f_v^\varepsilon z^- = \lambda^\varepsilon w^-, & \\ DM^\varepsilon z^- + \varepsilon g_u^\varepsilon w^- + \varepsilon g_v^\varepsilon z^- = \delta\varepsilon\lambda^\varepsilon z^-, & \text{in } \Omega^{*-}, \\ \dfrac{\partial w^-}{\partial n} = 0 = \dfrac{\partial z^-}{\partial n} & \text{on } \partial\Omega^*, \\ w^- = \Theta^\varepsilon, \quad z^- = q^\varepsilon & \text{on } \Gamma^0, \end{cases} \tag{3.3$_-$}$$

$$\begin{cases} \varepsilon^2 M^\varepsilon w^+ + f_u^\varepsilon w^+ + f_v^\varepsilon z^+ = \lambda^\varepsilon w^+, & \\ DM^\varepsilon z^+ + \varepsilon g_u^\varepsilon w^+ + \varepsilon g_v^\varepsilon z^+ = \delta\varepsilon\lambda^\varepsilon z^+, & \text{in } \Omega^{*+}, \\ w^+ = \Theta^\varepsilon, \quad z^+ = q^\varepsilon & \text{on } \Gamma^0, \end{cases} \tag{3.3$_+$}$$

where

$$\Theta^\varepsilon \equiv \sum_{k=0}^m \varepsilon^k \Theta_k(s), \quad q^\varepsilon \equiv \sum_{k=0}^m \varepsilon^k q_k(s),$$

$$\lambda^\varepsilon \equiv \sum_{k=1}^m \varepsilon^k \lambda_k.$$

Θ^ε, q^ε, and λ^ε are unknown boundary data and they are determined later by the C^1-matching condition.

We construct the matched asymptotic solutions $(w_\pm^m(x,\varepsilon), z_\pm^m(x,\varepsilon))$ of $(3.3)_\pm$ in a parallel way as in Ikeda [6]. More precisely we have a solution $(w_\pm^m(x,\varepsilon), z_\pm^m(x,\varepsilon))$ in the form

$$\begin{cases} w_\pm^m(x,\varepsilon) = W_m^\pm(x,\varepsilon) + \Xi_m^\pm(x,\varepsilon) + \varepsilon^m r^\pm(x,\varepsilon) \\ \qquad\qquad + \varepsilon^m h'(v_0^\pm(x)) s^\pm(x,\varepsilon), \\ z_\pm^m(x,\varepsilon) = Z_m^\pm(x,\varepsilon) + \varepsilon^2 \Pi_m^\pm(x,\varepsilon) + \varepsilon^m s^\pm(x,\varepsilon), \end{cases} \tag{3.4$_\pm$}$$

where

$$W_m^{\pm}(x,\varepsilon)=\sum_{k=0}^{m} w_k^{\pm}(x)\varepsilon^k, \quad Z_m^{\pm}(x,\varepsilon)=\sum_{k=0}^{m} z_k^{\pm}(x)\varepsilon^k,$$

$$\Xi_m^{\pm}(x,\varepsilon)=\begin{cases}\omega\left(\dfrac{Y(x)}{d}\right)\displaystyle\sum_{k=0}^{m}\zeta_k^{\pm}\left(S(x),\frac{Y(x)}{\varepsilon}\right)\varepsilon^k, & \\ & x\in U_d(\Gamma^0)\cap\Omega^{*\pm}, \\ 0, & x\in\Omega^{*\pm}\backslash U_d(\Gamma^0),\end{cases}$$

$$\Pi_m^{\pm}(x,\varepsilon)=\begin{cases}\omega\left(\dfrac{Y(x)}{d}\right)\displaystyle\sum_{k=0}^{m}\eta_k^{\pm}\left(S(x),\frac{Y(x)}{\varepsilon}\right)\varepsilon^k, & \\ & x\in U_d(\Gamma^0)\cap\Omega^{*\pm}, \\ 0, & x\in\Omega^{*\pm}\backslash U_d(\Gamma^0).\end{cases}$$

$w_k^{\pm}$, $z_k^{\pm}$, $\zeta_k^{\pm}$, and $\eta_k^{\pm}$ satisfy the following equations and relations:
$k=0$

$$\begin{cases} f_u^{0\pm}w_0^{\pm}+f_v^{0\pm}z_0^{\pm}=0, \\ DM_0 z_0^{\pm}=0, \end{cases} \quad \text{in } \Omega^{*\pm}, \tag{3.5$_0$}$$

$$z_0^{\pm}=q_0 \text{ on } \Gamma^0, \quad \frac{\partial z_0^-}{\partial n}=0 \text{ on } \partial\Omega^*, \tag{3.6$_0$}$$

$$\begin{cases} \ddot{\zeta}_0^{\pm}+\tilde{f}_u^{0\pm}\zeta_0^{\pm}=-\tilde{f}_u^{0\pm}w_0^{\pm}(s,0)-\tilde{f}_v^{0\pm}z_0^{\pm}(s,0), \\ \qquad\qquad \xi\in I^{\mp}, s\in\Gamma^0, \\ D\ddot{\eta}_0^{\pm}=0, \end{cases} \tag{3.7$_0$}$$

$$\begin{aligned} &\zeta_0^{\pm}(s,0)=\Theta_0(s)-w_0^{\pm}(s,0), \quad \zeta_0^{\pm}(s,\mp\infty)=0, \\ &\eta_0^{\pm}(s,\mp\infty)=0=\dot{\eta}_0^{\pm}(s,\mp\infty), \end{aligned} \tag{3.8$_0$}$$

$k = 1$

$$\begin{cases} f_u^{0\pm} w_1^{\pm} + f_v^{0\pm} z_1^{\pm} = \lambda_1 w_0^{\pm} - f_u^{1\pm} w_0^{\pm} - f_v^{1\pm} z_0^{\pm}, \\ DM_0 z_1^{\pm} = -DM_1 z_0^{\pm} - g_u^{0\pm} w_0^{\pm} - g_v^{0\pm} z_0^{\pm}, \end{cases} \quad \text{in } \Omega^{*\pm}, \tag{3.5$_1$}$$

$$z_1^{\pm} = q_1 \text{ on } \Gamma^0, \quad \frac{\partial z_1^-}{\partial n} = 0 \text{ on } \partial\Omega^*, \tag{3.6$_1$}$$

$$\begin{cases} \ddot{\zeta}_1^{\pm} + \tilde{f}_u^{0\pm} \zeta_1^{\pm} = \lambda_1 (\tilde{W}_0^{\pm} + \zeta_0^{\pm}) - \tilde{M}_1 \zeta_0^{\pm} - \tilde{f}_u^{1\pm} (\tilde{W}_0^{\pm} + \zeta_0^{\pm}) \\ \qquad -\tilde{f}_v^{1\pm} \tilde{Z}_0^{\pm} - \tilde{f}_u^{0\pm} \tilde{W}_1^{\pm} - \tilde{f}_v^{0\pm} \tilde{Z}_1^{\pm}, \\ \qquad\qquad \xi \in I^{\mp}, s \in \Gamma^0, \\ D\ddot{\eta}_1^{\pm} = -\tilde{g}_u^{0\pm} (\tilde{W}_0^{\pm} + \zeta_0^{\pm}) - \tilde{g}_v^{0\pm} \tilde{Z}_0^{\pm} \\ \qquad -g_u(h_{\pm}(v^*), v^*) w_0^{\pm}(s, 0) - g_v(h_{\pm}(v^*), v^*) z_0^{\pm}(s, 0), \end{cases} \tag{3.7$_1$}$$

$$\begin{aligned} &\zeta_1^{\pm}(s, 0) = \Theta_1(s) - w_1^{\pm}(s, 0), \quad \zeta_1^{\pm}(s, \mp\infty) = 0, \\ &\eta_1^{\pm}(s, \mp\infty) = 0 = \dot{\eta}_1^{\pm}(s, \mp\infty), \end{aligned} \tag{3.8$_1$}$$

$k \geq 2$

$$\begin{cases} f_u^{0\pm} w_k^{\pm} + f_v^{0\pm} z_k^{\pm} \\ \quad = -\sum_{i+j=k-2} M_i w_j^{\pm} - \sum_{i+j=k, i\geq 1} (f_u^{i\pm} w_j^{\pm} + f_v^{i\pm} z_j^{\pm}) \\ \qquad + \sum_{i+j=k, i\geq 1} \lambda_i w_j^{\pm}, \qquad \text{in } \Omega^{*\pm}, \\ DM_0 z_k^{\pm} = -D \sum_{i+j=k, i\geq 1} M_i z_j^{\pm} - \sum_{i+j=k-1} (g_u^{i\pm} w_j^{\pm} + g_v^{i\pm} z_j^{\pm}) \\ \qquad + \delta \sum_{i+j=k-1} \lambda_i z_j^{\pm}, \end{cases} \tag{3.5$_k$}$$

$$z_k^{\pm} = q_k(s) - \eta_{k-2}^{\pm}(s, 0) \text{ on } \Gamma^0, \quad \frac{\partial z_k^-}{\partial n} = 0 \text{ on } \partial\Omega^*, \tag{3.6$_k$}$$

$$
(3.7)_k \quad \begin{cases}
\ddot{\zeta}_k^{\pm} + \tilde{f}_u^{0\pm}\zeta_k^{\pm} = -\sum_{i+j=k, i\geq 1} \tilde{M}_i \zeta_j^{\pm} - \sum_{i+j=k} (\tilde{f}_u^{i\pm}\tilde{W}_j^{\pm} + \tilde{f}_v^{i\pm}\tilde{Z}_j^{\pm}) \\
\qquad - \sum_{i+j=k, i\geq 1} \tilde{f}_u^{i\pm}\zeta_j^{\pm} - \sum_{i+j=k-2} \tilde{f}_v^{i\pm}\eta_j^{\pm} \\
\qquad + \sum_{i+j=k-1} \lambda_i(\tilde{W}_j^{\pm} + \zeta_j^{\pm}) - \tilde{P}_{k-2}^{\pm}, \\
\qquad\qquad \xi \in I^{\mp}, s \in \Gamma^0, \\
D\ddot{\eta}_k^{\pm} = -D\sum_{i+j=k, i\geq 1} \tilde{M}_i \eta_j^{\pm} - \sum_{i+j=k-1} \tilde{g}_u^{i\pm}(\tilde{W}_j^{\pm} + \zeta_j^{\pm}) \\
\qquad - \sum_{i+j=k-1} \tilde{g}_v^{i\pm}\tilde{Z}_j^{\pm} - \sum_{i+j=k-3} \tilde{g}_v^{i\pm}\eta_j^{\pm} \\
\qquad + \delta \sum_{i+j=k-1} \lambda_i \tilde{Z}_j^{\pm} + \delta \sum_{i+j=k-3} \lambda_i \eta_j^{\pm} - \tilde{Q}_k^{\pm},
\end{cases}
$$

$$
\begin{aligned}
&\zeta_k^{\pm}(s,0) = \Theta_k(s) - w_k^{\pm}(s,0), \quad \zeta_k^{\pm}(s,\mp\infty) = 0, \\
&\eta_k^{\pm}(s,\mp\infty) = 0 = \dot{\eta}_k^{\pm}(s,\mp\infty),
\end{aligned} \qquad (3.8)_k
$$

where

$$
f_u^{i\pm} \equiv \frac{1}{i!}\frac{d^i}{d\varepsilon^i} f_u \left(\sum_{k=0}^{m} \varepsilon^k u_k^{\pm}(x), \sum_{k=0}^{m} \varepsilon^k v_k^{\pm}(x) \right) \Bigg|_{\varepsilon=0},
$$

$$
\begin{aligned}
\tilde{f}_u^{i\pm} \equiv \frac{1}{i!}\frac{d^i}{d\varepsilon^i} f_u \Bigg(&\sum_{k=0}^{m} \varepsilon^k u_k^{\pm}(s,\varepsilon\xi) + \sum_{k=0}^{m} \varepsilon^k \phi_k^{\pm}(s,\xi), \\
&\sum_{k=0}^{m} \varepsilon^k v_k^{\pm}(s,\varepsilon\xi) + \varepsilon^2 \sum_{k=0}^{m} \varepsilon^k \psi_k^{\pm}(s,\xi) \Bigg) \Bigg|_{\varepsilon=0},
\end{aligned}
$$

$$
\tilde{W}_i^{\pm} \equiv \frac{1}{i!}\frac{d^i}{d\varepsilon^i} \sum_{k=0}^{m} \varepsilon^k w_k^{\pm}(s,\varepsilon\xi) \Bigg|_{\varepsilon=0}, \quad \tilde{Z}_i^{\pm} \equiv \frac{1}{i!}\frac{d^i}{d\varepsilon^i} \sum_{k=0}^{m} \varepsilon^k z_k^{\pm}(s,\varepsilon\xi) \Bigg|_{\varepsilon=0},
$$

$$
\tilde{P}_i^{\pm} \equiv \frac{1}{i!}\frac{d^i}{d\varepsilon^i} \left(\widetilde{\sum_{k=0}^{m} \varepsilon^k M^{\varepsilon} w_k^{\pm}} \right) \Bigg|_{\varepsilon=0}, \quad \tilde{Q}_i^{\pm} \equiv \frac{1}{i!}\frac{d^i}{d\varepsilon^i} \left(D \widetilde{\sum_{k=0}^{m} \varepsilon^k M^{\varepsilon} z_k^{\pm}} \right) \Bigg|_{\varepsilon=0},
$$

$f_v^{i\pm}$, $g_u^{i\pm}$, $\tilde{f}_v^{i\pm}$, $\tilde{g}_v^{i\pm}$, and the others are similarly defined. Here we used the following notation

$$\tilde{u} = \tilde{u}(s,\xi) \equiv u(s,\varepsilon\xi)$$

for the function u defined on a tubular neighborhood $U_d(\Gamma^0)$. $\tilde{P}_k^{\pm}$ and $\tilde{Q}_k^{\pm}$ depend on $w_0^{\pm}, z_0^{\pm}, \dots, w_k^{\pm}, z_k^{\pm}$.

Now we shall show that $\zeta_k^{\pm}$ and $\eta_k^{\pm}$ are uniquely determined. First we define a functional space. Since the definition domain of $\zeta_k^{\pm}$ and $\eta_k^{\pm}$ is semi-infinite, the inhomogeneous terms of their equations must have some decaying property for solvability.

Definition 3.1 *Let $\mathcal{E}^{\pm}$ be the set of functions $E^{\pm}(s,\xi,\varepsilon)$ defined on $\Gamma^0 \times I^{\mp} \times [0,\varepsilon_0)$ with the property that for each C^{∞} linear differential operator D of any order in the variables s and ξ, there exist positive constants $C_{\pm}$ and K (possibly depending on D and $E^{\pm}$, but not on s, ξ, and ε) with $|DE^{\pm}| \le Ke^{-C_{\pm}|\xi|}$.*

We see that $\dot{\phi}_0^{\pm}$ (>0) are fundamental solutions of the equation for $\zeta_k^{\pm}$. In fact, the boundary value problem

$$\begin{cases} \ddot{\phi}_0^{\pm} + f(h_{\pm}(v^*) + \phi_0^{\pm}, v^*) = 0 \\ \phi_0^{\pm}(0) = \alpha^* - h_{\pm}(v^*), \quad \phi_0^{\pm}(\mp\infty) = 0 \end{cases}$$

has a unique monotone increasing solution $\phi_0^{\pm}(\xi) \in \mathcal{E}^{\pm}$. More precisely, the boundary value problem

$$\begin{cases} \ddot{\Phi} + f(\Phi, v^*) = 0 \\ \Phi(0) = \alpha^*, \quad \Phi(\mp\infty) = h_{\pm}(v^*) \end{cases}$$

has a unique solution that is bounded and uniformly continuous on $\mathbf{R}$, and continuously differentiable. By using this function Φ, $\phi_0^{\pm}$ are represented as

$$\phi_0^{\pm}(\xi) \equiv \Phi(\xi) - h_{\pm}(v^*)$$

(see Lemma 6 of Ikeda [6]). Since $\dot{\phi}_0^{+}$ and $\dot{\phi}_0^{-}$ (moreover, their derivatives) are continuous at $\xi = 0$, we omit the superscript of $\dot{\phi}_0^{\pm}$. Then, it is obvious that $p(\xi) = \dot{\phi}_0(\xi)$ satisfies

$$\ddot{p} + \tilde{f}_u^{0\pm} p = 0,$$

where $\tilde{f}_u^{0\pm} \equiv f_u(h_{\pm}(v^*) + \phi_0^{\pm}, v^*)$.

In a similar way as in Ikeda [6], we can prove that the right hand side of $(3.7)_k$ $(k \ge 1)$ belongs to $\mathcal{E}^{\pm}$. Then we have

Lemma 3.2 *For $s \in \Gamma^0$, the boundary value problem*

$$\begin{cases} \partial_\xi^2 p^\pm(s,\xi) + f_u(h_\pm(v^*) + \phi_0^\pm(\xi), v^*)p^\pm(s,\xi) = R^\pm(s,\xi), \quad \xi \in I^\mp \\ p^\pm(s,0) = p_0^\pm(s), \quad p^\pm(s,\mp\infty) = 0, \end{cases}$$

has a unique solution $p^\pm(s,\xi) \in \mathcal{E}^\pm$ for $p_0^\pm(s) \in C^\infty(\Gamma^0)$ and $R^\pm(s,\xi) \in \mathcal{E}^\pm$.

Proof. See Lemma 9 of Ikeda [6]. □

Noting the fact that $\dot{\phi}_0$ is a fundamental solution of the equation for $\zeta_k^\pm(s,\xi)$, we see that the solutions $\zeta_k^\pm(s,\xi)$ and $\eta_k^\pm(s,\xi)$ of $(3.7)_k$ and $(3.8)_k$ have the following expressions:

$$\begin{aligned} \zeta_k^\pm(s,\xi) &= \frac{\dot{\phi}_0(\xi)}{\dot{\phi}_0(0)}(\Theta_k(s) - w_k^\pm(s,0)) + \dot{\phi}_0(\xi)\int_0^\xi (\dot{\phi}_0(t))^{-2} \\ &\qquad \times \int_{\mp\infty}^t \{\text{the right hand side of the} \\ &\qquad\qquad \text{equation for } \zeta_k^\pm\}\dot{\phi}_0(z)dzdt, \\ \eta_k^\pm(s,\xi) &= \int_{\mp\infty}^\xi \int_{\mp\infty}^y \{\text{the right hand side of the} \\ &\qquad\qquad \text{equation for } \eta_k^\pm\}dtdy \quad (1 \le k \le m). \end{aligned}$$

By using the same argument as in Ikeda [6], we obtain the following proposition.

Proposition 3.3 *There are $\varepsilon_0 > 0$ and $C > 0$ such that for any $\varepsilon \in (0,\varepsilon_0)$, $(3.3)_\pm$ have a solution $(w_\pm^m(x,\varepsilon), z_\pm^m(x,\varepsilon))$ of the form $(3.4)_\pm$ and satisfy*

$$\|w_\pm^m(x,\varepsilon) - (W_m^\pm(x,\varepsilon) + \Xi_m^\pm(x,\varepsilon))\|_{C_{\varepsilon,0}^{2+\alpha}(\overline{\Omega}^{*\pm})} \le C\varepsilon^{m+1-\alpha},$$

$$\|z_\pm^m(x,\varepsilon) - (Z_m^\pm(x,\varepsilon) + \varepsilon^2\Pi_m^\pm(x,\varepsilon))\|_{C_{\varepsilon,0}^{2+\alpha}(\overline{\Omega}^{*\pm})} \le C\varepsilon^{m+1-\alpha}$$

for any $\alpha \in (0,1)$.

Let

$$w^m(x,\varepsilon) = \begin{cases} w_+^m(x,\varepsilon) & x \in \Omega^{*+}, \\ w_-^m(x,\varepsilon) & x \in \Omega^{*-}. \end{cases}$$

$$z^m(x,\varepsilon) = \begin{cases} z_+^m(x,\varepsilon) & x \in \Omega^{*+}, \\ z_-^m(x,\varepsilon) & x \in \Omega^{*-}. \end{cases}$$

In order that $(w^m(x,\varepsilon), z^m(x,\varepsilon))$ become solutions of (3.1), they must satisfy the C^1-matching conditions in each order $O(\varepsilon^m)$. That is,

$$\dot{\zeta}_0^+(s,0) = \dot{\zeta}_0^-(s,0), \quad \frac{\partial z_0^+}{\partial \nu}(s,0) = \frac{\partial z_0^-}{\partial \nu}(s,0), \tag{3.9$_0$}$$

$k \geq 1$

$$\begin{aligned} &\dot{\zeta}_k^+(s,0) + \frac{\partial w_{k-1}^+}{\partial \nu}(s,0) = \dot{\zeta}_k^-(s,0) + \frac{\partial w_{k-1}^-}{\partial \nu}(s,0), \\ &\frac{\partial z_k^+}{\partial \nu}(s,0) + \dot{\eta}_{k-1}^+(s,0) = \frac{\partial z_k^-}{\partial \nu}(s,0) + \dot{\eta}_{k-1}^-(s,0). \end{aligned} \tag{3.9$_k$}$$

By using the above relations, we have

Theorem 3.4 *There exist an $\varepsilon_0 > 0$ and an integer $n_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and integer $n \in [1, n_0]$, the asymptotic form of the principal parts of the eigenvalue λ^ε and the eigenfunction $(w^{m(n)}(x,\varepsilon), z^{m(n)}(x,\varepsilon))$ are, respectively, given by the following forms*:

$$\lambda^\varepsilon \approx \varepsilon^2 \lambda_2^{(n)},$$

$$w^{m(n)}(x,\varepsilon) \approx \begin{cases} \omega\left(\frac{Y(x)}{d}\right)\frac{\dot{\phi}_0(Y(x)/\varepsilon)}{\dot{\phi}_0(0)}\Theta_0^{(n)}(S(x)), & x \in U_d(\Gamma^0), \\ 0, & x \in \Omega^* \backslash U_d(\Gamma^0), \end{cases} \tag{3.10}$$

$$z^{m(n)}(x,\varepsilon) \approx -\varepsilon^2 \frac{[g]}{D\dot{\phi}_0(0)} K_N^*(\delta_{\Gamma^0} \otimes \Theta_0^{(n)}). \tag{3.11}$$

Here, $(\lambda_2^{(n)}, \Theta_0^{(n)})$ are determined by the following eigenvalue problem on Γ^0:

$$L^*\Theta_0^{(n)} + \frac{1}{D}c_2[g]J'(v^*)\left\langle K_N^*(\delta_{\Gamma^0} \otimes \Theta_0^{(n)}), \delta_{\Gamma^0}\right\rangle = \lambda_2^{(n)}\Theta_0^{(n)}$$

with

$$\int_{\Gamma^0} \Theta_0^{(n)} dS = 0,$$

where

$$L^* \equiv \Delta^{\Gamma^0} + \frac{1}{2}H_1(s) - c_1 P_3(s) + \hat{\Lambda}(s),$$

$K_N^* \equiv (-\Delta)^{-1}$ *with the Neumann boundary conditions on* $\partial\Omega^*$, $c_1 > 0$, $c_2 > 0$. *The definition of* $H_1(s)$ *and* $P_3(s)$ *are given in Lemma* 2.3, *and* $\hat{\Lambda}(s)$ *is a smooth bounded function of* $s \in \Gamma^0$.

In order to prove this theorem, we prepare the next lemma.

Lemma 3.5 *Let* $\Theta_0 \in C^\infty(\Gamma^0)$. *Then the followings hold;*

(i) *The conditions in* $(3.9)_0$ *determine* $w_0^\pm$ *and* $z_0^\pm$ *(hence* $q_0(s)$*), that is,*

$$w_0^\pm \equiv 0, \quad z_0^\pm \equiv 0 \quad (q_0(s) \equiv 0).$$

Then, $\zeta_0^\pm$ *and* $\eta_0^\pm$ *are given by*

$$\zeta_0^\pm(s,\xi) = \frac{\dot{\phi}_0(\xi)}{\dot{\phi}_0(0)}\Theta_0(s), \quad \eta_0^\pm \equiv 0. \tag{3.12}$$

(ii) *The conditions in* $(3.9)_1$ *determine* λ_1, $w_1^\pm$, *and* $z_1^\pm$ *(hence* $q_1(s)$*), that is,*

$$\lambda_1 = 0, \quad w_1^\pm \equiv 0, \quad z_1^\pm \equiv 0 \quad (q_1(s) \equiv 0).$$

(iii) $z_2^\pm$ *is determined by the conditions in* $(3.9)_2$. *That is,*

$$z_2 = -\frac{[g]}{D\dot{\phi}_0(0)} K_N^*(\delta_{\Gamma^0} \otimes \Theta_0) \tag{3.13}$$

provided that $\int_{\Gamma^0} \Theta_0 ds = 0$, *where* $[g] \equiv g(h_+(v^*), v^*) - g(h_-(v^*), v^*)$, $K_N^* \equiv (-\Delta)^{-1}$ *with the Neumann boundary conditions on* $\partial\Omega^*$. *Then,* q_2 *is given by*

$$q_2(s) = -\frac{[g]}{D\dot{\phi}_0(0)} \left\langle K_N^*(\delta_{\Gamma^0} \otimes \Theta_0), \delta_{\Gamma^0} \right\rangle, \tag{3.14}$$

where $\langle\ ,\ \delta_{\Gamma^0}\rangle$ *denotes the trace operator on* Γ^0. *In fact, from the property of Neumann function, the trace of* z_2 *on* Γ^0 *is well-defined as a continuous function (see Friedman* [3]*).*

In the following of this paper, we omit the superscript $\pm$ of $\zeta_0^\pm$ since ζ_0^+ and ζ_0^- (moreover, their derivatives) are continuous at $\xi = 0$.

Proof of Lemma 3.5

(i) Solving the second equation of $(3.5)_0$ and $(3.6)_0$ with C^1-matching condition for $z_0^{\pm}$ (see $(3.9)_0$), we see that $z_0^{\pm}$ must be equal to some constant $\hat{z}_0$ and $w_0^{\pm}$ is given by $\hat{w}_0^{\pm} = -f_v^{0\pm}\hat{z}_0/f_u^{0\pm}$. Then, the equation of $\zeta_0^{\pm}$ (see $(3.7)_0$) is rewritten as

$$\begin{cases} \ddot{\zeta}_0^{\pm} + \tilde{f}_u^{0\pm}\zeta_0^{\pm} = -\tilde{f}_u^{0\pm}\hat{w}_0^{\pm} - \tilde{f}_v^{0\pm}\hat{z}_0, \\ \zeta_0^{\pm}(s,0) = \Theta_0(s) - \hat{w}_0^{\pm}, \quad \zeta_0^{\pm}(s,\mp\infty) = 0, \end{cases} \qquad \xi \in I^{\mp}, \quad s \in \Gamma^0,$$

Then the derivative of $\zeta_0^{\pm}$ with respect to ξ at $(s,0)$ is computed as follows:

$$\begin{aligned} \dot{\zeta}_0^{\pm}(s,0) &= \frac{\zeta_0^{\pm}(s,0)}{\dot{\phi}_0(0)}\ddot{\phi}_0(0) + \frac{1}{\dot{\phi}_0(0)}\int_{\mp\infty}^0 [-\tilde{f}_u^{0\pm}\hat{w}_0^{\pm} - \tilde{f}_v^{0\pm}\hat{z}_0]\dot{\phi}_0 d\xi \\ &= \frac{1}{\dot{\phi}_0(0)}\Bigg[(\Theta_0(s) - \hat{w}_0^{\pm})\ddot{\phi}_0(0) \\ &\qquad - \hat{w}_0^{\pm}\int_{\mp\infty}^0 \tilde{f}_u^{0\pm}\dot{\phi}_0 d\xi - \hat{z}_0\int_{\mp\infty}^0 \tilde{f}_v^{0\pm}\dot{\phi}_0 d\xi\Bigg] \\ &= \frac{1}{\dot{\phi}_0(0)}\left[\Theta_0(s)\ddot{\phi}_0(0) - \hat{z}_0\int_{h_{\pm}(v^*)}^{\alpha^*} f_v(u,v^*)du\right]. \end{aligned} \tag{3.15}$$

Here we used the fact that

$$\int_{\mp\infty}^0 \tilde{f}_u^{0\pm}\dot{\phi}_0 d\xi = -\ddot{\phi}_0(0) \quad \text{and} \quad \zeta_0^{\pm}(s,0) = \Theta_0(s) - \hat{w}_0^{\pm}.$$

By using (3.15), the first relation of $(3.9)_0$ is rewritten as

$$\begin{aligned} 0 &= \dot{\zeta}_0^{+}(s,0) - \dot{\zeta}_0^{-}(s,0) \\ &= \frac{\hat{z}_0}{\dot{\phi}_0(0)}\int_{h_-(v^*)}^{h_+(v^*)} f_v(u,v^*)du. \end{aligned}$$

From which, we obtain $\hat{z}_0 = 0$ since $J'(v^*) = \int_{h_-(v^*)}^{h_+(v^*)} f_v(u,v^*)du \neq 0$ (see (A.2)). This yields $w_0^{\pm} \equiv 0 \equiv z_0^{\pm}$ and $q_0(s) \equiv 0$. Then we see that $\zeta_0^{\pm}$ is represented as in (3.12). On the other hand, noting the second equation of $(3.7)_0$ and $(3.8)_0$, we easily see that $\eta_0^{\pm} \equiv 0$.

(ii) By using the results of (i), $(3.5)_1$ and C^1-matching condition for

$z_1^{\pm}$ (see $(3.9)_1$) are rewritten as

$$\begin{cases} f_u^{0\pm} w_1^{\pm} + f_v^{0\pm} z_1^{\pm} = 0, \\ DM_0 z_1^{\pm} = 0, \end{cases} \quad \text{in } \Omega^{*\pm}, \tag{3.16}$$

$$\frac{\partial z_1^+}{\partial \nu}(s,0) = \frac{\partial z_1^-}{\partial \nu}(s,0). \tag{3.17}$$

From (3.16), $(3.6)_1$, and (3.17), we see that $z_1^{\pm}$ must be equal to some constant $\hat{z}_1$ and $w_1^{\pm}$ is given by $\hat{w}_1^{\pm} = -f_v^{0\pm}\hat{z}_1/f_u^{0\pm}$. Then, the C^1-matching condition of $\zeta_1^{\pm}$ becomes

$$\zeta_1^+(s,0) = \zeta_1^-(s,0) \tag{3.18}$$

(see $(3.9)_1$).

Here, we prove that (3.18) is equivalent to the next equation

$$\Theta_0 \lambda_1 \int_{-\infty}^{\infty} \dot{\phi}_0^2 d\xi = -\hat{z}_1 \dot{\phi}_0(0) J'(v^*). \tag{3.19}$$

If (3.19) is satisfied, we can conclude that $\lambda_1 = 0$ and $\hat{z}_1 = 0$ because Θ_0 is arbitrary and $\int_{-\infty}^{\infty} \dot{\phi}_0^2 d\xi$, $\dot{\phi}_0(0)$ and $J'(v^*)$ are non-zero. Hence $w_1^{\pm} \equiv 0$.

First note that $\zeta_1^{\pm}$ satisfies

$$\ddot{\zeta}_1^{\pm} + \tilde{f}_u^{0\pm}\zeta_1^{\pm} = \frac{\Theta_0}{\dot{\phi}_0(0)}[\lambda_1\dot{\phi}_0 + \hat{H}^{\pm}] - \tilde{f}_u^{0\pm}\hat{w}_1^{\pm} - \tilde{f}_v^{0\pm}\hat{z}_1,$$

where

$$\hat{H}^{\pm} = -(N-1)H_0\ddot{\phi}_0 - \tilde{f}_u^{1\pm}\dot{\phi}_0.$$

Then the derivative of $\zeta_1^{\pm}$ with respect to ξ at $(s,0)$ is computed as follows:

$$\begin{aligned} \dot{\zeta}_1^{\pm}(s,0) &= \frac{\zeta_1^{\pm}(s,0)}{\dot{\phi}_0(0)}\ddot{\phi}_0(0) + \frac{\Theta_0(s)}{(\dot{\phi}_0(0))^2}\int_{\mp\infty}^{0}[\lambda_1\dot{\phi}_0 + \hat{H}^{\pm}]d\xi \\ &\quad - \frac{\hat{w}_1^{\pm}}{\dot{\phi}_0(0)}\int_{\mp\infty}^{0}\tilde{f}_u^{0\pm}\dot{\phi}_0 d\xi - \frac{\hat{z}_1}{\dot{\phi}_0(0)}\int_{\mp\infty}^{0}\tilde{f}_v^{0\pm}\dot{\phi}_0 d\xi \\ &= \frac{\Theta_1(s)}{\dot{\phi}_0(0)}\ddot{\phi}_0(0) + \frac{\Theta_0(s)}{(\dot{\phi}_0(0))^2}\int_{\mp\infty}^{0}[\lambda_1\dot{\phi}_0 + \hat{H}^{\pm}]d\xi \\ &\quad - \frac{\hat{z}_1}{\dot{\phi}_0(0)}\int_{h_{\pm}(v^*)}^{\alpha^*} f_v(u,v^*)du. \end{aligned} \tag{3.20}$$

Here we used the fact that

$$\int_{\mp\infty}^{0} \tilde{f}_u^{0\pm}\dot{\phi}_0 d\xi = -\ddot{\phi}_0(0) \quad \text{and} \quad \zeta_1^{\pm}(s,0) = \Theta_1(s) - \hat{w}_1^{\pm}.$$

By using (3.20), the first relation of (3.18) is rewritten as

$$\begin{aligned} 0 &= \dot{\zeta}_1^{+}(s,0) - \dot{\zeta}_1^{-}(s,0) \\ &= \frac{\Theta_0(s)}{(\dot{\phi}_0(0))^2}\Big[\lambda_1 \int_{-\infty}^{+\infty} (\dot{\phi}_0(t))^2 dt \\ &\quad + \int_{-\infty}^{0} \hat{H}^{+}\dot{\phi}_0(t)dt + \int_{0}^{+\infty} \hat{H}^{-}\dot{\phi}_0(t)dt\Big] \\ &\quad - \frac{\hat{z}_1}{\dot{\phi}_0(0)} \int_{h_+(v^*)}^{h_-(v^*)} f_v(u,v^*)du. \end{aligned} \tag{3.21}$$

In order to compute right hand side of (3.21), we note that $\phi_1^{\pm}$ satisfies

$$\begin{aligned} \ddot{\phi}_1^{\pm} + \tilde{f}_u^{0\pm}\phi_1^{\pm} &= -\tilde{M}_1\phi_0^{\pm} - \tilde{f}_u^{0\pm}\{u_1^{\pm}(s,0) + u_{0\tau}^{\pm}(s,0)\xi\} \\ &\quad - \tilde{f}_v^{0\pm}\{v_1^{\pm}(s,0) + v_{0\tau}^{\pm}(s,0)\xi\} \end{aligned}$$

and then, $p^{\pm} \equiv \dot{\phi}_1^{\pm}$ satisfies

$$\ddot{p}^{\pm} + \tilde{f}_u^{0\pm}p^{\pm} = \hat{H}^{\pm}.$$

From which we obtain

$$\dot{p}^{\pm}(s,0) = \frac{\ddot{\phi}_0(0)}{\dot{\phi}_0(0)}p^{\pm}(s,0) + \frac{1}{\dot{\phi}_0(0)}\int_{\mp\infty}^{0} \hat{H}^{\pm}(s,\xi)\dot{\phi}_0(\xi)d\xi. \tag{3.22}$$

Combining (3.22) with the fact that $\dot{\phi}_1^{+}(s,0) = \dot{\phi}_1^{-}(s,0)$ and $\ddot{\phi}_1^{+}(s,0) = \ddot{\phi}_1^{-}(s,0)$, we obtain the following relation:

$$\int_{-\infty}^{0} \hat{H}^{+}(s,\xi)\dot{\phi}_0(\xi)d\xi + \int_{0}^{\infty} \hat{H}^{-}(s,\xi)\dot{\phi}_0(\xi)d\xi = 0. \tag{3.23}$$

Using (3.21) and (3.23), we have (3.19).

(iii) Using the result of (i) and (ii), we can rewrite the equations for

$z_2^{\pm}$ as follows:

$$\begin{cases} D\Delta z_2^{\pm} = 0 \quad \text{in} \ \ \Omega^{*\pm}, \\ z_2^{+} = q_2 = z_2^{-}, \quad \dfrac{\partial z_2^{+}}{\partial \nu} - \dfrac{\partial z_2^{-}}{\partial \nu} = -\{\dot{\eta}_1^{+}(s,0) - \dot{\eta}_1^{-}(s,0)\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on} \ \ \Gamma^0, \\ \dfrac{\partial z_2^{-}}{\partial n} = 0 \quad \text{on} \ \ \partial\Omega^{*}. \end{cases} \tag{3.24}$$

First we compute the C^1-matching condition for $z_2^{\pm}$. Integrating the equation for $\eta_1^{\pm}$ in $(3.7)_1$, we have

$$\begin{aligned} \dot{\eta}_1^{\pm}(s,0) &= -\frac{\Theta_0}{D\dot{\phi}_0(0)} \int_{\mp\infty}^{0} g_u(h_{\pm}(v^*) + \phi_0^{\pm}(\xi), v^*)\dot{\phi}_0(\xi)d\xi \\ &= -\frac{\Theta_0}{D\dot{\phi}_0(0)} \int_{h_{\pm}(v^*)}^{\alpha} g_u(u, v^*)du. \end{aligned}$$

This yields

$$\dot{\eta}_1^{-}(s,0) - \dot{\eta}_1^{+}(s,0) = -\frac{[g]}{D\dot{\phi}_0(0)}\Theta_0.$$

Now we rewrite (3.24) in a weak form:

$$B^*(z_2, \phi) = -\frac{[g]}{D\dot{\phi}_0(0)} \langle \delta_{\Gamma^0} \otimes \Theta_0, \phi \rangle,$$

where

$$B^*(z_2, \phi) \equiv \int_{\Omega} \nabla z_2 \cdot \nabla \phi.$$

For a given $h \in H^{-1}(\Omega^*)$, we consider the equation for $z \in H^1(\Omega^*)$:

$$B^*(z, \phi) = \langle h, \phi \rangle \qquad \text{for any} \quad \phi \in H^1(\Omega^*).$$

Then we can define the mapping K_N^* as

$$K_N^*(h) = z; \ H^{-1}(\Omega^*) \to H^1(\Omega^*)$$

provided that $\int_{\Omega^*} h dx = 0$. Using this operator, we derive (3.13) provided that

$$\int_{\Omega^*} \delta_{\Gamma^0} \otimes \Theta_0 dx = \int_{\Gamma^0} \Theta_0 dS,$$

where dS is surface element of Γ^0. This completes the proof of Lemma 3.5. $\square$

Using Lemma 3.5, we can conclude that the principal part of $(w^\varepsilon, z^\varepsilon)$ is given by (3.10) and (3.11). The remaining problem is to determine λ_2 and Θ_0. Concerning this, we have

Proposition 3.6 *Θ_0 and λ_2 are determined by the following eigenvalue problem on Γ^0 :*

$$\left(\Delta^{\Gamma^0} + \frac{1}{2}H_1(s) - c_1 P_3(s)\right)\Theta_0 + \hat{\Lambda}(s)\Theta_0 + \frac{1}{D}c_2[g]J'(v^*)\left\langle K_N^*(\delta_{\Gamma^0}\otimes\Theta_0), \delta_{\Gamma^0}\right\rangle = \lambda_2\Theta_0 \tag{3.25}$$

with

$$\int_{\Gamma^0}\Theta_0 dS = 0,$$

where

$$c_1 = \int_{-\infty}^{\infty}(\ddot{\phi}_0)^2 d\xi \Big/ \int_{-\infty}^{\infty}(\dot{\phi}_0)^2 d\xi > 0, \quad c_2 = 1\Big/\int_{-\infty}^{\infty}(\dot{\phi}_0)^2 d\xi > 0,$$

$$\hat{\Lambda}(s) = \left[\int_{-\infty}^{0}\tilde{f}_u^{2+}(\dot{\phi}_0)^2 d\xi + \int_0^{\infty}\tilde{f}_u^{2-}(\dot{\phi}_0)^2 d\xi - \int_{-\infty}^{0}R_1^+[\dot{\phi}_0\zeta_1^{\dagger+}](s,\xi)d\xi - \int_0^{\infty}R_1^-[\dot{\phi}_0\zeta_1^{\dagger-}](s,\xi)d\xi\right] \Big/ \int_{-\infty}^{\infty}(\dot{\phi}_0)^2 d\xi,$$

$R_1^\pm[\zeta^\pm] = -\tilde{M}_1\zeta^\pm - \tilde{f}_u^{1\pm}\zeta^\pm$ for $\zeta^\pm \in \mathcal{E}^\pm$, and the definitions of $H_1(s)$ and $P_3(s)$ are given in Lemma 2.3.

Proof of Proposition 3.6 First we rewrite the equation of $\zeta_2^\pm$.

$$\ddot{\zeta}_2^\pm + \tilde{f}_u^{0\pm}\zeta_2^\pm = \lambda_2\zeta_0 - \tilde{M}_2\zeta_0 - \tilde{f}_u^{2\pm}\zeta_0 - \tilde{M}_1\zeta_1^\pm - \tilde{f}_u^{1\pm}\zeta_1^\pm - \tilde{f}_u^{0\pm}w_2^\pm(s,0) - \tilde{f}_v^{0\pm}q_2(s), \tag{3.26}$$

$$\zeta_2^\pm(s,0) = \Theta_2(s) - w_2^\pm(s,0), \quad \zeta_2^\pm(s,\mp\infty) = 0.$$

Then

$$\begin{aligned}
\dot{\zeta}_2^{\pm}(s,0) &= \frac{\ddot{\phi}_0(0)}{\dot{\phi}_0(0)}\zeta_2^{\pm}(s,0) + \frac{1}{\dot{\phi}_0(0)}\int_{\mp\infty}^{0} \\
&\qquad\qquad \{\text{the right hand side of (3.26)}\}\dot{\phi}_0(\xi)d\xi \\
&= \frac{\ddot{\phi}_0(0)}{\dot{\phi}_0(0)}\{\Theta_2(s) - w_2^{\pm}(s,0)\} \\
&\quad + \frac{1}{\dot{\phi}_0(0)}\Bigg[\int_{\mp\infty}^{0}\{\lambda_2\zeta_0 - \tilde{M}_2\zeta_0 - \tilde{f}_u^{2\pm}\zeta_0 \\
&\qquad\qquad - \tilde{M}_1\zeta_1^{\pm} - \tilde{f}_u^{1\pm}\zeta_1^{\pm}\}\dot{\phi}_0(\xi)d\xi \\
&\quad - w_2^{\pm}(s,0)\int_{h_{\pm}(v^*)}^{\alpha} f_u(u,v^*)du \\
&\qquad\qquad - q_2(s)\int_{h_{\pm}(v^*)}^{\alpha} f_v(u,v^*)du\Bigg] \\
&= \frac{\ddot{\phi}_0(0)}{\dot{\phi}_0(0)}\Theta_2(s) + \frac{1}{\dot{\phi}_0(0)}\Bigg[\int_{\mp\infty}^{0}\{\lambda_2\zeta_0 - \tilde{M}_2\zeta_0 - \tilde{f}_u^{2\pm}\zeta_0 \\
&\quad - \tilde{M}_1\zeta_1^{\pm} - \tilde{f}_u^{1\pm}\zeta_1^{\pm}\}\dot{\phi}_0(\xi)d\xi - q_2(s)\int_{h_{\pm}(v^*)}^{\alpha} f_v(u,v^*)du\Bigg].
\end{aligned}$$

Using the above equations and the fact that $w_1^{\pm} \equiv 0$, we can rewrite the first relation of $(3.9)_2$ as follows:

$$\begin{aligned}
0 &= \dot{\zeta}_2^{+}(s,0) - \dot{\zeta}_2^{-}(s,0) \\
&= \frac{1}{\dot{\phi}_0(0)}\int_{-\infty}^{\infty}\{\lambda_2\zeta_0 - \tilde{M}_2\zeta_0\}\dot{\phi}_0(\xi)d\xi \\
&\quad - \frac{1}{\dot{\phi}_0(0)}\left[\int_{-\infty}^{0}\tilde{f}_u^{2+}\zeta_0\dot{\phi}_0(\xi)d\xi + \int_0^{\infty}\tilde{f}_u^{2-}\zeta_0\dot{\phi}_0(\xi)d\xi\right] \\
&\quad + \frac{1}{\dot{\phi}_0(0)}\Bigg[\int_{-\infty}^{0}(-\tilde{M}_1\zeta_1^{+} - \tilde{f}_u^{1+}\zeta_1^{+})\dot{\phi}_0(\xi)d\xi \\
&\qquad\qquad + \int_0^{\infty}(-\tilde{M}_1\zeta_1^{-} - \tilde{f}_u^{1-}\zeta_1^{-})\dot{\phi}_0(\xi)d\xi\Bigg] \\
&\qquad\qquad + \frac{1}{\dot{\phi}_0(0)}J'(v^*)q_2(s). \qquad (3.27)
\end{aligned}$$

By using the definition of $\tilde{M}_2$ (see (2.14)) and by integrating by parts, the first term of the right hand side of (3.27) (which we call Part 1 simply and similarly Part k for k-th term) becomes

$$\begin{aligned}\text{Part1} &= \frac{1}{(\dot{\phi}_0(0))^2}\left[\left(\lambda_2 - \Delta^{\Gamma^0} - \frac{1}{2}H_1(s)\right)\Theta_0 \int_{-\infty}^{\infty}(\dot{\phi}_0)^2 d\xi \right. \\ &\quad \left. + P_3(s)\Theta_0 \int_{-\infty}^{\infty}(\ddot{\phi}_0)^2 d\xi\right].\end{aligned}$$

Concerning Part 2, we have

$$\text{Part 2} = -\frac{1}{(\dot{\phi}_0(0))^2}\left[\int_{-\infty}^{0} \tilde{f}_u^{2+}(\dot{\phi}_0)^2 d\xi + \int_0^{\infty} \tilde{f}_u^{2-}(\dot{\phi}_0)^2 d\xi\right]\Theta_0(s).$$

In order to compute Part 3, we note that $\zeta_1^{\pm}$ is given by the following form:

$$\zeta_1^{\pm}(s,\xi) = \frac{1}{\dot{\phi}_0(0)}\Theta_1(s)\dot{\phi}_0(\xi) + \frac{1}{\dot{\phi}_0(0)}\Theta_0(s)\dot{\phi}_0(\xi)\zeta_1^{\dagger\pm}(s,\xi) \tag{3.28}$$

where

$$\begin{aligned}\zeta_1^{\dagger\pm}(s,\xi) &= \int_0^{\xi}(\dot{\phi}_0(t))^{-2}\int_{\mp\infty}^{t}\{-(N-1)H_0(s)\ddot{\phi}_0(z) \\ &\quad - \tilde{f}_u^{1\pm}\dot{\phi}_0(z)\}\dot{\phi}_0(z)dzdt.\end{aligned}$$

Substituting (3.28) into Part 3, we can compute Part 3 as follows:

$$\begin{aligned}\text{Part 3} &= \frac{1}{(\dot{\phi}_0(0))^2}\left[\Theta_1(s)\left\{\int_{-\infty}^{0}\hat{H}^{+}(s,\xi)\dot{\phi}_0(\xi)d\xi \right.\right. \\ &\quad \left. + \int_0^{\infty}\hat{H}^{-}(s,z)\dot{\phi}_0(\xi)d\xi\right\} \\ &\quad + \Theta_0(s)\left\{\int_{-\infty}^{0}R_1^{+}[\dot{\phi}_0\zeta_1^{\dagger+}](s,\xi)d\xi \right. \\ &\quad \left.\left. + \int_0^{\infty}R_1^{-}[\dot{\phi}_0\zeta_1^{\dagger-}](s,\xi)d\xi\right\}\right] \\ &= \frac{1}{(\dot{\phi}_0(0))^2}\left[\int_{-\infty}^{0}R_1^{+}[\dot{\phi}_0\zeta_1^{\dagger+}](s,\xi)d\xi \right. \\ &\quad \left. + \int_0^{\infty}R_1^{-}[\dot{\phi}_0\zeta_1^{\dagger-}](s,\xi)d\xi\right]\Theta_0(s).\end{aligned}$$

Here we use the property of $\hat{H}^{\pm}$ (see (3.23)) to cancel Θ_1. Combining (3.14) with the above results, we can derive (3.25). This completes the proof of Proposition 3.6. □

Proof of Theorem 3.4 This is a direct consequence of Lemma 3.5 and Proposition 3.6. □

Appendix

Derivation of (2.14).

For $\tilde{u} = \tilde{u}(\tilde{s}, \xi)$, $\tilde{M}^{\varepsilon}$ is defined by

$$\begin{aligned} \tilde{M}^{\varepsilon}\tilde{u} &\equiv \Delta_{(s,y)}\tilde{u}\left(s, \frac{\tau(s,y,\varepsilon)}{\varepsilon}\right) \\ &\equiv \frac{1}{\sqrt{g}}\sum_{i=1}^{N-1}\frac{\partial}{\partial s^i}\left(\sqrt{g}\sum_{j=1}^{N-1}g^{ij}\frac{\partial}{\partial s^j}\tilde{u}\left(s, \frac{\tau(s,y,\varepsilon)}{\varepsilon}\right)\right) \qquad (1) \\ &\quad + \frac{\partial^2}{\partial y^2}\tilde{u}\left(s, \frac{\tau(s,y,\varepsilon)}{\varepsilon}\right) + (N-1)H(s,y)\frac{\partial}{\partial y}\tilde{u}\left(s, \frac{\tau(s,y,\varepsilon)}{\varepsilon}\right), \end{aligned}$$

where $y = \varepsilon\xi + \omega(\varepsilon\xi/d)\gamma(s,\varepsilon)$. Then,

$$\begin{aligned} &\frac{1}{\sqrt{g}}\sum_{i=1}^{N-1}\frac{\partial}{\partial s^i}\left(\sqrt{g}\sum_{j=1}^{N-1}g^{ij}\frac{\partial}{\partial s^j}\tilde{u}\left(s, \frac{\tau(s,y,\varepsilon)}{\varepsilon}\right)\right) \\ &\quad = \frac{1}{2g}\sum_{i=1}^{N-1}g_{s^i}\left(\sum_{j=1}^{N-1}g^{ij}\frac{\partial}{\partial s^j}\tilde{u}\right) \\ &\qquad + \sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\left(g^{ij}_{s^i}\frac{\partial}{\partial s^i}\tilde{u} + g^{ij}\frac{\partial^2}{\partial s^i \partial s^j}\tilde{u}\right), \qquad (2) \end{aligned}$$

Moreover

the first term of the last row in (2)

$$= \frac{1}{2g}\left[\sum_{i=1}^{N-1}g_{s^i}\sum_{j=1}^{N-1}g^{ij}\tilde{u}_{\tilde{s}^j} + \sum_{i=1}^{N-1}g_{s^i}\sum_{j=1}^{N-1}g^{ij}\frac{1}{\varepsilon}\tau_{s^j}\tilde{u}_{\xi}\right], \qquad (3)$$

and

the second term of the last row in (2)

$$\begin{aligned} &= \sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\left[g^{ij}_{s^i}\tilde{u}_{\tilde{s}^j} + g^{ij}\tilde{u}_{\tilde{s}^j\tilde{s}^i}\right] \\ &\quad + \sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\left[g^{ij}_{s^i}\frac{1}{\varepsilon}\tau_{s^j}\tilde{u}_{\xi} + g^{ij}\left(\frac{1}{\varepsilon}\tau_{s^j s^i}\tilde{u}_{\xi} + \frac{1}{\varepsilon^2}\tau_{s^j}\tau_{s^i}\tilde{u}_{\xi\xi}\right)\right] \end{aligned}$$

$$+\sum_{i=1}^{N-1}\sum_{j=1}^{N-1} g^{ij}\left(\frac{1}{\varepsilon}\tau_{s^i}\tilde{u}_{\tilde{s}^j\xi}+\frac{1}{\varepsilon}\tau_{s^j}\tilde{u}_{\tilde{s}^i\xi}\right), \tag{4}$$

here we use the notation $\tilde{u}_{\tilde{s}^i}=\frac{\partial}{\partial \tilde{s}^i}\tilde{u}$, and so on. We are interested in the order $O(1)$-term of (2). After some computation, we have

$$\tau_{s^i}=-\varepsilon\partial_{s^i}\gamma_1+O(\varepsilon^2),\quad \tau_{s^is^j}=-\varepsilon\partial_{s^is^j}\gamma_1+O(\varepsilon^2)$$

(note that the relation $\tau=\varepsilon\xi$). Then the dominant term of (2) is computed as

$$\begin{aligned}
&\frac{1}{\sqrt{G}}\Bigg[\frac{1}{2\sqrt{G}}\sum_{i=1}^{N-1}G_{\tilde{s}^i}\left(\sum_{j=1}^{N-1}G^{ij}\tilde{u}_{\tilde{s}^j}\right)\\
&\qquad+\sqrt{G}\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\left(G^{ij}_{\tilde{s}^i}\tilde{u}_{\tilde{s}^j}+G^{ij}\tilde{u}_{\tilde{s}^j\tilde{s}^i}\right)\Bigg]\\
&-\frac{1}{\sqrt{G}}\sum_{i=1}^{N-1}\frac{G_{\tilde{s}^i}}{2\sqrt{G}}\left(\sum_{j=1}^{N-1}G^{ij}\partial_{s^j}\gamma_1\tilde{u}_\xi\right)\\
&+\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\left[-G^{ij}_{\tilde{s}^i}\partial_{s^j}\gamma_1\tilde{u}_\xi+G^{ij}\left(-\partial_{s^is^j}\gamma_1\tilde{u}_\xi+\partial_{s^j}\gamma_1\partial_{s^i}\gamma_1\tilde{u}_{\xi\xi}\right)\right]\\
&-\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}G^{ij}\left(\partial_{s^i}\gamma_1\tilde{u}_{\tilde{s}^j\xi}+\partial_{s^j}\gamma_1\tilde{u}_{\tilde{s}^i\xi}\right),
\end{aligned}\tag{5}$$

here, we used the notation

$$G=G(s)\equiv g(s,0),$$

i.e., these are the dominant term of

$$g\left(s,\varepsilon\xi+\omega\left(\frac{\varepsilon\xi}{d}\right)\gamma(s,\varepsilon)\right)$$

with respect to ε regarded as a function in s and ξ. Similarly G^{ij} and others are defined. Then, the first row of (5) is equal to

$$\frac{1}{\sqrt{G}}\sum_{i=1}^{N-1}\frac{\partial}{\partial\tilde{s}^i}\left(\sqrt{G}\sum_{j=1}^{N-1}G^{ij}\frac{\partial}{\partial\tilde{s}^j}\tilde{u}\right). \tag{6}$$

Noting that G^{ij} is the contravariant metric tensor for the manifold Γ_0 of dimension $N-1$ and $G=\det(G^{ij})$, we see that (6) is Laplace-Beltrami's operator defined on Γ_0. The last row of (5) is a linear combination of the differential operator in s and ξ those coefficients are independent of ξ.

On the other hand, applying the implicit function theorem to (2.5), we see that

$$\tau_y = 1 + O(\varepsilon), \quad \tau_{yy} = O(\varepsilon^2).$$

Then the third row of (1) is expanded as

$$\frac{1}{\varepsilon^2}\frac{\partial^2}{\partial \xi^2}\tilde{u} + \frac{1}{\varepsilon}(N-1)H_0\frac{\partial}{\partial \xi}\tilde{u} - H_1(s)(\xi + \gamma_1)\frac{\partial}{\partial \xi}\tilde{u} + O(\varepsilon), \tag{7}$$

here we used the following expansion:

$$H(s, \varepsilon\xi + \omega(\frac{\varepsilon\xi}{d})\gamma(s,\varepsilon)) = H_0(s) - \varepsilon H_1(s)(\xi + \gamma_1) + O(\varepsilon^2).$$

Using (5), (6) and (7), we obtain the expansions in (2.14).

Note added in Proof. The author has recently succeeded in smplifying the representation of L^* in Theorem A as $L^* \equiv \Delta^{\Gamma^0} + H_1(s) + c_2 J'(v^*)\frac{\partial v_1}{\partial \nu}|_{\Gamma^0}$. This result enables us to construct the stationary interfacial patterns and to study the stability property of them. For the details, see the forthcoming paper.

Acknowledgments The author expresses his sincere gratitude to Professor Yasumasa Nishiura, Hokkaido University, for valuable discussions and continuous encouragement. He is greatly indebted to Professor Masayasu Mimura, University of Tokyo, for beneficial conversations. He would like to thank Professor Shyozo Niizeki, Kochi University, for his long-term encouragement. Special thanks goes to Professor Kunimochi Sakamoto, Hiroshima University, for variable comments and suggestions to improve the article.

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