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Solutions of second order homogeneous Dirichlet systems

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Abstract. In this paper we establish some existence theorems for second order nonlinear systems of the form $y''=f(t,y,y')$, $y(0)=y(1)=0$. We also give two uniqueness results.

Key words: Nonlinear systems, *a priori* estimates.

1. Introduction

In this paper we consider the following homogeneous system

$$y''=f(t,y,y'),\quad y(0)=y(1)=0, \tag{$\mathcal{D}_h$}$$

where $f:[0,1]\times\mathbb{R}^m\times\mathbb{R}^m\rightarrow\mathbb{R}^m$ is continuous.

By a solution to the above problem we mean a function $y\in C^2([0,1],\mathbb{R}^m)$.

The existence of solutions to $(\mathcal{D}_h)$ has been studied extensively in recent years (see [3] when $m=1$, [9] when $m\geq 1$ and their references). In any case growth conditions are imposed on f in order to obtain *a priori* estimates. Then the transversality theorem [5] is applied and the existence of a solution is established. The more general nonlinearities treated in [3] include all the nonlinearities previously studied in this setting.

The purpose of this paper is to improve and complement the results of [3] and [9]. The proofs use in a decisive manner the theory of positive operators in finite dimensions (see [4]).

We shall denote by $|x|$ the euclidean norm of $x\in\mathbb{R}^m$ and by $||A||$ the spectral norm of an $m\times m$ matrix A. Finally, we denote by $||y||_p$ the L^p norm of $y\in L^p((0,1),\mathbb{R}^m))$.

In Section 2 we provide some preliminary results from the theory of nonnegative matrices. The existence theorems are presented in Section 3. Finally the uniqueness question is examined in Section 4.

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2. Preliminaries

The following results are needed in the sequel. We refer the reader to [1] for proofs. We consider the proper cone

$$\mathbb{R}^m_+ = \{x = (x_1, \ldots, x_m) \in \mathbb{R}^m;\ x_j \geq 0,\ j = 1, \ldots, m\}.$$

Definition 1 An $m \times m$ matrix A is called $\mathbb{R}^m_+$-monotone if

$$Ax \in \mathbb{R}^m_+ \Rightarrow x \in \mathbb{R}^m_+.$$

An $m \times m$ matrix $N = (n_{jk})_{1 \leq j,k \leq m}$ is nonnegative if $n_{jk} \geq 0$ for $j, k = 1, \ldots, m$.

Theorem 1 ([1] p. 113). *An $m \times m$ matrix A is $\mathbb{R}^m_+$-monotone if and only if it is nonsingular and A^{-1} is nonnegative.*

Theorem 2 ([1] p. 113). *Let $A = \alpha I - N$ where $\alpha \in \mathbb{R}$ and N is an $m \times m$ nonnegative matrix. Then the following are equivalent:*

(i) *The matrix A is $\mathbb{R}^m_+$-monotone;*

(ii) *$\rho(N) < \alpha$ where $\rho(N)$ denotes the spectral radius of N.*

3. Existence theorems

We first establish the following theorem.

Theorem 3 *Let $f \in C([0,1] \times \mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^m)$. Assume that f has the decomposition*

$$f(t, y, p) = g(t, y, p) + h(t, y, p)$$

where $g = (g_1, \ldots, g_m)$ and $h = (h_1, \ldots, h_m)$ satisfy the following hypotheses:

(H1) *$y_j g_j(t, y, p) \geq 0$ for $j = 1, \ldots, m$ and $(t, y, p) \in [0,1] \times \mathbb{R}^m \times \mathbb{R}^m$;*

(H2) *There are constants a_{jk}, $b_{jk} \geq 0$ and $c_j \geq 0$, $j, k = 1, \cdots, m$ such that*

$$|h_j(t, y, p)| \leq \sum_{k=1}^{m} (a_{jk}|y_k| + b_{jk}|p_k|) + c_j \tag{1}$$

for $j = 1, \ldots, m$ and $(t, y, p) \in [0,1] \times \mathbb{R}^m \times \mathbb{R}^m$, with

$$\rho(M) < \pi^2 \tag{2}$$

where $M = (a_{jk} + \pi b_{jk})_{1 \le j,k \le m}$;

(H3) $|g(t,y,p)| \le A(t,y)w(|p|^2)$ *for* $(t,y,p) \in [0,1] \times \mathbb{R}^m \times \mathbb{R}^m$, *where* $A(t,y)$ *is bounded on bounded subsets of* $[0,1] \times \mathbb{R}^m$ *and* $w \in C([0,\infty),(0,\infty))$ *is a nondecreasing function such that*

$$\int_0^\infty \frac{ds}{w(s)} = \infty.$$

Then the Dirichlet system $(\mathcal{D}_h)$ *has a solution* $y \in C^2([0,1],\mathbb{R}^m)$.

Proof. We introduce the problems

$$y'' = \lambda f(t,y,y'), \quad y(0) = y(1) = 0, \tag{$3)_\lambda$}$$

where $\lambda \in [0,1]$ is the Leray-Schauder homotopy parameter.

We first prove that there exists a constant $C > 0$ such that for any $\lambda \in [0,1]$ and any solution y of $(3)_\lambda$ we have

$$\|y\|_\infty \le C, \quad \|y'\|_\infty \le C, \quad \|y''\|_\infty \le C. \tag{4}$$

Multiplying the jth differential equation in $(3)_\lambda$ by y_j, integrating from 0 to 1 and using (H1) and (1) we obtain

$$\begin{aligned}
\int_0^1 (y_j')^2\,dt &= -\lambda \int_0^1 y_j f_j(t,y,y')\,dt \\
&= -\lambda \int_0^1 y_j g_j(t,y,y')\,dt - \lambda \int_0^1 y_j h_j(t,y,y')\,dt \\
&\le \int_0^1 |y_j h_j(t,y,y')|\,dt \\
&\le \sum_{k=1}^m \left(a_{jk} \int_0^1 |y_j y_k|\,dt + b_{jk} \int_0^1 |y_j y_k'|\,dt\right) + c_j \int_0^1 |y_j|\,dt
\end{aligned}$$

for $j = 1,\dots,m$ with $f = (f_1,\dots,f_m)$. Using the Schwarz inequality we get

$$\|y_j'\|_2^2 \le \sum_{k=1}^m (a_{jk}\|y_j\|_2\|y_k\|_2 + b_{jk}\|y_j\|_2\|y_k'\|_2) + c_j\|y_j\|_2. \tag{5}$$

Since $y_j(0) = y_j(1) = 0$ the Wirtinger's inequality (see [6]) gives

$$\pi\|y_j\|_2 \le \|y_j'\|_2, \quad j = 1,\dots,m. \tag{6}$$

From (5) and (6) we deduce that

$$||y_j'||_2 \leq \sum_{k=1}^{m} \left(\frac{a_{jk}}{\pi^2} + \frac{b_{jk}}{\pi} \right) ||y_k'||_2 + \frac{c_j}{\pi}, \quad j = 1, \dots, m. \tag{7}$$

Let x and c denote the vectors

$$x = (||y_j'||_2)_{1 \leq j \leq m} \quad \text{and} \quad c = \left(\frac{c_j}{\pi} \right)_{1 \leq j \leq m}.$$

(7) can be written

$$c - (I - \pi^{-2} M) x \in \mathbb{R}_+^m.$$

(2) and theorem 2 imply that $I - \pi^{-2} M$ is $\mathbb{R}_+^m$-monotone. Hence using theorem 1 we obtain

$$(I - \pi^{-2} M)^{-1} c - x \in \mathbb{R}_+^m. \tag{8}$$

From (6) and (8) we get

$$||y_j||_2 \leq C_1, \quad ||y_j'||_2 \leq C_1, \quad j = 1, \dots, m \tag{9}$$

for some positive constant C_1. Since $y_j(0) = 0$, we can write

$$\begin{aligned} |y_j(t)| &= \left| \int_0^t y_j'(s)\, ds \right| \leq \int_0^1 |y_j'(s)|\, ds \\ &\leq ||y_j'||_2 \leq C_1 \end{aligned}$$

for $j = 1, \dots, m$ and $t \in [0, 1]$, so that

$$||y||_\infty \leq \sqrt{m} C_1. \tag{10}$$

We claim that for each solution y of $(3)_\lambda$ there exists $t_0 \in [0, 1]$ such that

$$|y'(t_0)| \leq \sqrt{m} C_1. \tag{11}$$

Indeed, by (9) we have

$$\int_0^1 |y'(t)|^2\, dt \leq m C_1^2, \tag{12}$$

hence there exists $t_0 \in [0, 1]$ such that (11) is satisfied.

Using (H2), (H3) and (10) we can find a constant $A > 0$ such that

$$|y''(t)| \leq A(w(|y'(t)|^2) + 1 + |y'(t)|), \quad 0 \leq t \leq 1,$$

and, since

$$|y'(t)| \leq \frac{1}{2}(1+|y'(t)|^2) \leq 1+|y'(t)|^2, \quad 0 \leq t \leq 1,$$

we get

$$|y''(t)| \leq A(w(|y'(t)|^2)+|y'(t)|^2+2), \quad 0 \leq t \leq 1. \tag{13}$$

Define $z(t) = |y'(t)|^2$, $t \in [0,1]$. By (13) we have

$$\begin{aligned} |z'(t)| &= 2|y'(t).y''(t)| \leq 2|y'(t)||y''(t)| \\ &\leq 2A|y'(t)|(w(z(t))+2+z(t)), \end{aligned} \tag{14}$$

for $t \in [0,1]$. (14) implies that

$$\frac{z'(t)}{w(z(t))+z(t)+2} \leq 2A|y'(t)|, \quad 0 \leq t \leq 1.$$

An integration from t_0 to $t \geq t_0$ yields

$$\begin{aligned} \int_{z(t_0)}^{z(t)} \frac{ds}{w(s)+s+2} &\leq 2A \int_{t_0}^{t} |y'(s)|\, ds \\ &\leq 2A \left(\int_0^1 |y'(s)|^2\, ds \right)^{1/2} \leq 2\sqrt{m}AC_1. \end{aligned} \tag{15}$$

Since $w \in C([0,\infty),(0,\infty))$ is nondecreasing and

$$\int_0^\infty \frac{ds}{w(s)} = \infty,$$

we easily have (see [2]) that

$$\int_0^\infty \frac{ds}{w(s)+s+2} = \infty,$$

and by (11) and (15) we obtain the existence of a constant $C > 0$ independent of t_0 such that

$$z(t) = |y'(t)|^2 \leq C, \quad t_0 \leq t \leq 1.$$

(14) also implies that

$$\frac{-z'(t)}{w(z(t))+z(t)+2} \leq 2A|y'(t)|, \quad 0 \leq t \leq 1.$$

An integration from $t \leq t_0$ to t_0 yields

$$\int_{z(t_0)}^{z(t)} \frac{ds}{w(s)+s+2} \leq 2A \int_t^{t_0} |y'(s)|\,ds$$
$$\leq 2A \left(\int_0^1 |y'(s)|^2\,ds \right)^{1/2} \leq 2\sqrt{m} A C_1.$$

As before we conclude that

$$z(t) = |y'(t)|^2 \leq C, \quad 0 \leq t \leq t_0,$$

for some constant C independent of t_0. Therefore there exists a constant $C_2 > 0$ such that

$$||y'||_\infty \leq C_2. \tag{16}$$

From $(3)_\lambda$, (10) and (16) we deduce that

$$|y''(t)| \leq C_3, \quad 0 \leq t \leq 1,$$

where

$$C_3 = \sup\{|f(t,y,p)|;\, t \in [0,1],\, |y| \leq \sqrt{m} C_1,\, |p| \leq C_2\},$$

and (4) is proved.

Now let $C_0^2 = \{y \in C^2([0,1],\mathbb{R}^m); y(0) = y(1) = 0\}$. Define the linear operator

$$L : C_0^2 \to C([0,1],\mathbb{R}^m), \quad Ly = \frac{d^2y}{dt^2},$$

and the family of maps $(0 \leq \lambda \leq 1)$

$$T_\lambda : C^1([0,1],\mathbb{R}^m) \to C([0,1],\mathbb{R}^m),$$
$$T_\lambda v(t) = \lambda f(t, v(t), v'(t)), \quad 0 \leq t \leq 1.$$

$j : C_0^2 \to C^1([0,1],\mathbb{R}^m)$ denotes the completely continuous embedding. L is invertible and

$$L^{-1}v(t) = -\int_0^1 G(t,s)v(s)\,ds, \quad 0 \leq t \leq 1,$$

where G is the Green's function

$$G(t,s) = \begin{cases} s(1-t) & \text{if} \quad 0 \leq s \leq t \leq 1 \\ t(1-s) & \text{if} \quad 0 \leq t \leq s \leq 1. \end{cases}$$

Let

$$K = \{y \in C_0^2;\ ||y||_{\max} = \max\{||y||_\infty, ||y'||_\infty, ||y''||_\infty\} \le C+1\}$$

where C is the constant in (4). We can define a compact homotopy

$$H_\lambda : K \to C_0^2, \quad H_\lambda = L^{-1}T_\lambda j, \quad 0 \le \lambda \le 1.$$

Since the fixed points of H_λ are the solutions of $(3)_\lambda$, the choice of C implies that the homotopy H_λ is fixed point free on the boundary of K. We have $H_0 \equiv 0$. Thus we can apply the topological transversality theorem [5] to obtain that H_1 has a fixed point. This shows that there is a solution to $(\mathcal{D}_h)$. □

Example 1. Let $a, b \in \mathbb{R}$ be such that

$$\pi|b| < a + \pi^2 \quad \text{when} \quad a < 0,$$

and

$$|b| < \pi \quad \text{when} \quad a \ge 0.$$

Then the homogeneous Dirichlet problem

$$\begin{aligned} y'' &= y^3 y'^2 \ln(1+y'^2) + ay + by' + \sin(tyy'), \\ &y(0) = y(1) = 0, \end{aligned} \tag{17}$$

has a solution by theorem 3.

We shall show that the results of [3] do not apply to problem (17).

(i) Suppose that

$$\begin{aligned} g(t,y,p) + h(t,y,p) &= y^3 p^2 \ln(1+p^2) + ay + bp + \sin(typ), \\ &(t,y,p) \in [0,1] \times \mathbb{R}^2, \end{aligned}$$

with g, h satisfying the assumptions of theorem 1 in [3]. There exist constants $0 \le \alpha, \beta < 1$ and $M_0 > 0$ such that the following condition holds:

$$|h(t,y,p)| \le M_0(|y|^\alpha + |p|^\beta), \quad (t,y,p) \in [0,1] \times \mathbb{R}^2. \tag{C}$$

Let k be the function defined by

$$\begin{aligned} k(t,y,p) &= g(t,y,p) - y^3 p^2 \ln(1+p^2) \\ &= ay + bp + \sin(typ) - h(t,y,p) \end{aligned}$$

for $(t,y,p) \in [0,1] \times \mathbb{R}^2$.

Assume first that $b \neq 0$. Since $yg(t,y,p) \geq 0$ for $(t,y,p) \in [0,1] \times \mathbb{R}^2$, we deduce that $g(t,0,p) = k(t,0,p) = 0$ for $(t,p) \in [0,1] \times \mathbb{R}$. Thus $h(t,0,p) = bp$ for $(t,p) \in [0,1] \times \mathbb{R}$ and (C) cannot be satisfied: take $y = 0$ and $p \to \infty$.

Now suppose that $a < 0$. Since $yg(t,y,p) \geq 0$ for $(t,y,p) \in [0,1] \times \mathbb{R}^2$, we deduce that

$$yk(t,y,0) = \left(a - \frac{h(t,y,0)}{y}\right) y^2 \geq 0$$

for $(t,y) \in [0,1] \times (\mathbb{R} \setminus \{0\})$. With the help of condition (C) we obtain

$$\lim_{|y| \to \infty} \left(a - \frac{h(t,y,0)}{y}\right) = a < 0, \quad t \in [0,1],$$

and we reach a contradiction.

We conclude that we cannot apply theorem 1 of [3] to problem (17) when $a < 0$ or $b \neq 0$.

Finally assume that there exists a constant $m \in (0, \pi^2 - \frac{1}{2})$ such that

$$yg(t,y,p) \geq -my^2, \quad (t,y,p) \in [0,1] \times \mathbb{R}^2.$$

We easily deduce that $g(t,0,p) = k(t,0,p) = 0$ and $h(t,0,p) = bp$ for $(t,p) \in [0,1] \times \mathbb{R}$. Again condition (C) cannot be satisfied if $b \neq 0$. Therefore remark 1 of [3] does not apply when $b \neq 0$.

(ii) If $a < 0$, we have $yf(t,y,0) = ay^2 < 0$ for $y \in \mathbb{R} \setminus \{0\}$ and we cannot use theorem 2 of [3].

The next example shows that theorem 3 complements some results obtained in [9] (theorem 6.1, corollaries 6.2 and 6.3).

Example 2. The homogeneous Dirichlet system

$$\begin{aligned}
y_1'' &= y_1^5 + y_1^3((y_1')^2 + (y_2')^2)\ln(1 + (y_1')^2 + (y_2')^2) + y_1 y_2^2 \\
&\qquad - \frac{\pi^2}{6}\left(y_1 + y_2 + \frac{1}{\pi}y_1' + \frac{1}{\pi}y_2'\right) + \cos(ty_1'y_2'), \\
y_2'' &= y_2^7 + y_2^5((y_1')^2 + (y_2')^2)\ln(1 + (y_1')^2 + (y_2')^2) + y_1^4 y_2 \\
&\qquad - \frac{\pi^2}{6}\left(y_1 + y_2 + \frac{1}{\pi}y_1' + \frac{1}{\pi}y_2'\right) + \cos t(y_1' + y_2'), \\
y(0) &= y(1) = 0
\end{aligned}$$

has a solution by theorem 3.

Theorem 4 *Let $f \in C([0,1] \times \mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^m)$. Assume that $f = g + h$ with g satisfying* (H1). *Assume moreover that the following conditions hold:*

(H2)′ *h satisfies* (H2) *with $b_{jk} = 0$ for $k \geq j+1$, $j = 1, \ldots, m-1$ (if $m \geq 2$);*

(H4) *There are functions $A_j(t, y, p_1, \ldots, p_{j-1}) \geq 0$ (for $j = 1$, A_1 is independent of the p variables) which are bounded on bounded subsets of $[0,1] \times \mathbb{R}^m \times \mathbb{R}^{j-1}$ and satisfy*

$$|g_j(t, y, p)| \leq A_j(t, y, p_1, \ldots, p_{j-1}) w_j(p_j^2),$$

for $j = 1, \ldots, m$ and $(t, y, p) \in [0,1] \times \mathbb{R}^m \times \mathbb{R}^m$, where $w_j \in C([0,\infty), (0,\infty))$ are nondecreasing functions such that

$$\int_0^\infty \frac{ds}{w_j(s)} = \infty, \quad j = 1, \ldots, m.$$

Then the Dirichlet system $(\mathcal{D}_h)$ has a solution $y \in C^2([0,1], \mathbb{R}^m)$.

Proof. As in the proof of theorem 3 we consider the family of problems $(3)_\lambda$ with $0 \leq \lambda \leq 1$. Again we shall prove that there exists a constant $C > 0$ such that for any $\lambda \in [0,1]$ and any solution y of $(3)_\lambda$ the estimates (4) hold. The same arguments lead to (9) and (10). By (1) with (H2)′, (H4) with $j = 1$ and (10) there exists a constant $A > 0$ such that

$$|y_1''(t)| \leq A(w_1(y_1'(t)^2) + |y_1'(t)| + 1), \quad 0 \leq t \leq 1,$$

and, since

$$|y_1'(t)| \leq \frac{1}{2}(1 + y_1'(t)^2) \leq 1 + y_1'(t)^2, \quad 0 \leq t \leq 1,$$

we get

$$|y_1''(t)| \leq A(w_1(y_1'(t)^2) + y_1'(t)^2 + 2), \quad 0 \leq t \leq 1. \tag{18}$$

Since y_1' vanishes at least once in $(0,1)$, each connected component of $\{t \in [0,1];\ y_1'(t) \neq 0\}$ is included in some interval $[a,b] \subset [0,1]$ such that $|y_1'(t)| > 0$ in (a,b) and $y_1'(a) = 0$ or $y_1'(b) = 0$. Assume that $y_1'(t) > 0$ in (a,b) and $y_1'(a) = 0$. Define $z(t) = y_1'(t)$, $t \in [a,b]$. By (18) we have

$$z'(t) \leq A(w_1(z(t)^2) + z(t)^2 + 2), \quad a \leq t \leq b,$$

thus

$$\frac{(z(t)^2)'}{w_1(z(t)^2) + z(t)^2 + 2} \leq 2Az(t), \quad a \leq t \leq b. \tag{19}$$

Integrating (19) from a to $t \in [a,b]$ and using (9) we obtain

$$\int_0^{z(t)^2} \frac{ds}{w_1(s) + s + 2}\, ds \leq 2A \int_a^t y_1'(s)\, ds$$
$$\leq 2A \int_0^1 |y_1'(s)|\, ds \leq 2AC_1.$$

As in the proof of theorem 3 we obtain the existence of a constant $C > 0$ independent of a and b such that

$$0 \leq z(t) = y_1'(t) \leq C, \quad a \leq t \leq b.$$

Since each case can be handled in the same way, we get

$$|y_1'(t)| \leq C, \quad 0 \leq t \leq 1$$

for some constant C. Using (1) with (H2)$'$, (H4), (10) and an induction argument we deduce that each component y_j' is bounded and we obtain a constant C such that

$$||y'||_\infty \leq C.$$

Then we conclude as in the proof of theorem 3. □

Example 3. The homogeneous Dirichlet system

$$y_1'' = y_1^5 + y_1^3(y_1')^2 \ln(1 + (y_1')^2) + y_1\left(y_2^2 + \frac{(y_2')^2}{1 + (y_2')^2}\right)$$
$$- \frac{\pi^2}{6}\left(y_1 + 2y_2 + \frac{1}{\pi}y_1'\right) + \cos(ty_1'y_2'),$$

$$y_2'' = y_2^7 + y_2^5((y_1')^4 + (y_2')^2) \ln(1 + (y_1')^4 + (y_2')^2) + y_1^4 y_2$$
$$- \frac{\pi^2}{6}\left(y_1 + y_2 + \frac{1}{\pi}y_1' + \frac{1}{\pi}y_2'\right) + \cos t(y_1' + y_2'),$$

$$y(0) = y(1) = 0,$$

has a solution by theorem 4. It is easily seen that the results of [9] do not apply.

Now we give a result which complements theorem 3.

Theorem 5 *Let $f \in C([0,1] \times \mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^m)$. Assume that $f = g + h$ satisfies* (H1) *and* (H2). *Assume moreover that the following condition holds*:

(H5) $|g(t,y,p)| \leq A(t,y)|p|^{2+\alpha} + B(t,y)$ *for* $(t,y,p) \in [0,1] \times \mathbb{R}^m \times \mathbb{R}^m$, *where $\alpha \in (0,1)$ and $A(t,y)$, $B(t,y)$ are bounded on bounded subsets of* $[0,1] \times \mathbb{R}^m$.
Then the Dirichlet system $(\mathcal{D}_h)$ has a solution $y \in C^2([0,1], \mathbb{R}^m)$.

Proof. Again we consider the family of problems $(3)_\lambda$ with $0 \leq \lambda \leq 1$. We shall prove that there exists a constant $C > 0$ such that for any $\lambda \in [0,1]$ and any solution y of $(3)_\lambda$ the estimates (4) hold. Arguing as in the proof of theorem 3 we are led to (10)–(12). Now using (1), (H5), (10) and (11) we can write

$$
\begin{aligned}
|y'(t)| &= \left|y'(t_0) + \int_{t_0}^t y''(s)\,ds\right| = \left|y'(t_0) + \lambda \int_{t_0}^t f(s, y(s), y'(s))\,ds\right|, \\
&\leq |y'(t_0)| + \left|\int_{t_0}^t f(s, y(s), y'(s))\,ds\right| \\
&\leq C\left(1 + \int_0^1 |y'(s)|^{2+\alpha}\,ds + \int_0^1 |y'(s)|\,ds\right) \\
&\leq C\left\{1 + \left(\int_0^1 |y'(s)|^2\,ds\right)||y'||_\infty^\alpha + \left(\int_0^1 |y'(s)|^2\,ds\right)^{1/2}\right\}
\end{aligned}
$$

for $0 \leq t \leq 1$, where $C > 0$ is a constant independent of t_0. With the help of (12) we obtain

$$
|y'(t)| \leq C(1 + ||y'||_\infty^\alpha), \quad 0 \leq t \leq 1,
$$

for another constant C. Therefore

$$
||y'||_\infty \leq C(1 + ||y'||_\infty^\alpha). \tag{20}
$$

Since $\alpha < 1$, (20) implies that there exists a constant $C > 0$ such that

$$
||y'||_\infty \leq C.
$$

Then we conclude as in the proof of theorem 3. □

Example 4. Let a, b be as in example 1 and let $\beta \in (0, 1/2)$. Then the homogeneous Dirichlet problem

$$y'' = y^3 + yy'^2(1+y'^2)^\beta + ay + by' + \sin ty', \quad y(0) = y(1) = 0,$$

has a solution by theorem 5. Clearly, we cannot apply the results of [3].

The next theorem complements theorem 4.

Theorem 6 *Let $f \in C([0,1] \times \mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^m)$. Assume that $f = g + h$ satisfies* (H1) *and* (H2)$'$. *Assume moreover that the following condition holds*:

(H6) *There are functions $A_j(t, y, p_1, \dots, p_{j-1}), B_j(t, y, p_1, \dots, p_{j-1}) \geq 0$ (for $j = 1$, A_1, B_1 are independent of the p variables) which are bounded on bounded subsets of $[0,1] \times \mathbb{R}^m \times \mathbb{R}^{j-1}$ and satisfy*

$$|g_j(t,y,p)| \leq A_j(t,y,p_1,\dots,p_{j-1})|p_j|^{2+\alpha} + B_j(t,y,p_1,\dots,p_{j-1}),$$

for $j = 1, \dots, m$ and $(t,y,p) \in [0,1] \times \mathbb{R}^m \times \mathbb{R}^m$, where $\alpha \in (0,1)$.
Then the Dirichlet system $(\mathcal{D}_h)$ has a solution $y \in C^2([0,1], \mathbb{R}^m)$.

Proof. Consider the family of problems $(3)_\lambda$ with $0 \leq \lambda \leq 1$. Let us prove that there exists a constant $C > 0$ such that for any $\lambda \in [0,1]$ and any solution y of $(3)_\lambda$ the estimates (4) hold. Arguing as in the proof of theorem 3 we are led to (9) and (10). Let $t_1 \in (0,1)$ be such that $y_1'(t_1) = 0$. Using (1) with (H2)$'$, (H6) with $j = 1$ and (10) we can write

$$\begin{aligned}
|y_1'(t)| &= \Big|\int_{t_1}^t y_1''(s)\,ds\Big| = \Big|\lambda \int_{t_1}^t f_1(s, y(s), y'(s))\,ds\Big| \\
&\leq \int_0^1 |g_1(s,y(s),y'(s))|\,ds + \int_0^1 |h_1(s,y(s),y'(s))|\,ds \\
&\leq C\Big(1 + \int_0^1 |y_1'(s)|^{2+\alpha}\,ds + \int_0^1 |y_1'(s)|\,ds\Big) \\
&\leq C\Big\{1 + \Big(\int_0^1 |y_1'(s)|^2\,ds\Big) ||y_1'||_\infty^\alpha + \Big(\int_0^1 |y_1'(s)|^2\,ds\Big)^{1/2}\Big\}
\end{aligned}$$

for $0 \leq t \leq 1$, where $C > 0$ is a constant. With the help of (9) we obtain

$$|y_1'(t)| \leq C(1 + ||y_1'||_\infty^\alpha), \quad 0 \leq t \leq 1,$$

for another constant C. Therefore

$$||y_1'||_\infty \leq C(1 + ||y_1'||_\infty^\alpha). \tag{21}$$

Since $\alpha < 1$, (21) implies that there exists a constant $C > 0$ such that

$$||y_1'||_\infty \leq C.$$

Using (1) with (H2)$'$, (H6), (10) and an induction argument we deduce that each component y_j' is bounded and we obtain a constant C such that

$$||y'||_\infty \leq C.$$

Again we conclude as in the proof of theorem 3. $\square$

Our next theorem extends some results obtained in [9] (theorem 6.1 and corollaries 6.2 and 6.3).

Theorem 7 *Let $f \in C([0,1] \times \mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^m)$. Assume that $f = g + h$ with g satisfying* (H3). *Assume also that the following conditions hold*:

(H7) *$y.g(t,y,p) \geq 0$ for $(t,y,p) \in [0,1] \times \mathbb{R}^m \times \mathbb{R}^m$;*

(H8) *There are constants $A, B, C \geq 0$ such that*

$$|h(t,y,p)| \leq A|y| + B|p| + C \tag{22}$$

for $(t,y,p) \in [0,1] \times \mathbb{R}^m \times \mathbb{R}^m$, with

$$A + \pi B < \pi^2. \tag{23}$$

Then the Dirichlet system $(\mathcal{D}_h)$ has a solution $y \in C^2([0,1], \mathbb{R}^m)$.

Proof. As before we shall prove that the estimates (4) hold for any $\lambda \in [0,1]$ and any solution y of $(3)_\lambda$. Taking the dot product of both sides of the differential equation in $(3)_\lambda$ with y, integrating from 0 to 1 and using (H7) and (22) we obtain

$$\begin{aligned}\int_0^1 |y'|^2\,dt &= -\lambda \int_0^1 y.g(t,y,y')\,dt - \lambda \int_0^1 y.h(t,y,y')\,dt \\ &\leq \int_0^1 |y||h(t,y,y')|\,dt \\ &\leq A\int_0^1 |y|^2\,dt + B\int_0^1 |y||y'|\,dt + C\int_0^1 |y|\,dt.\end{aligned}$$

Now by virtue of the Schwarz inequality and the Wirtinger's inequality we obtain

$$||y'||_2 \leq \left(\frac{A}{\pi^2} + \frac{B}{\pi}\right) ||y'||_2 + \frac{C}{\pi}.$$

Using (23) we deduce that

$$||y'||_2 \leq C\left(\pi - \frac{A}{\pi} - B\right)^{-1}.$$

Then we use the same arguments as in the proof of theorem 3. $\square$

Remark 1. Let $f \in C([0,1] \times \mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^m)$. Assume that $f = g + h$ satisfies (H7) and (H8). Clearly, if we assume also that g verifies (H5) we can establish the existence of solutions to $(\mathcal{D}_h)$.

Example 5. Let $\omega = \frac{1}{2}\sqrt{2(3+\sqrt{5})}$. Let $\mu \in \mathbb{R} \setminus \{0\}$ and $\theta \in (-\frac{1}{2\omega}, \frac{1}{1+\omega})$. Then the homogeneous Dirichlet system

$$\begin{aligned}
y_1'' &= y_1^5 + y_1^3((y_1')^2 + (y_2')^2)\ln(1 + (y_1')^2 + (y_2')^2) + \mu y_1 y_2^2 \\
&\quad + \theta\pi^2\left(y_1 + y_2 + \frac{1}{\pi}y_1' + \frac{1}{\pi}y_2'\right) + \sin(ty_1'y_2'), \\
y_2'' &= y_2^7 + y_2^5((y_1')^2 + (y_2')^2)\ln(1 + (y_1')^2 + (y_2')^2) - \mu y_1^2 y_2 \\
&\quad + \theta\pi^2\left(y_2 + \frac{1}{\pi}y_2'\right) + \sin t(y_1' + y_2'), \\
y(0) &= y(1) = 0,
\end{aligned}$$

has a solution by theorem 7. Indeed, define the matrices L, N and the vector r

$$L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad r = \begin{pmatrix} \sin(tp_1p_2) \\ \sin t(p_1 + p_2) \end{pmatrix}.$$

Suppose first that $\theta < 0$. Then we take $h(t,y,p) = \theta\pi^2 Ly + \theta\pi Lp + r$ and we have

$$|h(t,y,p)| \leq |\theta|\pi||L||(\pi|y| + |p|) + \sqrt{2},$$

with

$$||L|| = \sqrt{\rho(LL^*)} = \sqrt{\frac{3+\sqrt{5}}{2}} = \omega.$$

Now, if $\theta \geq 0$ we take $h(t,y,p) = \theta\pi^2 Ny + \theta\pi Lp + r$ and we have

$$|h(t,y,p)| \leq \theta\pi(\pi||N||\,|y| + ||L||\,|p|) + \sqrt{2},$$

with

$$||N|| = \sqrt{\rho(NN^*)} = 1.$$

The result follows.

Since $\mu \neq 0$, theorem 3 does not apply.

If $\mu = 0$ and $\theta \in (-\frac{1}{2}, 1)$, then the above problem has a solution by theorem 3.

4. Uniqueness results

When $m = 1$ and $f(t, y, p)$ is strictly increasing in y for each fixed $(t, p) \in [0, 1] \times \mathbb{R}$, then uniqueness holds for the solution of $(\mathcal{D}_h)$ (see [7]).

When $m \geq 1$ some results are given in [8]. In the particular case where f is independent of $p \in \mathbb{R}^m$, uniqueness for the solution of $(\mathcal{D}_h)$ holds under a simple monotonicity condition (see [8]).

We give below two uniqueness results.

Theorem 8 *Let $f \in C([0, 1] \times \mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^m)$. Assume that f has the decomposition*

$$f_j(t, y, p) = g_j(t, y_j) + h_j(t, y, p), \quad j = 1, \ldots, m,$$

where g and h satisfy the following conditions:

(H9) *For $j = 1, \ldots, m$ and $t \in [0, 1]$, $s \to g_j(t, s)$ is nondecreasing;*

(H10) *There are constants a_{jk}, $b_{jk} \geq 0$, $j, k = 1, \ldots, m$ such that*

$$|h_j(t, y, p) - h_j(t, z, q)| \\ \leq \sum_{k=1}^{m} a_{jk}|y_k - z_k| + b_{jk}|p_k - q_k|, \quad j = 1, \ldots, m, \tag{24}$$

for $(t, y, p), (t, z, q) \in [0, 1] \times \mathbb{R}^m \times \mathbb{R}^m$, with $M = (a_{jk} + \pi b_{jk})_{1 \leq j,k \leq m}$ satisfying (2).

Then the Dirichlet system $(\mathcal{D}_h)$ has at most one solution $y \in C^2([0, 1], \mathbb{R}^m)$.

Proof. Let y and $z \in C^2([0, 1], \mathbb{R}^m)$ be two solutions of $(\mathcal{D}_h)$. For $j = 1, \ldots, m$ we have

$$y_j'' - z_j'' = g_j(t, y_j) - g_j(t, z_j) + h_j(t, y, y') - h_j(t, z, z'). \tag{25}$$

Multiplying (25) by $y_j - z_j$, integrating from 0 to 1 and using (H9) and (24) we obtain

$$\begin{aligned}\int_0^1 (y_j' - z_j')^2\,dt &= -\int_0^1 (y_j - z_j)(g_j(t,y_j) - g_j(t,z_j))\,dt \\ &\qquad - \int_0^1 (y_j - z_j)(h_j(t,y,y') - h_j(t,z,z'))\,dt \\ &\le \int_0^1 |y_j - z_j||h_j(t,y,y') - h_j(t,z,z')|\,dt \\ &\le \sum_{k=1}^m \Big(a_{jk} \int_0^1 |y_j - z_j||y_k - z_k|\,dt \\ &\qquad + b_{jk} \int_0^1 |y_j - z_j||y_k' - z_k'|\,dt \Big)\end{aligned}$$

for $j = 1, \ldots, m$. Now, arguing as in the first part of the proof of theorem 3 we get

$$||y_j' - z_j'||_2 \le \sum_{k=1}^m \left(\frac{a_{jk}}{\pi^2} + \frac{b_{jk}}{\pi} \right) ||y_k' - z_k'||_2, \quad j = 1, \ldots, m.$$

By (2) we can use successively theorem 2 and theorem 1 and we deduce that

$$||y_j' - z_j'||_2 = 0, \quad j = 1, \ldots, m$$

which easily implies that $y_j = z_j$ for $j = 1, \ldots, m$. □

Example 6. (i) Let a, b be as in example 1. Then $y \equiv 0$ is the unique solution in $C^2[0,1]$ of the homogeneous Dirichlet problem

$$y'' = t(1+y)^3 + ay + by' - t, \quad y(0) = y(1) = 0.$$

If $a < 0$, we cannot apply the uniqueness results of [7].

(ii) Let $\theta \in (-\frac{1}{2}, 1)$. The homogeneous Dirichlet system

$$y_1'' = y_1^3 + \theta\pi^2 \left(y_1 + y_2 + \frac{1}{\pi} y_1' + \frac{1}{\pi} y_2' \right) + \sin t$$

$$y_2'' = y_2^5 + \theta\pi^2 \left(y_2 + \frac{1}{\pi} y_2' \right) + \cos t$$

$$y(0) = y(1) = 0,$$

has a unique solution $y \in C^2([0,1], \mathbb{R}^2)$ by theorems 3 and 8. When $\theta \in (-\frac{1}{2}, 0]$ the uniqueness results given in [8] do not apply.

Theorem 9 *Let $f \in C([0,1] \times \mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^m)$. Assume that f has the decomposition*

$$f(t,y,p) = g(t,y,p) + h(t,y,p)$$

with g satisfying (H7). *Assume also that the following conditions hold:*

(H11) $(y-z).(g(t,y,p) - g(t,z,p)) \geq 0$ *for* $(t,y,p), (t,z,p) \in [0,1] \times \mathbb{R}^m \times \mathbb{R}^m$;

(H12) *There are constants $A, B \geq 0$ such that*

$$|h(t,y,p) - h(t,z,q)| \leq A|y-z| + B|p-q|$$

for $(t,y,p), (t,z,q) \in [0,1] \times \mathbb{R}^m \times \mathbb{R}^m$, with $A + \pi B < \pi^2$;

(H13) *$p \to g(t,y,p)$ is continuously differentiable and*

$$\left\| \frac{\partial g}{\partial p}(t,y,p) \right\| \leq R(t,y), \quad (t,y,p) \in [0,1] \times \mathbb{R}^m \times \mathbb{R}^m,$$

with

$$D = \sup\left\{ R(t,y) : t \in [0,1],\ |y| \leq C\left(\pi - \frac{A}{\pi} - B\right)^{-1} \right\} < \pi - \frac{A}{\pi} - B,$$

where $C = \sup_{t\in[0,1]} |h(t,0,0)|$.
Then the homogeneous Dirichlet system $(\mathcal{D}_h)$ has at most one solution $y \in C^2([0,1], \mathbb{R}^m)$.

Proof. We first note that (H12) implies that

$$\begin{aligned} |h(t,y,p)| &\leq |h(t,y,p) - h(t,0,0)| + |h(t,0,0)| \\ &\leq A|y| + B|p| + C \end{aligned} \tag{26}$$

for $(t,y,p) \in [0,1] \times \mathbb{R}^m \times \mathbb{R}^m$. Since $A + \pi B < \pi^2$, using (H7) and (26) we can argue as in the proof of theorem 7 to get

$$||y'||_2 \leq C\left(\pi - \frac{A}{\pi} - B\right)^{-1}$$

for any solution $y \in C^2([0,1], \mathbb{R}^m)$ of $(\mathcal{D}_h)$. Since $y(0) = 0$, we can write

$$|y(t)| = \left| \int_0^t y'(s)\, ds \right| \leq ||y'||_2.$$

Therefore

$$||y||_\infty \le C\left(\pi - \frac{A}{\pi} - B\right)^{-1} \tag{27}$$

for any solution $y \in C^2([0,1],\mathbb{R}^m)$ of $(\mathcal{D}_h)$.

Now let $y, z \in C^2([0,1],\mathbb{R}^m)$ be two solutions of $(\mathcal{D}_h)$. We have

$$y'' - z'' = g(t,y,y') - g(t,z,z') + h(t,y,y') - h(t,z,z'). \tag{28}$$

Taking the dot product of both sides of (28) with $y-z$, integrating from 0 to 1 and using (H11), (H12), (26), the Schwarz inequality and the Wirtinger's inequality we obtain

$$\begin{aligned} ||y'-z'||_2^2 &= -\int_0^1 (y-z).(g(t,y,y') - g(t,z,y'))\,dt \\ &\quad - \int_0^1 (y-z).(g(t,z,y') - g(t,z,z'))\,dt \\ &\quad - \int_0^1 (y-z).(h(t,y,y') - h(t,z,z'))\,dt \\ &\le -\int_0^1 (y-z).(g(t,z,y') - g(t,z,z'))\,dt \\ &\quad + A\int_0^1 |y-z|^2\,dt + B\int_0^1 |y-z||y'-z'|\,dt \\ &\le \left(\frac{A}{\pi^2} + \frac{B}{\pi}\right)||y'-z'||_2^2 - \int_0^1 (y-z).(g(t,z,y') \\ &\quad - g(t,z,z'))\,dt, \end{aligned}$$

which implies

$$\left(1 - \frac{A}{\pi^2} - \frac{B}{\pi}\right)||y'-z'||_2^2 \le \int_0^1 |y-z||g(t,z,y') - g(t,z,z')|\,dt. \tag{29}$$

Now, by (H13) and (27) we can write

$$\begin{aligned} |g(t,z,y') - g(t,z,z')| &= \left|\int_0^1 \frac{\partial g}{\partial p}(t,z,sy' + (1-s)z')(y'-z')\,ds\right| \\ &\le |y'-z'|\int_0^1 \left\|\frac{\partial g}{\partial p}(t,z,sy' + (1-s)z')\right\|\,ds \\ &\le |y'-z'|R(t,z) \le D|y'-z'|. \end{aligned} \tag{30}$$

From (29) and (30), using always the same arguments we obtain

$$\begin{aligned}\left(1-\frac{A}{\pi^2}-\frac{B}{\pi}\right)||y'-z'||_2^2 &\le D\int_0^1 |y-z||y'-z'|\,dt\\ &\le D||y-z||_2||y'-z'||_2\\ &\le \frac{D}{\pi}||y'-z'||_2^2.\end{aligned}$$

Using (H13) we deduce that $||y'-z'||_2 = 0$, which easily implies that $y = z$. $\square$

Example 7. (i) Let a, b be as in example 1 and let μ be such that

$$0 < \mu < \left(\pi - \frac{|a|}{\pi} - |b|\right)^4, \quad \text{when } \ a < 0,$$

and

$$0 < \mu < (\pi - |b|)^4, \quad \text{when } \ a \ge 0.$$

Then the homogeneous Dirichlet problem

$$y'' = y^5 + t\mu y^3(1 + y'^2)^{1/2} + ay + by' + \sin t, \quad y(0) = y(1) = 0,$$

has a unique solution $y \in C^2[0,1]$ by theorems 3 (or 7) and 9. If $a < 0$, we cannot apply the uniqueness results of [7].

(ii) Let $P = (a_{jk})_{1\le j,k\le 2}$, $Q = (b_{jk})_{1\le j,k\le 2}$ and $\mu_j, j = 1,2$ be such that

$$||P|| + \pi||Q|| < \pi^2,$$

$$0 < \mu_1 < \sqrt{\sigma}\left(\pi - \frac{||P||}{\pi} - ||Q||\right)^4 \quad \text{and}$$

$$0 < \mu_2 < \sqrt{1-\sigma}\left(\pi - \frac{||P||}{\pi} - ||Q||\right)^6,$$

where $\sigma \in (0,1)$. Then the homogeneous Dirichlet system

$$\begin{aligned} y_1'' &= y_1^5 + \mu_1 t y_1^3(1 + (y_1')^2 + (y_2')^2)^{1/2}\\ &\qquad + a_{11}y_1 + a_{12}y_2 + b_{11}y_1' + b_{12}y_2' + \sin t,\\ y_2'' &= y_2^7 + \mu_2 t y_2^5(1 + (y_1')^2 + (y_2')^2)^{1/2}\\ &\qquad + a_{21}y_1 + a_{22}y_2 + b_{21}y_1' + b_{22}y_2' + \cos t,\\ y(0) &= y(1) = 0,\end{aligned}$$

has a unique solution $y \in C^2([0,1],\mathbb{R}^2)$ by theorems 7 and 9. Indeed, taking

$$g(t,y,p) = \begin{pmatrix} y_1^5 + \mu_1 t y_1^3 (1+|p|^2)^{1/2} \\ y_2^7 + \mu_2 t y_2^5 (1+|p|^2)^{1/2} \end{pmatrix},$$

it is easily seen that

$$\begin{aligned} \left\| \frac{\partial g}{\partial p}(t,y,p) \right\| &= \sqrt{\rho\left(\left(\frac{\partial g}{\partial p}\right) \left(\frac{\partial g}{\partial p}\right)^* \right)} \\ &= \frac{t|p|}{(1+|p|^2)^{1/2}} (\mu_1^2 y_1^6 + \mu_2^2 y_2^{10})^{1/2} \\ &\le (\mu_1^2 |y|^6 + \mu_2^2 |y|^{10})^{1/2}, \end{aligned}$$

thus condition (H13) is satisfied. If we choose $a_{11} \le 0$ or $a_{22} \le 0$, the uniqueness results given in [8] do not apply.

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