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Preconditioning cubic collocation method for elliptic equations

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Abstract. In this paper we provide the preconditioning results of the linear system generated by the cubic spline collocation discretization with penalty terms for an elliptic equation with Neumann boundary conditions. Moreover, we show that the linear system of an elliptic equation with Dirichlet or mixed boundary conditions can be directly derived from the linear system of the same equation with Neumann boundary conditions.

Key words: C^1-cubic spline, preconditioning collocation method.

1. Introduction

In this paper, we will provide preconditioning results of the cubic spline collocation discretization with penalty terms for a positive definite second order elliptic boundary value problem with Neumann boundary conditions.

In the course of patching collocation method introduced by Orszag in [O] for elliptic boundary value problems, Funaro has provided a method to handle different types of boundary conditions in [F2]. The idea is to collocate an equation at both the interior nodes and the boundary nodes of a given interval for the equations to be solved at the boundary points. In this paper, we will deal with the preconditioning cubic spline collocation method for the Neumann problem following the ideas provided in [F2] and [KP].

Let us consider an equation

$$-u'' + cu = f \quad \text{in} \quad (0,1) \tag{1.1}$$

with the Neumann boundary conditions

$$u'(0) = 0, \quad u'(1) = 0, \tag{1.2}$$

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where f and c are continuous functions and

$$0 < m \le c(x) \le M < \infty \quad \text{for} \quad 0 \le x \le 1.$$

Let us introduce the space $\mathcal{S}_{\Delta,3}$ of C^1-cubic spline functions with C^1-cubic Lagrange spline basis $\{\psi_i\}_{i=0}^{2N+1}$ with respect to a given partition $\Delta := \{t_i\}_{i=0}^{N}$ of the unit interval I where $\psi_i(\xi_j) = \delta_{ij}$ for given collocation points $\{\xi_j\}_{j=0}^{2N+1}$ such that, for $j = 1, 2, \ldots, N$,

$$\xi_0 = 0, \quad t_{j-1} < \xi_{2j-1} < \xi_{2j} < t_j, \quad \xi_{2N+1} = 1.$$

Notice that $\{\psi_i\}_{i=1}^{2N}$ can be chosen from [KK] or [KS] in the sense that ψ_i satisfies the Dirichlet boundary conditions, *i.e.*, $\psi_i(0) = 0$ and $\psi_i(1) = 0$, and in order to handle the Neumann boundary conditions ψ_0 and ψ_{2N+1} can be added by following the ideas in the proceeding references.

Concerning the numerical approximation, we look for $u \in \mathcal{S}_{\Delta,3}$, and we collocate the Eq. (1.1) at the interior points ξ_j, *i.e.*,

$$(-u'' + c\,u)(\xi_j) = f(\xi_j), \quad j = 1, 2, \ldots, 2N. \tag{1.3}$$

For the equation to be solved at the boundary points 0 and 1, we will impose the following conditions as penalty terms:

$$\begin{aligned}(-u'' + c\,u)(0) - w_0^{-1}u'(0) &= f(0),\\ (-u'' + c\,u)(1) + w_{2N+1}^{-1}u'(1) &= f(1)\end{aligned} \tag{1.4}$$

where $\{w_i\}_{i=0}^{2N+1}$ are the quadrature weights relative to the points $\{\xi_i\}_{i=0}^{2N+1}$.

The analogous scheme for the pseudospectral approximation was showned and analyzed in [F2]. Hence, stability and convergence for our approximation can be obtained by adapting the theory in [F2].

The purpose of this paper is to give preconditioning results for the linear system obtained from the proceeding schemes (1.3) and (1.4).

Multiplying both sides of the linear system $B_N\,U_h = F_h$ obtained from the scheme (1.3) and (1.4) by the weights matrix W_N, we have a symmetric linear system such that $\tilde{B}_N\,U_h := W_N\,B_N\,U_h = W_N\,F_h$. It is well-known that the condition number of $\tilde{B}_N$ increase like N^2. Here, we shall study the preconditioning which is important for the sucessful application of conjugate gradient method.

As a preconditioner we take a stiffness matrix β_N associated with the same Eq. (1.1) on the space $\mathcal{S}_{\Delta,1}$ of the continuous piecewise linear functions which break at the collocation points. Now, we have a preconditioned linear system such that

$$Q_N\, U_h := \beta_N^{-1}\, \tilde{B}_N\, U_h = \beta_N^{-1}\, W_N\, F_h.$$

Following the ideas in [KP], we will show that the matrix Q_N has all positive bounded eigenvalues independent of mesh size on a quasi-uniform mesh. In [KP], using Hermite cubic splines, Kim and Parter gave preconditioning results for the same problem (1.1) immediately, where they didn't collocate at boundary points. Indeed, our work is quite similar to the work in [KP] so that we will quote their results for the proofs of our results.

One major difference is that we handle the Neumann boundary conditions by a different way. The other difference is that we can directly derive one collocation matrix B_N^d associated with Dirichlet boundary conditions, *i.e.*, $u(0) = 0$, $u(1) = 0$, taking the interior $(2N) \times (2N)$ elements of B_N and the other collocation matrix B_N^m associated with mixed boundary conditions, *i.e.*, $u(0) = 0$, $u'(1) = 0$, eliminating the first row and the first column from B_N. Of course, we can obtain the preconditioning results for the Eq. (1.1) with Dirichlet boundary conditions or mixed boundary conditions using the matrix B_N^d or B_N^m. Moreover, using the tensor products we can also extend the preconditioning strategy to a positive definite second order elliptic partial differential equation.

The rest of the paper is organized as follows. In section 2, we give some preliminary ideas, notations, etc. In section 3, we will give one dimensional preconditioning result for the Neumann problem. And our discussions are extended to a positive definite second order elliptic partial differential equation with Neumann boundary conditions in section 4. We give the computational results about one dimensional problem in section 5.

2. Preliminary

Let I be the unit interval and let $\Omega := I \times I$ be the unit square. Let Δ be a partition of I such as

$$\Delta : 0 = t_0 < t_1 < \cdots < t_N = 1, \quad h_i := t_i - t_{i-1}, \quad I_i = [t_{i-1}, t_i].$$

Let h and ρ be the mesh size and the bounded global mesh ratio of the partition Δ, respectively, such that

$$h := \max_{1\le i\le N}\{h_i\} \quad \text{and} \quad 1 \le \rho := \max_{1\le i,j\le N} \frac{h_i}{h_j} \le \gamma. \tag{2.1}$$

Throughout this paper, the work is based on the quasi-uniform mesh.

Using Legendre-Gauss points and Legendre-Gauss-Radau points (see [CHQZ]), we define the collocation points $\{\xi_i\}_{i=0}^{2N+1}$ with the corresponding quadrature wights $\{w_i\}_{i=0}^{2N+1}$ such as

$$\left\{\begin{array}{ll} \xi_0 &= 0 \\ \xi_1 &= \frac{h_1}{2}\left(\frac{6-\sqrt{6}}{5}\right) \\ \xi_2 &= \frac{h_1}{2}\left(\frac{6+\sqrt{6}}{5}\right) \\ \xi_{2i-1} &= \frac{h_i}{2}\left(1-\frac{1}{\sqrt{3}}\right)+t_{i-1} \\ \xi_{2i} &= \frac{h_i}{2}\left(1+\frac{1}{\sqrt{3}}\right)+t_{i-1} \\ \xi_{2N-1} &= \frac{h_N}{2}\left(\frac{4-\sqrt{6}}{5}\right)+t_{N-1} \\ \xi_{2N} &= \frac{h_N}{2}\left(\frac{4+\sqrt{6}}{5}\right)+t_{N-1} \\ \xi_{2N+1} &= 1 \end{array}\right. \qquad \left\{\begin{array}{ll} w_0 &= \frac{h_1}{9} \\ w_1 &= \frac{h_1}{2}\left(\frac{5^3(4+\sqrt{6})}{9(58+12\sqrt{6})}\right) \\ w_2 &= \frac{h_1}{2}\left(\frac{5^3(4-\sqrt{6})}{9(58-12\sqrt{6})}\right) \\ w_{2i-1} &= \frac{h_i}{2} \\ w_{2i} &= \frac{h_i}{2} \\ w_{2N-1} &= \frac{h_N}{2}\left(\frac{5^3(4-\sqrt{6})}{9(58-12\sqrt{6})}\right) \\ w_{2N} &= \frac{h_N}{2}\left(\frac{5^3(4+\sqrt{6})}{9(58+12\sqrt{6})}\right) \\ w_{2N+1} &= \frac{h_N}{9}. \end{array}\right. \tag{2.2}$$

Let $\mathcal{S}_{\Delta,3}$ be the space of all C^1-cubic splines for the partition Δ, *i.e.*,

$$\mathcal{S}_{\Delta,3} = \{u \in C^1[0,1] : u|_{I_i} \text{ is a cubic polynomial for each } i\}$$

and let $\{\psi_i\}_{i=0}^{2N+1}$ be the C^1-cubic Lagrange spline basis for the space $\mathcal{S}_{\Delta,3}$ which satisfy $\psi_i(\xi_j) = \delta_{i,j}$ for $i, j = 0, 1, \ldots, 2N+1$.

Let $\mathcal{S}_{\Delta,1}$ be the space of continuous piecewise linear functions with basis $\{\phi_i\}_{i=0}^{2N+1}$ which break at the collocation points $\{\xi_i\}_{i=0}^{2N+1}$, *i.e.*, $\phi_i(\xi_j) = \delta_{i,j}$.

Let $\pi := \Delta \times \Delta$ be a partition of Ω and let us denote by two dimensional function spaces $\mathcal{S}_{\pi,3} := \mathcal{S}_{\Delta,3} \otimes \mathcal{S}_{\Delta,3}$ and $\mathcal{S}_{\pi,1} := \mathcal{S}_{\Delta,1} \otimes \mathcal{S}_{\Delta,1}$.

Let $(\cdot\,,\cdot)_I$ and $\|\cdot\|$ denote by the usual L_2 inner product and the corresponding norm:

$$(u,v)_I = \int_I u\,v\,dt \quad \text{and} \quad \|u\|_0 = \sqrt{(u,u)_I}.$$

We will use the usual Sobolev norm. Let $H^1(I)$ be a Sobolev space

$$H^1(I) = \{v \in L_2(I) : v' \in L_2(I)\}.$$

The inner product and its associated norm on $H^1(I)$ are given by

$$(u, v)_{H^1(I)} = \int_I \Big(u'\, v' + u\, v\Big) dt, \quad \|u\|_1 = \sqrt{(u, u)_{H^1(I)}}.$$

Define a quadratic form $[\cdot, \cdot]_N$ on $\mathcal{S}_{\Delta,3} \times \mathcal{S}_{\Delta,3}$ by

$$[u, v]_N := \sum_{i=0}^{2N+1} w_i u(\xi_i) v(\xi_i) \quad \text{for} \quad u,\, v \in \mathcal{S}_{\Delta,3}.$$

Let us define the cubic spline interpolation operator I_N as

$$(I_N u)(t) = \sum_{i=0}^{2N+1} u(\xi_i)\, \psi_i(t) \in \mathcal{S}_{\Delta,3} \quad \text{for} \quad u \in \mathcal{S}_{\Delta,1}.$$

3. Preconditioning on 1D case

Theorem 3.1 *For $\{\psi_i\}_{i=0}^{2N+1}$, there exists a positive constant C_1, independent of mesh size, such that*

$$\max_{t_{j-1} \le t \le t_j} |\psi_i(t)| \le C_1 \left(\frac{1}{7}\right)^{|j-i^*|} \tag{3.1}$$

where i^* denotes the largest integer less than or equal to $(i+1)/2$.

Proof. Using Theorem 1 in [KS], we can easily check that there is a positive constant C_1 which satisfies the estimate (3.1) for ψ_i $(i = 1, 2, \ldots, 2N)$ and in addition, for ψ_0 and ψ_{2N+1}. □

Using Theorem 3.1 and repeating the appropriate modifications of the works in [KP], we have the following lemma:

Lemma 3.1 *There exist positive constants C_2, C_3 and C_4, independent of mesh size, such that*

$$C_2 \|u\|_0^2 \le [u, u]_N \le C_3 \|u\|_0^2 \quad \textit{for every} \quad u \in \mathcal{S}_{\Delta,3} \tag{3.2}$$

and

$$\|u'\|_0 \le \|(I_N u)'\|_0 \le C_4 \, \|u'\|_0 \quad \textit{for every} \quad u \in \mathcal{S}_{\Delta,1}. \tag{3.3}$$

Lemma 3.2 *For every $u \in \mathcal{S}_{\Delta,3}$, we have*

$$\|u'\|_0^2 \le [-u'', u]_N + u'(1)u(1) - u'(0)u(0) \le \frac{5}{3}\|u'\|_0^2. \tag{3.4}$$

Proof. Denote that $[f,g]_{I_j}$ and $(f,g)_{I_j}$ are the restrictions of $[f,g]_N$ and $(f,g)_I$ on I_j, respectively. Since $f''g$ is a polynomial of degree ≤ 4 for $f, g \in \mathcal{S}_{\Delta,3}$ on subintervals I_1 and I_N, by Legendre-Gauss-Radau integration, we have

$$-[f'', g]_{I_1} + f'(t_1)g(t_1) - f'(0)g(0) \; = \; (f', g')_{I_1}, \tag{3.5}$$

$$-[f'', g]_{I_N} + f'(1)g(1) - f'(t_{N-1})g(t_{N-1}) \; = \; (f', g')_{I_N}. \tag{3.6}$$

By Lemma 3.1 in [CP] and [DD], there is a positive constant P such that, for $k = 2, 3, \dots, N-1$,

$$\begin{aligned} -[f'', g]_{I_k} &+ f'(t_k)g(t_k) - f'(t_{k-1})g(t_{k-1}) \\ &= (f', g')_{I_k} + \frac{2}{3}\, P\, f_k^{(3)} g_k^{(3)} h_k^5 \end{aligned} \tag{3.7}$$

where $f_k^{(3)}$ is the third derivative of f on I_k.

Combining (3.5), (3.6) and (3.7), we have

$$\begin{aligned} -[f'', g]_N &+ f'(1)g(1) - f'(0)g(0) \\ &= (f', g')_I + \frac{2}{3}\, P \sum_{k=2}^{N-1} f_k^{(3)} g_k^{(3)} h_k^5. \end{aligned} \tag{3.8}$$

From Lemma 3.2 in [DD], we observe that

$$P \sum_{k=2}^{N-1} f_k^{(3)} g_k^{(3)} h_k^5 \le (f', g')_I. \tag{3.9}$$

Setting $f = u$ and $g = u$ and combining (3.8) and (3.9), we have

$$(u', u')_I \le -[u'', u]_N + u'(1)u(1) - u'(0)u(0) \le \frac{5}{3}(u', u')_I,$$

so it completes the proof. □

Define two bilinear forms $a_N(\cdot\,,\cdot)$ and $b_N(\cdot\,,\cdot)$ on $\mathcal{S}_{\Delta,3} \times \mathcal{S}_{\Delta,3}$ as

$$a_N(f, g) \; = \; f'(1)g(1) - f'(0)g(0) := f'\, g'|_0^1, \tag{3.10}$$

$$b_N(f,g) \;=\; [-f'',g]_N + [c\,f,g]_N + a_N(f,g). \tag{3.11}$$

Using (3.8), we can rewrite the bilinear form b_N as

$$b_N(f,g) = (f',g')_I + [cf,g]_N + \frac{2}{3}P\sum_{i=2}^{N-1} f_i^{(3)} g_i^{(3)} h_i^5 \tag{3.12}$$

which implies that $b_N(\cdot\,,\cdot)$ is symmetric.

Theorem 3.2 *For every $u \in \mathcal{S}_{\Delta,1}$, there are positive constants C_5, C_6 and C_7, independent of mesh size, such that*

$$\|u'\|_0^2 \le [-(I_Nu)'', I_Nu]_N + a_N(I_Nu, I_Nu) \le C_5\|u'\|_0^2 \tag{3.13}$$

and

$$C_6\,\|u\|_0^2 \le [I_Nu, I_Nu]_N \le C_7\,\|u\|_0^2. \tag{3.14}$$

Proof. From (3.4), we have

$$\|(I_Nu)'\|_0^2 \le [-(I_Nu)'', I_Nu]_N + a_N(I_N\,u, I_N\,u) \le \frac{5}{3}\,\|(I_Nu)'\|_0^2.$$

Hence, by (3.3), we have one of the conclusion (3.13) with $C_5 = 5\,C_4\,/3$.

Once the estimate (3.2) is done, applying the arguments in Theorem 4.1 and Theorem 4.2 in [KP] word by word to a quasi-uniform mesh, we have another conclusion (3.14). □

Theorem 3.3 *For every $u \in \mathcal{S}_{\Delta,1}$, there are positive constants C_8 and C_9, independent of mesh size, such that*

$$C_8\,\|u\|_1^2 \le b_N(I_Nu, I_Nu) \le C_9\,\|u\|_1^2. \tag{3.15}$$

Proof. Since $0 < m \le c(t) \le M < \infty$, using(3.14) we have

$$mC_6\|u\|_0^2 \le [c\,I_Nu, I_Nu]_N \le MC_7\|u\|_0^2. \tag{3.16}$$

Now, combining (3.13) and (3.16) we have the conclusion (3.15). □

With

$$u = \sum_{j=0}^{2N+1} u(\xi_j)\psi_j \in \mathcal{S}_{\Delta,3},$$

we collocate the Eq. (1.1) at the interior collocation points ξ_i $(i = 1, 2, \ldots, 2N)$ such as

$$\sum_{j=0}^{2N+1} \Big[-u(\xi_j)\psi_j''(\xi_i) + c(\xi_j)u(\xi_i)\psi_j(\xi_i) \Big] = f(\xi_i). \tag{3.17}$$

To impose boundary conditions, we can choice the following conditions

$$\begin{aligned} -u''(0) + c(0)u(0) - w_0^{-1}\, u'(0) &= f(0) \\ -u''(1) + c(1)u(1) + w_{2N+1}^{-1}\, u'(1) &= f(1). \end{aligned} \tag{3.18}$$

Then, combining the schemes (3.17) and (3.18) we have the collocation matrix B_N for the Eq. (1.1) with the Neumann boundary conditions (1.2) such that

$$B_N\, U_h = F_h$$

where $U_h = (u(\xi_0), u(\xi_1), \ldots, u(\xi_{2N+1}))^t$, $F_h = (f(\xi_0), f(\xi_1), \ldots, f(\xi_{2N+1}))^t$ and the elements of B_N are represented as

$$\begin{aligned} B_N(0,j) &= -\psi_j''(0) + c(0)\,\delta_{0,j} - w_0^{-1}\,\psi_j'(0) \\ B_N(i,j) &= -\psi_j''(\xi_i) + c(\xi_i)\,\delta_{i,j} \\ B_N(2N+1,j) &= -\psi_j''(1) + c(1)\,\delta_{2N+1,j} + w_{2N+1}^{-1}\,\psi_j'(1) \end{aligned}$$

for $i = 1, 2 \cdots, 2N$ and $j = 0, 1, \ldots, 2N+1$.

Let $\tilde{B}_N := W_N\, B_N$ be the symmetric matrix where W_N is a weights matrix. Comparing $\tilde{B}_N(i,j)$ with $b_N(\psi_j, \psi_i)$ in (3.11) we have

$$\tilde{B}_N(i,j) = [W_N\, B_N](i,j) = b_N(\psi_j, \psi_i) \tag{3.19}$$

so that $\tilde{B}_N$ is symmetric.

Consider the stiffness matrix β_N associated with the finite element discretization of the Eq. (1.1) in the space $\mathcal{S}_{\Delta,1}$ as preconditioner:

$$\beta_N(i,j) = (\phi_i', \phi_j')_I + (c\phi_i, \phi_j)_I. \tag{3.20}$$

The condition $m \le c(t) \le M$ for $t \in I$ implies that $U_h^t\, \beta_N\, U_h$ is equivalent to $\|u\|_1^2$ in $\mathcal{S}_{\Delta,1}$, in the sense that

$$\begin{aligned} \min\{1, m\}\|u\|_1^2 &\le U_h^t \beta_N U_h \\ &\le \max\{1, M\}\|u\|_1^2 \quad \text{for } \; u \in \mathcal{S}_{\Delta,1}. \end{aligned} \tag{3.21}$$

Multiplying $\tilde{B}_N$ by β_N^{-1}, we have a preconditioned linear system such that

$$Q_N\, U_h := \beta_N^{-1}\, \tilde{B}_N\, U_h = \beta_N^{-1}\, W_N\, F_N. \tag{3.22}$$

Now we will prove the main results that the preconditioned matrix Q_N has all positive bounded eigenvalues, independent of mesh size.

Theorem 3.4 *For every $u \in \mathcal{S}_{\Delta,1}$, there are two positive constants C_{10} and C_{11}, independent of mesh size, such that*

$$C_{10}\,(U_h^t \beta_N U_h) \le U_h^t\, \tilde{B}_N\, U_h \le C_{11}\,(U_h^t \beta_N U_h) \tag{3.23}$$

with $U_h = (u(\xi_0), u(\xi_1), \ldots, u(\xi_{2N+1}))^t$.

Moreover the eigenvalues $\{\lambda_j\}_{j=0}^{2N+1}$ of the matrix $Q_N := (\beta_N)^{-1}\tilde{B}_N$ satisfy

$$C_{10} \le \lambda_j \le C_{11} \quad \textit{for} \;\; j = 0, 1, \ldots, 2N+1. \tag{3.24}$$

Proof. Since U_h represents both u and $I_N u$, we have

$$\begin{aligned} b_N(I_N u, I_N u) &= \sum_{i,j} u(\xi_i) u(\xi_j)\, b_N(\psi_i, \psi_j) \\ &= \sum_{i,j} u(\xi_i) u(\xi_j)\, \tilde{B}_N(i,j) = U_h^t \tilde{B}_N U_h. \end{aligned}$$

By (3.15), we have

$$C_8\, \|u\|_1^2 \le U_h^t \tilde{B}_N U_h \le C_9\, \|u\|_1^2 .$$

Combining this estimate and (3.21), we have one of the conclusions (3.23). Now if λ is one of the eigenvalues of Q_N and U_h is the corresponding eigenvector, then

$$Q_N\, U_h = (\beta_N)^{-1}\, \tilde{B}_N\, U_h = \lambda\, U_h \quad \text{or} \quad \tilde{B}_N U_h = \lambda\, \beta_N\, U_h.$$

Hence, we have $U_h^t\, \tilde{B}_N\, U_h = \lambda\, U_h^t\, \beta_N\, U_h$ which implies the other conclusion (3.24). □

Remark. From the linear system of order $(2N+2) \times (2N+2)$ for the Eq. (1.1) with a Neumann boundary conditions (1.2), we can directly derive the two linear systems for the same Eq. (1.1) with Dirichlet boundary conditions $u(0) = 0$, $u(1) = 0$ and mixed boundary conditions $u(0) = 0$, $u'(1) = 0$, and we can also investigate preconditioning results for these problems by a similar way as the case of Neumann problem.

1) For the case of Dirichlet boundary value problem, we have a linear system of order $(2N)\times(2N)$ such as

$$B_N^d\,U_h^d = F_h^d \quad \text{or} \quad (\beta_N^d)^{-1}\,W_N^d\,B_N^d\,U_h^d = (\beta_N^d)^{-1}\,W_N^d\,F_h^d$$

where B_N^d, W_N^d and β_N^d are $(2N)\times(2N)$ matrices whose elements are given by

$$B_N^d(i,j) = B_N(i,j), \quad W_N^d = W_N(i,j), \quad \beta_N^d(i,j) = \beta_N(i,j), \quad (i,j = 1,2,\ldots,2N),$$

and $U_h^d := (u(\xi_1), u(\xi_2), \ldots, u(\xi_{2N}))^t$, $F_h^d := (f(\xi_1), f(\xi_2), \ldots, f(\xi_{2N}))^t$.

2) For the case of mixed boundary value problem, we have a linear system of order $(2N+1)\times(2N+1)$ such as

$$B_N^m\,U_h^m = F_h^m \quad \text{or} \quad (\beta_N^m)^{-1}\,W_N^m\,B_N^m\,U_h^m = (\beta_N^m)^{-1}\,W_N^m\,F_h^m$$

where B_N^m, W_N^m and β_N^m are $(2N+1)\times(2N+1)$ matrices whose elements are given by

$$B_N^m(i,j) = B_N(i,j), \quad W_N^m = W_N(i,j), \quad \beta_N^m(i,j) = \beta_N(i,j), \quad (i,j = 1,2,\ldots,2N+1),$$

and $U_h^m := (u(\xi_1), u(\xi_2), \ldots, u(\xi_{2N+1}))^t$, $F_h^m := (f(\xi_1), f(\xi_2), \ldots, f(\xi_{2N+1}))^t$.

4. Preconditioning on 2D case

Let L_π be a differential operator such that

$$L_\pi := -\Delta u + (c_1(x) + c_2(y))u \quad \text{in } \Omega \tag{4.1}$$

with Neumann boundary condition

$$\frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega$$

where $\partial\Omega$ is the boundary of Ω, $\frac{\partial u}{\partial \mathbf{n}}$ denotes the outward normal derivative of u on $\partial\Omega$ and, c_1 and c_2 are continuous functions such that

$$0 < m \le c_1(x),\, c_2(y) \le M < \infty \quad \text{for } 0 \le x,\ y \le 1.$$

Let us decompose the operator $L_\pi := L_x + L_y$ such that

$$\begin{aligned}(L_x\,u)(x,y) &= -u_{xx}(x,y) + c_1(x)\,u(x,y),\\ (L_y\,u)(x,y) &= -u_{yy}(x,y) + c_2(y)\,u(x,y).\end{aligned}$$

Ordering the collocation points $P_\mu := (\xi_i, \xi_j)$ $(\mu = j + i(2N+2))$ and following the ideas in (3.19), we can obtain the symmetric matrix representations $\tilde{B}_{N_x} \otimes W_N$ and $W_N \otimes \tilde{B}_{N_y}$ of the cubic spline collocation discretization of L_x and L_y, respectively (see [PR] and [WM]). Combining these matrices we have the symmetric matrix representation $\tilde{B}_{N^2}$ of the cubic spline collocation discretization of L_π in the space $\mathcal{S}_{\pi,3}$ such that

$$\tilde{B}_{N^2} = \tilde{B}_{N_x} \otimes W_N + W_N \otimes \tilde{B}_{N_y}. \tag{4.2}$$

Furthermore we have the stiffness matrix β_{N^2} associated with the finite element discretization of L_π in the space $\mathcal{S}_{\pi,1}$ such that

$$\beta_{N^2} = \beta_{N_x} \otimes M_N + M_N \otimes \beta_{N_y} \tag{4.3}$$

where $M_N(i,j) = (\phi_i, \phi_j)_I$ and

$$\begin{aligned} \beta_{N_x}(i,j) &= (\phi_i', \phi_j')_I + (c_1\phi_i, \phi_j)_I, \\ \beta_{N_y}(i,j) &= (\phi_i', \phi_j')_I + (c_2\phi_i, \phi_j)_I. \end{aligned}$$

Note that $\tilde{B}_{N^2}$ and β_{N^2} are symmetric and positive definite.

Lemma 4.1 *There are positive constants T_1 and T_2, independent of mesh size, such that for all nonzero $U_\pi \in \mathbb{R}^{(2N+2)^2}$*

$$T_1 \le \frac{U_\pi^t(\tilde{B}_{N_x} \otimes W_N)U_\pi}{U_\pi^t(\beta_{N_x} \otimes M_N)U_\pi} \le T_2 \tag{4.4}$$

and

$$T_1 \le \frac{U_\pi^t(W_N \otimes \tilde{B}_{N_y})U_\pi}{U_\pi^t(M_N \otimes \beta_{N_y})U_\pi} \le T_2. \tag{4.5}$$

Proof. Observe that

$$(\beta_{N_x} \otimes M_N)^{-1}(\tilde{B}_{N_x} \otimes W_N) = \beta_{N_x}^{-1}\tilde{B}_{N_x} \otimes M_N^{-1}W_N. \tag{4.6}$$

By Theorem 3.4 all eigenvalues λ_j of $\beta_{N_x}^{-1}\tilde{B}_{N_x}$ satisfy $C_{10} \le \lambda_j \le C_{11}$. By (3.14) in Theorem 3.2 we can show that all eigenvalues μ_j of $M_N^{-1}W_N$ satisfy $C_6 \le \mu_j \le C_7$. Therefore the eigenvalues $\nu_j = \lambda_j\,\mu_j$ of (4.6) satisfy $C_{10}C_6 \le \nu_j \le C_{11}C_7$ where the constants C_6, C_7, C_{10} and C_{11} are independent of mesh size.
Hence we have (4.4). Also, we can obtain the conclusion (4.5) by a similar way. □

Theorem 4.1 *Let* $Q_{N^2} := \beta_{N^2}^{-1} \tilde{B}_{N^2}$. *Then we have the following main results*:

For all nonzero $U_\pi \in \mathbb{R}^{(2N+2)^2}$,

$$T_1 \le \frac{U_\pi^t \, \tilde{B}_{N^2} \, U_\pi}{U_\pi^t \, \beta_{N^2} \, U_\pi} \le T_2. \tag{4.7}$$

Furthermore, the eigenvalues $\{\Lambda_\mu\}$ *of the matrix* Q_{N^2} *satisfy*

$$T_1 \le \Lambda_\mu \le T_2. \tag{4.8}$$

Proof. From (4.2) and (4.3), we can see that

$$U_\pi^t \tilde{B}_{N^2} U_\pi = U_\pi^t (\tilde{B}_{N_x} \otimes W_N) U_\pi + U_\pi^t (W_N \otimes \tilde{B}_{N_y}) U_\pi$$

and

$$U_\pi^t \beta_{N^2} U_\pi = U_\pi^t (\beta_{N_x} \otimes M_N) U_\pi + U_\pi^t (M_N \otimes \beta_{N_y}) U_\pi.$$

Using (4.4) and (4.5), we can easily obtain that

$$T_1 \, U_\pi^t \beta_{N^2} U_\pi \le U_\pi^t \tilde{B}_{N^2} U_\pi \le T_2 \, U_\pi^t \beta_{N^2} U_\pi.$$

This estimate completes the conclusion (4.7) which implies (4.8). □

5. Computational Results

Consider the following boundary-value problem:

$$Lu := -u'' + u = f \quad \text{in} \quad I \tag{5.1}$$

$$u'(0) = 0 \quad \text{and} \quad u'(1) = 0. \tag{5.2}$$

From (3.19), we have the symmetric matrix representation $\tilde{B}_N$ for the Eq. (5.1) with the Neumann boundary conditions (5.2) such that

$$\tilde{B}_N U_h := W_N \, B_N \, U_h = W_N \, F_h$$

where $U_h = (u(\xi_i))^t$ and $F_h = (f(\xi_i))^t$.

From (3.22), we have preconditioned algebraic linear system:

$$Q_N U_N := \beta_N^{-1} W_N B_N U_N = \beta_N^{-1} W_N F_N.$$

For $\tilde{B}_N$, we have all positive real eigenvalues epending on N. Unfortunately, the condition number

$$k(\tilde{B}_N) := \frac{\max\{\textit{Eigenvalues of } \tilde{B}_N\}}{\min\{\textit{Eigenvalues of } \tilde{B}_N\}}$$

increases like N^2. This results in a very slow convergence by iterative method (e.g., Conjugate Gradient method and Richardson method).

For Q_N based on the quasi uniform mesh, we have all positive bounded eigenvalues independent of N. Hence the condition number $k(Q_N)$ of Q_N turns out to be bounded, independent of N.

N	$\tilde{B}_N$	Q_N
8	2812	2.163556
16	11243	2.187724
32	44969	2.194010
64	179876	2.195612
128	719500	2.196017
256	2877997	2.196118

Table 1. Condition numbers of $\tilde{B}_N$ and Q_N

In Table 1, we report the condition numbers of $\tilde{B}_N$ and Q_N, respectively, where the work is based on the uniform mesh (i.e., $h = h_i$ for all i).

References

[BF] Bertoluzza S. and Funaro D., *Remarks about the Matrices relative to the Psedospectral Approximation of Neumann Problems.* Proceedings of the IMACS (International Symposium on Iterative Methods), Bruxells, 2/4 april 1991 (R. Beauwens & P. de Groen Eds., North Holland. Amsterdam 1992), 209–215.

[CHQZ] Canuto C., Hussaini M. Y., Quarteroni A. and Zang T.A., *Spectral Methods in Fluid Dynamics, Springer Series in Computational Physics.* Springer-Verlag, Berlin, New York, (1988).

[CP] Cerutti J. and Parter S.V., *Collocation methods for Parabolic Partial Differential Equations in one dimensional space.* Numer. Math. **26** (1974), 227–254.

[DD] Douglas J. and Dupont T., *Collocation Methods for Parabolic Equations in a Single Space Variable.* Lecture Notes in Mathematics, **385**, Springer-Verlag, (1974).

[F1] Funaro D., *A multidomain spectral approximation of elliptic equations.* Numerical Methods for P.D.E. **2** (1986), 187–205.

[F2] Funaro D., *Domain Decomposition Methods for Pseudo Spectral Approximations.* Numer. Math. **58** (1988), 329–344.

[KK] Kim S.D. and Kim S.C., *Exponential decay of C^1-cubic splines vanishing at the local interior symmetric points.* (accepted to Numer. Math.).

[KP] Kim S.D. and Parter S. V., *Preconditioning Cubic Spline Collocation Discretizations of Elliptic Equations.* Numer. Math. **72** (1995), 39–72.

[KS] Kim S.D. and Shin B.C., *On C^1 cubic Lagrange splines vanishing at local unsymmetric interior points in a nonuniform mesh.* (submit to Houston Jour. of Math.).

[O] Orszag S.A., *Spectral methods for problems in complex geometries.* J. Comput. Phys. **37** (1980), 70–92.

[PR] Parter S.V. and Rothman S. P., *Preconditioning second-order elliptic operators: Condition numbers and the distribution of the singular values.* Journal of Scientific Computation, **6** (1991), 1927–1957.

[WM] Wait R. and Mitchell A.R., *Finite Element Analysis and Applications.* John Wiley and Sons, (1995).

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