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Analysis of a family of strongly commuting self-adjoint operators with applications to perturbed Dirac operators

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Abstract. It is shown that a class of unitary transformations of the canonical momentum operator in a direct sum of $L^2(\mathbf{R}^d)$ is given by a class of operator-valued Lorentz transformations of perturbed canonical momentum operators. As an application, analysis of the quantum theory of spin-$\frac{1}{2}$ charged particles in an external electromagnetic field is given.

Key words: strongly commuting self-adjoint operators, d'Alembertian, Dirac operator, operator-valued Lorentz transformation, external field problem.

1. Introduction

In [A-T], we have developed an operator theory concerning a family of strongly commuting self-adjoint operators in $L^2(\mathbf{R}^d)$ which are associated with some objects in the d-dimentional Minkowski space $\mathbf{M}^d$. We have constructed a class of self-adjoint operators in $L^2(\mathbf{R}^d)$ (Section V of [A-T]) which may be regarded as perturbed d'Alembertians in the sense of unitary groups and whose unitary groups have integral-kernel representations in explicit forms. We have applied the theory to the external field problem of a charged spinless particle. Moreover, we clarified algebraic-analytic structures of the proper-time method which Schwinger had presented ([Sch]) for a class of vector potentials.

The construction of a class of strongly commuting self-adjoint operators on $L^2(\mathbf{R}^d)$ in [A-T] can be extended to the case of $\bigoplus^m L^2(\mathbf{R}^d)$ ($m = 2, 3, \ldots$). In this paper, we do it and show that results of [A-T] can be generalized. We apply the obtained results to the external field problem of a charged spin-$\frac{1}{2}$ particle.

We use the same symbols as in [A-T]. We denote a vector in $\mathbf{M}^d$ as $x = (x^0, x^1, \ldots, x^{d-1}) = (x^\mu)_{\mu=0}^{d-1}$ and the metric tensor of $\mathbf{M}^d$ by $g =$

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$(g_{\mu\nu})_{\mu,\nu=0,\dots,d-1}$.

The indefinite inner product of $\mathbf{M}^d$ is given by

$$xy = g_{\mu\nu}x^\mu y^\nu = x^0y^0 - \sum_{j=1}^{d-1} x^j y^j.$$

We obey the Einstein's rule as to summation. We write $xx = x^2$.

We define

$$x_\mu = g_{\mu\nu}x^\nu, \quad \mu = 0, 1, \dots, d-1.$$

Then we can write $xy = x^\mu y_\mu$.

Let ∂_μ be the generalized partial differential operator in x^μ acting in $L^2(\mathbf{R}^d)$ and $p_\mu = i\partial_\mu$. We set $p = (p_0, \dots, p_{d-1})$.

It is known that a Green's function $S(x, y)$ for a relativistic spin-$\frac{1}{2}$ charged particle with charge e in an electromagnetic field satisfies

$$(\gamma^\mu(p_\mu - eA_\mu) + m)S(x, y) = \delta(x - y), \tag{1.1}$$

where $m > 0$ is the mass of the particle, $A = (A_0, \dots, A_{d-1})$ is the vector potential of the electromagnetic field and γ^μ's are matrices satisfying the following relations:

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \quad (\mu, \nu = 0, 1, \dots, d-1). \tag{1.2}$$

The operator $\gamma^\mu(p_\mu - eA_\mu) + m$ is called a Dirac operator.

We can see that, formally,

$$\begin{aligned}(\gamma^\mu(p_\mu - eA_\mu) + m)(-\gamma^\nu(p_\nu - eA_\nu) + m)\\ = -(p_\mu - eA_\mu)(p^\mu - eA^\mu) + \frac{e}{2}\sigma_{\mu\nu}F^{\mu\nu} + m^2,\end{aligned} \tag{1.3}$$

where

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu], \quad F_{\mu\nu} = \partial_\mu\partial_\nu - \partial_\nu A_\mu \tag{1.4}$$

and $[\,,\,]$ is the commutator:

$$[A, B] := AB - BA.$$

Thus if $S_1(x, y)$ is a Green's function of the operator (1.3), then

$$S(x, y) = (-\gamma^\mu(p_\mu - eA_\mu) + m)S_1(x, y) \tag{1.5}$$

is a Green's function of the Dirac operator. Considering the first term of (1.3) as a perturbed d'Alembertian, Vaidya and Hott ([V-H]) reduced the problem of finding $S(x,y)$ to the problem of the spinless case in [V-S-H] and they calculated algebraically a Green's function satisfying (1.1) for a special class of vector potentials by using the above idea.

In this paper, we extend results of operator calculus in [A-T] to a direct sum of $L^2(\mathbf{R}^d)$ (Section II) and give more general results (Section III).

In [A-T], we showed that an operator-valued Lorentz transformation of a perturbed momentum operator is unitarily equivalent to p on a dence domain. In the case of the direct sum of $L^2(\mathbf{R}^d)$, we can obtain a unitary equivalence between p and an operator-valued Lorentz transformation of a more generally perturbed momentum operator, that is, a momentum operator perturbed by a matrix operator.

We can compute the integral kernels of the unitary groups generated by d'Alembertians perturbed by the matrix operator (Section IV).

In the last section, we apply the results in Sections III and IV to the external field problem. We compute a Green's function for a spin-$\frac{1}{2}$ charged particle.

2. Operator calculus in the Minkowski space

We introduce a subset of $\mathbf{M}^d \times \mathbf{M}^d$

$$\mathbf{M}_0 = \{(a,b) \in \mathbf{M}^d \times \mathbf{M}^d \mid a \neq 0, b \neq 0, ab = 0\}. \tag{2.1}$$

Let $(a,b) \in \mathbf{M}_0$. We introduce a partial differential operator $\mathbf{p}_\mu$ on $\bigoplus^m L^2(\mathbf{R}^d)$

$$\mathbf{p}_\mu : (\psi_1(x), \ldots, \psi_m(x)) \mapsto (p_\mu\psi_1(x), \ldots, p_\mu\psi_m(x)). \tag{2.2}$$

Its domain $D(\mathbf{p}_\mu)$ is the following. (We denote operator domain by $D(\cdot)$.)

$$D(\mathbf{p}_\mu) = \bigoplus^m D(p_\mu).$$

We define a multiplication operator $\mathbf{x}_\mu$ on $\bigoplus^m L^2(\mathbf{R}^d)$

$$\mathbf{x}_\mu : (\psi_1(x), \ldots, \psi_m(x)) \mapsto (x_\mu\psi_1(x), \ldots, x_\mu\psi_m(x)) \tag{2.3}$$

$$D(\mathbf{x}_\mu) = \bigoplus^m D(x_\mu)$$

Two operators $\mathbf{p}_\mu$ and $\mathbf{x}_\mu$ are self-adjoint.

We introduce two operators $a^\mu \mathbf{x}_\mu$ and $b^\mu \mathbf{p}_\mu$ on $\bigoplus^m L^2(\mathbf{R}^d)$:

$$a^\mu \mathbf{x}_\mu : (\psi_1(x), \ldots, \psi_m(x)) \mapsto (a^\mu x_\mu \psi_1(x), \ldots, a^\mu x_\mu \psi_m(x)), \quad (2.4)$$

$$b^\mu \mathbf{p}_\mu : (\psi_1(x), \ldots, \psi_m(x)) \mapsto (b^\mu p_\mu \psi_1(x), \ldots, b^\mu p_\mu \psi_m(x)), \quad (2.5)$$

with

$$D(a^\mu \mathbf{x}_\mu) = \bigoplus^m D(ax),$$
$$D(ax) = \{\psi \in L^2(\mathbf{R}^d) \mid ax\psi \in L^2(\mathbf{R}^d)\}$$
$$D(b^\mu \mathbf{p}_\mu) = \bigoplus^m D(bp),$$
$$D(bp) = \left\{\psi \in L^2(\mathbf{R}^d) \mid \int_{\mathbf{R}^d} |b\xi\tilde{\psi}(\xi)|^2 d\xi < \infty\right\}$$

where $\tilde{\psi}(\xi) := (2\pi)^{-d/2}\int_{\mathbf{R}^d}\psi(x)e^{i\xi x}dx$ is the Fourier transform of ψ with ξx (ξx is the Minkowski inner product of ξ and x.). For convenience, we denote $a^\mu \mathbf{x}_\mu$ by $a\mathbf{x}$ and $b^\mu \mathbf{p}_\mu$ by $b\mathbf{p}$.

We have for all $(\psi_1, \ldots, \psi_m) \in \bigoplus^m L^2(\mathbf{R}^d)$

$$e^{isa\mathbf{x}}(\psi_1(x), \ldots, \psi_m(x)) = (e^{isax}\psi_1(x), \ldots, e^{isax}\psi_m(x)) \quad (2.6)$$

$$e^{itb\mathbf{p}}(\psi_1(x), \ldots, \psi_m(x)) = (\psi_1(x - tb), \ldots, \psi_m(x - tb)) \quad (2.7)$$

where $s, t \in \mathbf{R}$.

Two (not necessarily bounded) self-adjoint operators A and B in a Hilbert space are said to strongly commute if their spectral projections commute.

The following characterization is well known. (See, [R-S1], Theorem VIII.13.)

Lemma 2.1 *Let A and B be self-adjoint operators on a Hilbert space. Then the following* (1)–(3) *are equivalent.*

(1) *A and B strongly commute.*
(2) *For all $s, t \in \mathbf{R}$, $e^{itA}e^{isB} = e^{isB}e^{itA}$.*
(3) *For all $t \in \mathbf{R}$, $e^{itA}B \subset Be^{itA}$.*

We can easily show that $e^{isa\mathbf{x}}$ and $e^{itb\mathbf{p}}$ commute. Hence, by Lemma 2.1, we have the following lemma.

Lemma 2.2 *$a\mathbf{x}$ and $b\mathbf{p}$ strongly commute.*

Let $\mathbf{B}_{\rm real}(\mathbf{R}^d)$ be the set of real-valued Borel measurable functions on $\mathbf{R}^d$ which are almost everywhere finite with respect to the d-dimentional Lebesgue measure. Let $E_{a\mathbf{x}}(\cdot)$ and $E_{b\mathbf{p}}(\cdot)$ be the spectral measures of $a\mathbf{x}$ and $b\mathbf{p}$, respectively. Then there exists a unique two-dimentional spectral measure $E_{a\mathbf{x},b\mathbf{p}}(\cdot)$ such that $E_{a\mathbf{x},b\mathbf{p}}(B_1\times B_2) = E_{a\mathbf{x}}(B_1)E_{b\mathbf{p}}(B_2)$, ($B_1$ and B_2 are Borel sets in $\mathbf{R}$.)

Let $u \in \mathbf{B}_{\rm real}(\mathbf{R}^d)$. By functional calculus, we can define a self-adjoint operator $u(a\mathbf{x}, b\mathbf{p})$ on $\bigoplus^m L^2(\mathbf{R}^d)$ by

$$u(a\mathbf{x}, b\mathbf{p}) := \int_{\mathbf{R}^d} u(\lambda_1,\lambda_2)dE_{a\mathbf{x},b\mathbf{p}}(\lambda_1,\lambda_2).$$

We have for all $(\psi_1,\ldots,\psi_m) \in \bigoplus^m L^2(\mathbf{R}^d)$

$$\begin{aligned} &u(a\mathbf{x}, b\mathbf{p})(\psi_1(x),\ldots,\psi_m(x)) \\ &\qquad = (u(ax,bp)\psi_1(x),\ldots,u(ax,bp)\psi_m(x)) \end{aligned}$$

Definition Let $\mathbf{N}_0 = \{0,1,2,\cdots\}$. For $r\in\mathbf{N}_0$, we define the following sets of functions (cf. Section II of [A-T]).

(1) Let $C^r_{\rm real}(\mathbf{R}^d)$ be the set of r times continuously differentiable real-valued functions on $\mathbf{R}^d$.

(2) Let $\mathfrak{B}^r(\mathbf{R}^d)$ be the set of bounded functions u such that $u\in C^r_{\rm real}(\mathbf{R}^d)$ and the partial derivatives of u of order j ($j=0,1,\ldots,r$) is bounded on $\mathbf{R}^d$.

(3) A real-valued function $u = u(x_1,x_2)$ on $\mathbf{R}^2$ is in the set $\mathfrak{B}^{r,\infty}$ ($r\in\mathbf{N}_0$) if $u(\cdot,x_2)\in\mathfrak{B}^r(\mathbf{R}^d)$ for almost everywhere x_2 and the function $\partial_1^j u(x_1,x_2) := \partial^j/\partial x_1^j u(x_1,x_2)$ is bounded on $\mathbf{R}^2$ for all $j=0,\ldots,r$; $u\in\mathfrak{B}^{\infty,r}$ if the function $\tilde{u}(x_1,x_2) := u(x_2,x_1)$ is in $\mathfrak{B}^{r,\infty}$. We write $\partial_2^j u(x_1,x_2) := \partial^j/\partial x_2^j u(x_1,x_2)$

We denote by $C_0^\infty(\mathbf{R}^d)$ the set of infinitely differentiable functions on $\mathbf{R}^d$ with compact support. We denote by $L^\infty(\mathbf{R}^d)$ the set of essentially bounded Borel measurable functions on $\mathbf{R}^d$. The subset of real-valued functions in $L^\infty(\mathbf{R}^d)$ is denoted by $L^\infty_{\rm real}(\mathbf{R}^d)$.

Lemma 2.3 *Let $u\in\mathfrak{B}^{1,\infty}(\mathbf{R}^2)$ Then, for each $\mu = 0,1,\ldots,d-1$, $u(a\mathbf{x},b\mathbf{p})$ leaves $D(\mathbf{p}_\mu)$ invariant and*

$$[\mathbf{p}_\mu, u(a\mathbf{x},b\mathbf{p})] = ia_\mu\partial_1 u(a\mathbf{x},b\mathbf{p}) \tag{2.8}$$

on $D(\mathbf{p}_\mu)$.

If $u \in \mathfrak{B}^{\infty,1}(\mathbf{R}^2)$, *then,* $u(a\mathbf{x}, b\mathbf{p})$ *leaves* $D(\mathbf{x}_\mu)$ *invariant and*

$$[\mathbf{x}_\mu, u(a\mathbf{x}, b\mathbf{p})] = -ib_\mu \partial_2 u(a\mathbf{x}, b\mathbf{p})$$

on $D(\mathbf{x}_\mu)$.

Proof. We have only to show that (2.8) applied to each component of any vector in $D(\mathbf{p}_\mu)$ holds. This follows from Lemmas 2.3 and 2.4 in [A-T]. □

By functional caluculus, we get a unitary operator $e^{itu(a\mathbf{x},b\mathbf{p})}$ $(t \in \mathbf{R})$:

$$e^{itu(a\mathbf{x},b\mathbf{p})} : (\psi_1(x), \dots, \psi_m(x)) \mapsto (e^{itu(ax,bp)}\psi_1(x), \dots, e^{itu(ax,bp)}\psi_m(x))$$

We denote $(\psi_1(x), \dots, \psi_m(x))$ by $\boldsymbol{\psi}(x)$.

We say that a function $u \in \mathbf{B}_{\text{real}}(\mathbf{R}^2)$ is in the set $\mathfrak{C}^r(\mathbf{R}^2)$ if $u(\cdot, x_2) \in C^r_{\text{real}}(\mathbf{R})$ for almost everywhere $x_2 \in \mathbf{R}$ and there exists a sequence $\{u_k\}$ $(u_k \in \mathfrak{B}^{r,\infty}(\mathbf{R}^2))$ such that

$$\sup_{k \geq 1} |\partial_1^j u_k(x_1, x_2)| \leq C|\partial_1^j u(x_1, x_2)| \quad (C \text{ is a constant}), \tag{2.9}$$

$$\lim_{k \to \infty} \partial_1^j u_k(x_1, x_2) = \partial_1^j u(x_1, x_2) \tag{2.10}$$

for $j = 0, \dots, r$.

Theorem 2.4 *Let* $u \in \mathfrak{C}^1(\mathbf{R}^2)$ *and* $\boldsymbol{\psi} \in D(\mathbf{p}_\mu) \cap D(\partial_1 u(a\mathbf{x}, b\mathbf{p}))$. *Then for each* $\mu = 0, 1, \dots, d-1$, $e^{-iu(a\mathbf{x},b\mathbf{p})}\boldsymbol{\psi} \in D(\mathbf{p}_\mu)$, *and*

$$e^{iu(a\mathbf{x},b\mathbf{p})}\mathbf{p}_\mu e^{-iu(a\mathbf{x},b\mathbf{p})}\boldsymbol{\psi} = [\mathbf{p}_\mu + a_\mu \partial_1 u(a\mathbf{x}, b\mathbf{p})]\boldsymbol{\psi}, \tag{2.11}$$

where

$$D(\mathbf{p}_\mu) \cap D(\partial_1 u(a\mathbf{x}, b\mathbf{p})) = \bigoplus^m D(p_\mu) \cap D(\partial_1 u(ax, bp)).$$

Proof. In the same way as in the proof of Theorem 2.5 of [A-T], for each component ψ_j of $\boldsymbol{\psi}$, we can show that the following equation holds,

$$e^{iu(ax,bp)}p_\mu e^{iu(ax,bp)}\psi_j = [p_\mu + a_\mu \partial_1 u(ax, bp)]\psi_j \quad (j = 1, \dots, m). \tag{2.12}$$

To prove (2.12), for $u \in \mathfrak{C}^1(\mathbf{R})$, we first show that for $u \in \mathfrak{B}^{1,\infty}(\mathbf{R})$, (2.12) holds. Then, using the fact that for all $u \in \mathfrak{C}^1(\mathbf{R})$ there exists a sequence $\{u_k\}_k$ ($u_k \in \mathfrak{B}^{1,\infty}(\mathbf{R})$) convergeing to u and satisfying (2.9) and (2.10), we obtain the result. □

Theorem 2.5 *Let $u \in \mathfrak{B}^{2,\infty}(\mathbf{R}^2)$. Then, for each $\mu = 0, 1, \dots, d-1$, $e^{-iu(a\mathbf{x},b\mathbf{p})}$ leaves $D(\mathbf{p}_\mu{}^2) = \bigoplus^m D(p_\mu{}^2)$ invariant and the following equations holds on $D(\mathbf{p}_\mu{}^2)$:*

$$e^{iu(a\mathbf{x},b\mathbf{p})}\mathbf{p}_\mu^2 e^{-iu(a\mathbf{x},b\mathbf{p})}\boldsymbol{\psi} = [\mathbf{p}_\mu + a_\mu \partial_1 u(a\mathbf{x}, b\mathbf{p})]^2 \boldsymbol{\psi} \tag{2.13}$$

Proof. Let $\boldsymbol{\phi}$, $\boldsymbol{\psi} \in D(\mathbf{p}_\mu^2)$, by Theorem 2.4, we have

$$\begin{aligned}(\mathbf{p}_\mu^2\boldsymbol{\phi}, e^{-iu(a\mathbf{x},b\mathbf{p})}\boldsymbol{\psi}) &= (\mathbf{p}_\mu\boldsymbol{\phi}, \mathbf{p}_\mu e^{-iu(a\mathbf{x},b\mathbf{p})}\boldsymbol{\psi}) \\ &= (\mathbf{p}_\mu\boldsymbol{\phi}, e^{-iu(a\mathbf{x},b\mathbf{p})}\mathbf{p}_\mu\boldsymbol{\psi}) \\ &\quad + (\mathbf{p}_\mu\boldsymbol{\phi}, e^{-iu(a\mathbf{x},b\mathbf{p})}a_\mu\partial_1 u(a\mathbf{x}, b\mathbf{p})\boldsymbol{\psi})\end{aligned}$$

Since $\mathbf{p}_\mu\boldsymbol{\psi} \in D(\mathbf{p}_\mu)$, we obtain $e^{-iu(a\mathbf{x},b\mathbf{p})}\mathbf{p}_\mu\boldsymbol{\psi} \in D(\mathbf{p}_\mu)$. Using that $u(a\mathbf{x}, b\mathbf{p})$ and $\partial_1 u(a\mathbf{x}, b\mathbf{p})$ commute and $\partial_1 u \in \mathfrak{B}^{1,\infty}(\mathbf{R}^2)$, we obtain $e^{-iu(a\mathbf{x},b\mathbf{p})}a_\mu\partial_1 u(a\mathbf{x}, b\mathbf{p})\boldsymbol{\psi} \in D(\mathbf{p}_\mu)$. Hence $e^{-iu(a\mathbf{x},b\mathbf{p})}\boldsymbol{\psi} \in D(\mathbf{p}_\mu^2)$. Using (2.11), we have (2.13). □

Now let

$$\Box_m = -\mathbf{p}^2 = \partial_0\partial_0 - \partial_1\partial_1 - \cdots - \partial_{d-1}\partial_{d-1}, \tag{2.14}$$

$$D(\Box_m) = \bigoplus^m \bigcap_{\mu=0}^{d-1} D(p_\mu{}^2),$$

where

$$\partial_j\boldsymbol{\psi} := (\partial_j\psi_1(x), \dots, \partial_j\psi_m(x)). \tag{2.15}$$

The operator $\Box_m$ is a free d'Alembertian on $\bigoplus^m L^2(\mathbf{R}^d)$. Since it is essentially self-adjoint, we denote its closure by $\mathbf{H}_0$. For $u \in \mathfrak{C}^1(\mathbf{R}^2)$, we define

$$\begin{aligned}\Box_{m,u} &= -[\mathbf{p} + a\partial_1 u(a\mathbf{x}, b\mathbf{p})]^2 \\ &= [\boldsymbol{\partial}_0 - ia_0\partial_1 u(a\mathbf{x}, b\mathbf{p})]^2 - \sum_{j=1}^{d-1}[\boldsymbol{\partial}_j - ia_j\partial_1 u(a\mathbf{x}, b\mathbf{p})]^2,\end{aligned} \tag{2.16}$$

$$D(\Box_{m,u}) = \bigoplus^{m} \bigcap_{\mu=0}^{d-1} D([\mathbf{p}_\mu + a_\mu \partial_1 u(a\mathbf{x}, b\mathbf{p})]^2).$$

Using Theorem 2.5, we obtain the following theorem.

Theorem 2.6 *Let $u \in \mathfrak{B}^{2,\infty}(\mathbf{R}^2)$ Then, $\Box_{m,u}$ is essentially self-adjoint on $D(\Box_{m,u})$, and the following equation holds:*

$$e^{iu(a\mathbf{x},d\mathbf{p})}\mathbf{H}_0 e^{-iu(a\mathbf{x},b\mathbf{p})} = \overline{\Box_{m,u}}.$$

Let $M_d^{\mathrm{as}}(\mathbf{R})$ be the set of $d \times d$ real anti-symmetric matrices. For $f \in M_d^{\mathrm{as}}(\mathbf{R})$ and $q \in \mathbf{M}^d$ we define the following operator

$$Q_\mu(\mathbf{x}) = f_{\mu\nu}(\mathbf{x}^\nu - \mathbf{q}^\nu). \tag{2.17}$$

That is,

$$Q_\mu(\mathbf{x}) : (\psi_1(x), \ldots, \psi_m(x)) \\ \mapsto (f_{\mu\nu}(x^\nu - q^\nu)\psi_1(x), \ldots, f_{\mu\nu}(x^\nu - q^\nu)\psi_m(x)).$$

We denote $Q_\mu(\mathbf{x})$ by $\mathbf{Q}_\mu$. We can easily show that $\mathbf{Q}_\mu$ is essentially self-adjoint on $\bigoplus^m C_0^\infty(\mathbf{R}^d)$. $\mathbf{Q}_\mu$ leaves $\bigoplus^m \mathcal{S}(\mathbf{R}^d)$ invariant, and for $\mu, \nu = 0, 1, \ldots, d-1$,

$$[\mathbf{p}_\nu, \mathbf{Q}_\mu] = if_{\mu\nu}, \quad \mu, \nu = 0, 1, \ldots, d-1.$$

For $\mathbf{Q}$, we define an operator $\mathbf{L}_f$ as follows,

$$\mathbf{L}_f = \mathbf{Q}_\mu \mathbf{p}^\mu, \tag{2.18}$$

$$D(\mathbf{L}_f) = \bigoplus^{m} \bigcap_{\mu=0}^{d-1} D(Q_\mu p^\mu).$$

By the above commutation relations, the equation

$$\mathbf{p}^\mu \mathbf{Q}_\mu = \mathbf{Q}_\mu \mathbf{p}^\mu$$

holds on $D(\mathbf{L}_f)$. Hence, $\mathbf{L}_f$ is a symmetric operator on $\bigoplus^m L^2(\mathbf{R}^d)$.

Proposition 2.7 *$\mathbf{L}_f$ is essentially self-adjoint on $\bigoplus^m C_0^\infty(\mathbf{R}^d)$.*

Proof. Let

$$|x| = \sqrt{(x^0)^2 + (x^1)^2 + \cdots + (x^{d-1})^2}, \quad x = (x^0, \ldots, x^{d-1}),$$

$$\Delta = -\sum_{\mu=0}^{d-1} p_\mu^2.$$

Let $\mathbf{N}$ be the Hamiltonian of the harmonic oscillator operating on $\bigoplus^m L^2(\mathbf{R}^d)$;

$$\mathbf{N} : (\psi_1(x), \ldots, \psi_m(x)) \mapsto ((-\Delta + |x|^2)\psi_1(x), \ldots, (-\Delta + |x|^2)\psi_m(x)),$$

$$D(\mathbf{N}) = \bigoplus^m D(\Delta) \cap D(|x|^2).$$

We can see that $\mathbf{N}$ is essentially self-adjoint on $\bigoplus^m C_0^\infty(\mathbf{R}^d)$. We get for $\boldsymbol{\psi} \in \bigoplus^m C_0^\infty(\mathbf{R}^d)$,

$$\|\mathbf{L}_f \boldsymbol{\psi}\|^2 \leq C_1 \|(\mathbf{N}+1)\boldsymbol{\psi}\|^2, \tag{2.19}$$

$$|(\mathbf{L}_f \boldsymbol{\psi}, \mathbf{N}\boldsymbol{\psi}) - (\mathbf{N}\boldsymbol{\psi}, \mathbf{L}_f \boldsymbol{\psi})| \leq C_2 \|(\mathbf{N}+1)^{\frac{1}{2}} \boldsymbol{\psi}\|^2. \tag{2.20}$$

where C_1 and C_2 are positive constants. Hence, by Nelson's commutator theorem (for example, see [R-S2], Theorem X.37), we obtain the desired result. □

For each $a \in \mathbf{M}^d$

$$\mathcal{F}_a := \{f \in M_d^{\mathrm{as}}(\mathbf{R}) \mid a^\mu f_{\mu\nu} = 0,\ \nu = 0, 1, \ldots, d-1\}. \tag{2.21}$$

Proposition 2.8 *Let $a \in \mathbf{M}^d$.*

(1) *If $f \in \mathcal{F}_a$, then each $\mathbf{Q}_\mu$ and $a\mathbf{p}$ strongly commute.*

(2) *If $f \in \mathcal{F}_a$, then $\overline{\mathbf{L}_f}$ strongly commutes with $a\mathbf{x}$ and $a\mathbf{p}$.*

Proof. Similar to the proof of Proposition 3.2 in [A-T]. □

3. Lorentz transformations and operator calculus

Let $(a, b) \in \mathbf{M}_0$, $f \in \mathcal{F}_a \cap \mathcal{F}_b$ and $u \in \mathbf{B}_{\mathrm{real}}(\mathbf{R}^2)$. Then, by Proposition 2.8, $u(a\mathbf{x}, b\mathbf{p})$ and $\overline{\mathbf{L}_f}$ strongly commute. Putting

$$D_{f,u}^\infty := \bigcap_{j,k \in \mathbf{N}_0} D(u(a\mathbf{x}, b\mathbf{p})^j \overline{\mathbf{L}_f}^k). \tag{3.1}$$

The following operator

$$\mathbf{M}(u, \mathbf{L}_f) := \overline{[u(a\mathbf{x}, b\mathbf{p})\overline{\mathbf{L}_f}] \lceil D_{f,u}^\infty} \tag{3.2}$$

is self-adjoint. For $f \in M_d^{\mathrm{as}}(\mathbf{R})$ and $u \in L^\infty_{\mathrm{real}}(\mathbf{R}^2)$, we define the following bounded operator on the Hilbert space $\bigoplus_{\mu=0}^{d-1}[\bigoplus^m L^2(\mathbf{R}^d)]$

$$\mathbf{\Lambda}(f,u) := e^{-\tilde{f}u(a\mathbf{x},b\mathbf{p})} = \sum_{k=0}^{\infty} \frac{(-1)^k \tilde{f}^k u(a\mathbf{x},b\mathbf{p})^k}{k!}, \tag{3.3}$$

where

$$\tilde{f} = (f^\mu{}_\nu)_{\mu,\nu=0,\ldots,d-1}, \quad f^\mu{}_\nu = g^{\mu\lambda} f_{\lambda\nu}. \tag{3.4}$$

Note that each component of $(\mathbf{\Lambda}(f,u))$ is given as follows;

$$(\mathbf{\Lambda}(f,u))_{ij} = \sum_{k=0}^{\infty} \frac{(-1)^k [\tilde{f}^k]_{ij} u(a\mathbf{x},b\mathbf{p})^k}{k!} \quad (i,j=1,\ldots,m). \tag{3.5}$$

Lemma 3.1 *$\mathbf{\Lambda}(f,u)$ is a bounded Lorentz transformation on $\bigoplus_{\mu=0}^{d-1}[\bigoplus^m L^2(\mathbf{R}^d)]$,.*

Remark. A linear operator T represented as a matrix operator $T = (T^\mu{}_\nu)$ on $\mathcal{H} = \bigoplus_{\mu=0}^{d-1} \mathcal{H}_\mu$ (each $\mathcal{H}_\mu$ is a complex Hilbert space and $T^\mu{}_\nu$ is an operator from $\mathcal{H}_\nu$ to $\mathcal{H}_\mu$.) is called an operator-valued Lorentz transformation if

$$(T^\lambda{}_\mu)^* g_{\lambda\rho} T^\rho{}_\nu \subset g_{\mu\nu}$$

wehre $g = (g_{\mu\nu}) : \mathcal{H} \to \mathcal{H}$ is defined by

$$(g\psi)^0 = \psi^0, \quad (g\psi)^j = -\psi^j \quad (j = 1,\ldots,d-1)$$

To prove Lemma 3.1, we use the following proposition.

Proposition 3.2 *Let S be a symmetric operator on a Hilbert space $\mathcal{K}$ and $f \in M_d^{\mathrm{as}}(\mathbf{R})$. If S is bounded, then $e^{\tilde{f}S}$ is a bounded operator-valued Lorentz transformation on $\bigoplus_{\mu=0}^{d-1}\mathcal{K}$.*

(*For proof, see Proposition* 4.4 *of* [A-T].)

Proof of Lemma 3.1 Since $u(a\mathbf{x},b\mathbf{p})$ is a bounded symmetric operator on $\bigoplus^m L^2(\mathbf{R}^d)$, by Proposition 3.2, $e^{-\tilde{f}u(a\mathbf{x},b\mathbf{p})}$ is a bounded operator-valued Lorentz transformation on $\bigoplus_{\mu=0}^{d-1}[\bigoplus^m L^2(\mathbf{R}^d)]$. □

Lemma 3.3

(1) *If* $f \in \mathcal{F}_a$, *then*

$$\mathbf{\Lambda}(f,u)^{\mu}{}_{\nu}a^{\nu} = a^{\mu}, \mu = 0,1,\ldots,d-1 \tag{3.6}$$

on $\bigoplus^m L^2(\mathbf{R}^d)$.

(2) *Let* $u \in \mathfrak{B}^{1,\infty}(\mathbf{R}^2)$. *Then each component of* $\mathbf{\Lambda}(f,u)$ *leaves* $D(\mathbf{p}_\lambda)$ ($\lambda = 0,1,\ldots,d-1$) *invariant and for all* $\psi \in D(\mathbf{p}_\lambda)$,

$$\begin{aligned}&\mathbf{p}_\lambda \mathbf{\Lambda}(f,u)^{\mu}{}_{\nu}\psi \\ &\quad = -ia_\lambda \partial_1 u(a\mathbf{x}, b\mathbf{p})\mathbf{\Lambda}(f,u)^{\mu}{}_{\rho} f^{\rho}{}_{\nu}\psi + \mathbf{\Lambda}(f,u)^{\mu}{}_{\nu}\mathbf{p}_\lambda\psi.\end{aligned} \tag{3.7}$$

Proof. Similar to the proof of Lemma 5.1 in [A-T]. □

Theorem 3.4 *Let* $f \in \mathcal{F}_a \cap \mathcal{F}_b$ *and* $u \in \mathfrak{B}^{1,\infty}(\mathbf{R}^2)$. *Then for all* $\psi \in D(\mathbf{N})$, $e^{-i\mathbf{M}(u,\mathbf{L}_f)}\psi \in D(\mathbf{p}_\mu)$ ($\mu = 0,1,\ldots,d-1$) *and*

$$\begin{aligned}&e^{i\mathbf{M}(u,\mathbf{L}_f)}\mathbf{p}^{\mu}e^{-i\mathbf{M}(u,\mathbf{L}_f)}\psi \\ &\quad = \mathbf{\Lambda}(f,u)^{\mu}_{\nu}(\mathbf{p}^{\nu} + a^{\nu}\partial_1 u(a\mathbf{x}, b\mathbf{p})\overline{\mathbf{L}_f})\psi.\end{aligned} \tag{3.8}$$

Proof. Similar to the proof of Theorem 5.2 in [A-T]. □

Let $u, v \in \mathbf{B}_{\mathrm{real}}(\mathbf{R}^2)$ and $f \in \mathcal{F}_a \cap \mathcal{F}_b$. Then, by Proposition 2.8, $u(a\mathbf{x}, b\mathbf{p})$ and $\mathbf{M}(v, \mathbf{L}_f)$ strongly commute. Thus $u(a\mathbf{x}, b\mathbf{p}) + \mathbf{M}(v, \mathbf{L}_f)$ is essentially self-adjoint. We denote its closure by $\mathbf{M}(u; v, \mathbf{L}_f)$.

Theorem 3.5 *Let* $u, v \in \mathfrak{B}^{1,\infty}(\mathbf{R}^2)$ *and* $f \in \mathcal{F}_a \cap \mathcal{F}_b$. *Then* $e^{-i\mathbf{M}(u,v,\mathbf{L}_f)}\psi \in D(\mathbf{p}^{\mu})$ *for all* $\psi \in D(\mathbf{N})$ *and* $\mu = 0,1,\ldots,d-1$. *Moreover,*

$$\begin{aligned}&e^{i\mathbf{M}(u;v,\mathbf{L}_f)}\mathbf{p}^{\mu}e^{-i\mathbf{M}(u;v,\mathbf{L}_f)}\psi \\ &\quad = \mathbf{\Lambda}(f,v)^{\mu}{}_{\nu}\{\mathbf{p}^{\nu} + a^{\nu}(\partial_1 v(a\mathbf{x}, b\mathbf{p})\overline{\mathbf{L}_f} + \partial_1 u(a\mathbf{x}, b\mathbf{p})\}\psi.\end{aligned} \tag{3.9}$$

Proof. Similar to the proof of Theorem 5.5 in [A-T]. □

For $u, v \in \mathfrak{C}^1(\mathbf{R}^2)$, we define

$$\begin{aligned}\square_{m,f,u,v} &= -[\mathbf{p} + a(\partial_1 v(a\mathbf{x}, b\mathbf{p})\overline{\mathbf{L}_f} + \partial_1 u(a\mathbf{x}, b\mathbf{p})]^2 \\ &= [\boldsymbol{\partial}_0 - ia_0(\partial_1 v(a\mathbf{x}, b\mathbf{p})\overline{\mathbf{L}_f} + \partial_1 u(a\mathbf{x}, b\mathbf{p})]^2 \\ &\qquad - \sum_{j=1}^{d-1}[\boldsymbol{\partial}_j - ia_j(\partial_1 v(a\mathbf{x}, b\mathbf{p})\overline{\mathbf{L}_f} + \partial_1 u(a\mathbf{x}, b\mathbf{p})]^2,\end{aligned} \tag{3.10}$$

$$D(\square_{m,f,u,v}) = \bigoplus^{m} \bigcap_{\mu=0}^{d-1} D([\mathbf{p}_\mu + a_\mu(\partial_1 v(a\mathbf{x}, b\mathbf{p})\overline{\mathbf{L}_f} + \partial_1 u(a\mathbf{x}, b\mathbf{p})]^2.$$

Let $r, s \in \mathbf{N}_0$. We say that a function u on $\mathbf{R}^2$ is in the set $\mathfrak{B}^{r,s}(\mathbf{R}^2)$ if, for all $x_2 \in \mathbf{R}$, $u(\cdot, x_2) \in C^r_{\mathrm{real}}(\mathbf{R})$ and $\partial_1^j u \in \mathfrak{B}^{\infty,s}(\mathbf{R}^2) \cap C(\mathbf{R}^2)$ for $j = 0, \ldots, r$ with $\partial_2^k \partial_1^j u = \partial_1^j \partial_2^k u$, $j = 0, \ldots, r$, $k = 0, \ldots, s$, where $C(\mathbf{R}^2)$ denotes the space of continuous functions on $\mathbf{R}^2$.

Theorem 3.6 *Let $u, v \in \mathfrak{B}^{3,2}(\mathbf{R}^2)$ and $f \in \mathcal{F}_a \cap \mathcal{F}_b$. Then, for each $\mu = 0, 1, \ldots, d-1$, $e^{-i\mathbf{M}(u;v,\mathbf{L}_f)}$ maps $D(\mathbf{N}^2)$ into $D(\mathbf{p}^{\mu^2})$, and, for all $\psi \in D(\mathbf{N}^2)$,*

$$\begin{aligned} & e^{i\mathbf{M}(u;v,\mathbf{L}_f)}(\mathbf{p}^{\mu^2})e^{-i\mathbf{M}(u;v,\mathbf{L}_f)}\psi \\ &\quad = [\mathbf{\Lambda}(f,v)^\mu{}_\nu\{\mathbf{p}^\nu + a^\nu(\partial_1 v(a\mathbf{x}, b\mathbf{p})\overline{\mathbf{L}_f} + \partial_1 u(a\mathbf{x}, b\mathbf{p})\}]^2\psi. \end{aligned} \tag{3.11}$$

In particular, $D(\mathbf{N}^2) \subset D(\square_{m,f,u,v})$ and for $\psi \in D(\mathbf{N}^2)$

$$e^{i\mathbf{M}(u;v,\mathbf{L}_f)}\square_m e^{-i\mathbf{M}(u;v,\mathbf{L}_f)}\psi = \square_{m,f,u,v}\psi. \tag{3.12}$$

Proof. Similar to the proof of Theorem 5.10 in [A-T]. □

Let $w \in \mathbf{B}_{\mathrm{real}}(\mathbf{R}^2)$. Then we can define a self-adjoint operator $w(a\mathbf{x}, b\mathbf{p})$. Let K be an $m \times m$ Hermitian matrix. We can define a self-adjoint operator $Kw(a\mathbf{x}, b\mathbf{p})$ on $\bigoplus^m L^2(\mathbf{R}^d)$ by

$$(Kw(a\mathbf{x}, b\mathbf{p})\psi)_j = \sum_{k=1}^m K_{jk} w(ax, bp)\psi_k, \quad \psi \in D(\omega(a\mathbf{x}, b\mathbf{p}))$$

Theorem 3.7 *Let $w \in \mathfrak{C}^1(\mathbf{R}^2)$ and $\psi \in D(\mathbf{p}_\mu) \cap D(\partial_1 w(a\mathbf{x}, b\mathbf{p}))$. Then, for each $\mu = 0, 1, \ldots, d-1$, $e^{-iKw(a\mathbf{x}, b\mathbf{p})}\psi \in D(\mathbf{p}_\mu)$ and*

$$e^{iKw(a\mathbf{x}, b\mathbf{p})}\mathbf{p}_\mu e^{-iKw(a\mathbf{x}, b\mathbf{p})}\psi = \mathbf{p}_\mu\psi + a_\mu K\partial_1 w(a\mathbf{x}, b\mathbf{p})\psi. \tag{3.13}$$

Proof. The proof of this theorem is similar to that of Theorem 3.4. □

Let $u, v, w \in \mathbf{B}_{\mathrm{real}}(\mathbf{R}^2)$ and $f \in \mathcal{F}_a \cap \mathcal{F}_b$. Then, by Proposition 2.8, $u(a\mathbf{x}, b\mathbf{p})$ and $\mathbf{M}(w; v, \mathbf{L}_f)$ strongly commute. Thus, $Ku(a\mathbf{x}, b\mathbf{p}) + \mathbf{M}(w; v, \mathbf{L}_f)$ is essentially self-adjoint. We denote its closure by

$\mathbf{M}(u, K; w; v, \mathbf{L}_f)$. That is

$$\mathbf{M}(u, K; w; v, \mathbf{L}_f) := \overline{Ku(a\mathbf{x}, b\mathbf{p}) + \mathbf{M}(w; v, \mathbf{L}_f)}. \tag{3.14}$$

In particular, for $t \in \mathbf{R}$

$$e^{it\mathbf{M}(u,K;w;v,\mathbf{L}_f)} = e^{it\mathbf{M}(w;v,\mathbf{L}_f)} e^{itKu(a\mathbf{x},b\mathbf{p})} = e^{itKu(a\mathbf{x},b\mathbf{p})} e^{it\mathbf{M}(w;v,\mathbf{L}_f)}.$$

Theorem 3.8 *Let $u, v, w \in \mathfrak{B}^{1,\infty}(\mathbf{R}^2)$ and $f \in \mathcal{F}_a \cap \mathcal{F}_b$. Then $e^{-i\mathbf{M}(u,K;w;v\mathbf{L}_f)}\boldsymbol{\psi} \in D(\mathbf{p}^\mu)$ for all $\boldsymbol{\psi} \in D(\mathbf{N})$ and $\mu = 0, 1, \ldots, d-1$. Moreover,*

$$\begin{aligned} & e^{i\mathbf{M}(u,K;w;v,\mathbf{L}_f)} \mathbf{p}^\mu e^{-i\mathbf{M}(u,K;w;v,\mathbf{L}_f)} \boldsymbol{\psi} \\ & \quad = \boldsymbol{\Lambda}(f, v)^\mu{}_\nu \{\mathbf{p}^\nu + a^\nu(\partial_1 v(a\mathbf{x}, b\mathbf{p})\overline{\mathbf{L}_f} \\ & \qquad + \partial_1 w(a\mathbf{x}, b\mathbf{p}) + K\partial_1 u(a\mathbf{x}, b\mathbf{p}))\}\boldsymbol{\psi}. \end{aligned} \tag{3.15}$$

Proof. On $D(\mathbf{N})$ the following equation holds;

$$\begin{aligned} & e^{it\mathbf{M}(w;v,\mathbf{L}_f)} e^{itKu(a\mathbf{x},b\mathbf{p})} \mathbf{p}^\mu e^{-itKu(a\mathbf{x},b\mathbf{p})} e^{-it\mathbf{M}(w;v,\mathbf{L}_f)} \\ & \quad = e^{it\mathbf{M}(w;v,\mathbf{L}_f)} [\mathbf{p}_\mu + a_\mu K\partial_1 u(a\mathbf{x}, b\mathbf{p})] e^{-it\mathbf{M}(w;v,\mathbf{L}_f)} \\ & \quad = e^{it\mathbf{M}(w;v,\mathbf{L}_f)} \mathbf{p}_\mu e^{-it\mathbf{M}(w;v,\mathbf{L}_f)} + a_\mu K\partial_1 u(a\mathbf{x}, b\mathbf{p}) \end{aligned}$$

Then by Theorem 3.5, we can prove this theorem. □

For $u, v, w \in \mathfrak{C}^1(\mathbf{R}^2)$ and an $m \times m$ Hermitian matrix K, we define

$$\begin{aligned} & \Box_{m,K,f,u,v,w} \\ & \quad = -[\mathbf{p} + a(\partial_1 v(a\mathbf{x}, b\mathbf{p})\overline{\mathbf{L}_f} + \partial_1 w(a\mathbf{x}, b\mathbf{p}) + K\partial_1 u(a\mathbf{x}, b\mathbf{p}))]^2 \\ & \quad = [\boldsymbol{\partial}_0 - ia_0(\partial_1 v(a\mathbf{x}, b\mathbf{p})\overline{\mathbf{L}_f} + \partial_1 w(a\mathbf{x}, b\mathbf{p}) + K\partial_1 u(a\mathbf{x}, b\mathbf{p}))]^2 \\ & \qquad - \sum_{j=1}^{d-1} [\boldsymbol{\partial}_j - ia_j(\partial_1 v(a\mathbf{x}, b\mathbf{p})\overline{\mathbf{L}_f} \\ & \qquad + \partial_1 w(a\mathbf{x}, b\mathbf{p}) + K\partial_1 u(a\mathbf{x}, b\mathbf{p}))]^2, \end{aligned} \tag{3.16}$$

$$\begin{aligned} & D(\Box_{m,K,f,u,v,w}) \\ & \quad = \bigoplus^m \bigcap_{\mu=0}^{d-1} D([\mathbf{p}_\mu + a_\mu(\partial_1 v(a\mathbf{x}, b\mathbf{p})\overline{\mathbf{L}_f} \\ & \qquad + \partial_1 w(a\mathbf{x}, b\mathbf{p}) + K\partial_1 u(a\mathbf{x}, b\mathbf{p}))]^2). \end{aligned}$$

Theorem 3.9 *Let $u, v, w \in \mathfrak{B}^{3,2}(\mathbf{R}^2)$ and $f \in \mathcal{F}_a \cap \mathcal{F}_b$. Then, for each $\mu = 0, 1, \dots, d-1$, $e^{-i\mathbf{M}(u,K;w;v,\mathbf{L}_f)}$ maps $D(\mathbf{N}^2)$ into $D(\mathbf{p}^{\mu^2})$, and for all $\boldsymbol{\psi} \in D(\mathbf{N}^2)$,*

$$\begin{aligned} &e^{i\mathbf{M}(u,K;w;v,\mathbf{L}_f)}(\mathbf{p}^{\mu^2})e^{-i\mathbf{M}(u,K;w;v,\mathbf{L}_f)}\boldsymbol{\psi} \\ &\quad = [\mathbf{\Lambda}(f,v)^{\mu}{}_{\nu}\{\mathbf{p}^{\nu} + a^{\nu}(\partial_1 v(a\mathbf{x}, b\mathbf{p})\overline{\mathbf{L}_f} \\ &\qquad + \partial_1 w(a\mathbf{x}, b\mathbf{p}) + K\partial_1 u(a\mathbf{x}, b\mathbf{p}))\}]^2\boldsymbol{\psi}. \end{aligned} \tag{3.17}$$

In particular, $D(\mathbf{N}^2) \in D(\square_{m,K,f,u,v,w})$ and for $\boldsymbol{\psi} \in D(\mathbf{N}^2)$

$$e^{i\mathbf{M}(u,K;w;v,\mathbf{L}_f)}\square_m e^{-i\mathbf{M}(u,K;w;v,\mathbf{L}_f)}\boldsymbol{\psi} = \square_{m,K,f,u,v,w}\boldsymbol{\psi}. \tag{3.18}$$

The proof of of this theorem is similar to that of Theorem 3.6.
Let

$$\mathbf{H}_f(u,K;w,v) = e^{i\mathbf{M}(u,K;w;v,\mathbf{L}_f)}\mathbf{H}_0 e^{-i\mathbf{M}(u,K;w;v,\mathbf{L}_f)}$$

Corollary 3.10 *Let $u, v, w \in \mathfrak{B}^{3,2}(\mathbf{R}^2)$ and $f \in \mathcal{F}_a \cap \mathcal{F}_b$. Then $\mathbf{H}_f(u,K;w,v)$ is a self-adjoint extention of $\square_{m,K,f,u,v,w}\lceil_{D(\mathbf{N}^2)}$*

4. Integral kernels of the unitary groups generated by perturbed d'Alembertians

Let $\boldsymbol{\psi} \in \bigoplus^m L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$. Then, for each $\psi_j \in L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$ $(j = 1, 2, \dots, m)$,

$$(e^{isH_0}\psi_j)(x) = \int_{\mathbf{R}^d} \Delta_s(x,y)\psi_j(y)dy, \tag{4.1}$$

$$\Delta_s(x,y) = \frac{e^{i\epsilon(s)\pi(d-2)/4}}{2^d\pi^{d/2}|s|^{d/2}}e^{i(x-y)^2/4s}, \tag{4.2}$$

where $\epsilon(s)$ is the sign function.

Let $\mathcal{N}_d$ be the set of null vectors in $\mathbf{M}^d$, that is, each $x \in \mathcal{N}_d$ satisfies $x^2 = 0$. Let $H_f(u,v) := e^{iM(u;v,L_f)}H_0 e^{-iM(u;v,L_f)}$ ($M(u;v,L_f)$ and H_0 are $\mathbf{M}(u;v,\mathbf{L}_f)$ and $\mathbf{H}_0$ respectively in the case of $m = 1$.) and $\Phi_{v,f}(x,y;s)$ be a function on $\mathbf{R}^d \times \mathbf{R}^d \times \mathbf{R} \setminus \{0\}$ defined by

$$\begin{aligned} &\Phi_{v,f}(x,y;s) \\ &\quad := \frac{1}{2s}(y^{\mu} - q^{\mu})(1 - e^{-[u(ax,(by-bx)/2s)-u(ay,(by-bx)/2s)]f})_{\mu\nu}(x^{\nu} - q^{\nu}). \end{aligned} \tag{4.3}$$

Note that $e^{-[u(ax,(by-bx)/2s)-u(ay,(by-bx)/2s)]f}$ is a $d \times d$ matrix whose components are operators. Then we have the following theorem. (see Theorem 6.7 of [A-T].)

Theorem 4.1 *For each* $\psi_1, \ldots, \psi_m$, $u, v \in \mathbf{B}_{\mathrm{real}}(\mathbf{R}^2)$, $f \in \mathcal{F}_a \cap \mathcal{F}_b$, $s \in \mathbf{R} \setminus \{0\}$ *and* $a, b \in \mathcal{N}_d$, *we obtain*

$$\begin{aligned}&(e^{isH_f(u,v)}\psi_j)(x) \qquad (4.4)\\ &= \int_{\mathbf{R}^d} e^{i[u(ax,(by-bx)/2s)-u(ay,(by-bx)/2s)]+i\Phi_{v,f}(x,y;s)}\Delta_s(x,y)\psi_j(y)dy,\end{aligned}$$

Let $u, v, w \in \mathbf{B}_{\mathrm{real}}(\mathbf{R}^2)$. Now we can write

$$\begin{aligned}e^{is\mathbf{H}_f(u,K;w,v)} &= e^{i\mathbf{M}(u,K;w;v,\mathbf{L}_f)}e^{is\mathbf{H}_0}e^{-i\mathbf{M}(u,K;w;v,\mathbf{L}_f)}\\ &= e^{i\mathbf{M}(u,K;w;v,\mathbf{L}_f)}e^{is\mathbf{H}_0}e^{-i\mathbf{M}(u,K;w;v,\mathbf{L}_f)}e^{-is\mathbf{H}_0}e^{is\mathbf{H}_0}.\end{aligned}$$

Since the following equation hold on $\bigoplus^m \mathcal{S}(\mathbf{R}^d)$,

$$e^{is\mathbf{H}_0}\mathbf{x}^\mu e^{-is\mathbf{H}_0} = \mathbf{x}^\mu + 2s\mathbf{p}^\mu,$$

and $a\mathbf{x} + 2sa\mathbf{p}$ is essentially self-adjoint on $\bigoplus^m C_0^\infty(\mathbf{R}^d)$, by functional calculus, we obtain

$$\begin{aligned}e^{is\mathbf{H}_0}u(a\mathbf{x}, b\mathbf{p})e^{-is\mathbf{H}_0} &= u(\overline{a\mathbf{x} + 2sa\mathbf{p}}, b\mathbf{p})\\ e^{is\mathbf{H}_0}w(a\mathbf{x}, b\mathbf{p})e^{-is\mathbf{H}_0} &= w(\overline{a\mathbf{x} + 2sa\mathbf{p}}, b\mathbf{p}).\end{aligned}$$

Using the fact that $\overline{\mathbf{L}_f}$ strongly commute with $\mathbf{H}_0$ (see Lemma 6.6 in [A-T])

$$e^{is\mathbf{H}_0}v(a\mathbf{x}, b\mathbf{p})\overline{\mathbf{L}_f}e^{-is\mathbf{H}_0} = v(\overline{a\mathbf{x} + 2sa\mathbf{p}}, b\mathbf{p})\overline{\mathbf{L}_f}$$

Hence,

$$\begin{aligned}&e^{is\mathbf{H}_0}\mathbf{M}(u, K; w; v, \mathbf{L}_f)e^{-is\mathbf{H}_0} \qquad (4.5)\\ &= w(\overline{a\mathbf{x} + 2sa\mathbf{p}}, b\mathbf{p}) + Ku(\overline{a\mathbf{x} + 2sa\mathbf{p}}, b\mathbf{p}) + v(\overline{a\mathbf{x} + 2sa\mathbf{p}}, b\mathbf{p})\overline{\mathbf{L}_f},\end{aligned}$$

so that

$$\begin{aligned}&e^{is\mathbf{H}_f(u,K;w,v)}\\ &= e^{i\mathbf{M}(u,K;w;v,\mathbf{L}_f)}e^{-i[w(\overline{a\mathbf{x}+2sa\mathbf{p}},b\mathbf{p})+Ku(\overline{a\mathbf{x}+2sa\mathbf{p}},b\mathbf{p})+v(\overline{a\mathbf{x}+2sa\mathbf{p}},b\mathbf{p})\overline{\mathbf{L}_f}]}e^{is\mathbf{H}_0}\\ &= e^{iK[u(a\mathbf{x},b\mathbf{p})-u(\overline{a\mathbf{x}+2sa\mathbf{p}},b\mathbf{p})]}e^{is\mathbf{H}_f(w,v)}, \qquad (4.6)\end{aligned}$$

where

$$e^{is\mathbf{H}_f(w,v)} = e^{i[w(a\mathbf{x},b\mathbf{p})-w(\overline{a\mathbf{x}+2sa\mathbf{p}},b\mathbf{p})]}e^{i[v(a\mathbf{x},b\mathbf{p})-v(\overline{a\mathbf{x}+2sa\mathbf{p}},b\mathbf{p})\overline{\mathbf{L}_f}]}e^{is\mathbf{H}_0}.$$

Thus for all $\boldsymbol{\psi} \in \bigoplus^m L^2(\mathbf{R}^2)$,

$$e^{is\mathbf{H}_f(u,K;w,v)}\boldsymbol{\psi} = e^{iK[u(a\mathbf{x},b\mathbf{p})-u(\overline{a\mathbf{x}+2sa\mathbf{p}},b\mathbf{p})]}(e^{isH_f(w,v)}\psi_1(x),\dots,e^{isH_f(w,v)}\psi_m(x)). \tag{4.7}$$

Let $\lambda_1,\dots,\lambda_m(\lambda_1 \le \cdots \le \lambda_m)$ be eigenvalues of the Hermitian matrix K and V be a unitary matrix satisfying

$$K = V\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{pmatrix}V^{-1}. \tag{4.8}$$

We denote the above diagonal matrix by K_d. Then

$$e^{iK(u(a\mathbf{x},b\mathbf{p})-u(\overline{a\mathbf{x}+2sa\mathbf{p}},b\mathbf{p}))} = Ve^{iK_d(u(a\mathbf{x},b\mathbf{p})-u(\overline{a\mathbf{x}+2sa\mathbf{p}},b\mathbf{p}))}V^{-1}. \tag{4.9}$$

Since

$$e^{iK_d(u(a\mathbf{x},b\mathbf{p})-u(\overline{a\mathbf{x}+2sa\mathbf{p}},b\mathbf{p}))}:$$

$$\begin{pmatrix} \psi_1(x) \\ \vdots \\ \psi_m(x) \end{pmatrix} \mapsto \begin{pmatrix} e^{i\lambda_1(u(ax,bp)-u(\overline{ax+2sap},bp))}\psi_1(x) \\ \vdots \\ e^{i\lambda_m(u(ax,bp)-u(\overline{ax+2sap},bp))}\psi_m(x) \end{pmatrix},$$

we can write

$$\begin{aligned} e^{is\mathbf{H}_f(u,K;w,v)}\boldsymbol{\psi} &= Ve^{iK_d(u(a\mathbf{x},b\mathbf{p})-u(\overline{a\mathbf{x}+2sa\mathbf{p}},b\mathbf{p}))}V^{-1}\begin{pmatrix} e^{isH_f(w,v)}\psi_1(x) \\ \vdots \\ e^{isH_f(w,v)}\psi_m(x) \end{pmatrix} \\ &= V\begin{pmatrix} e^{i\lambda_1(u(ax,bp)-u(\overline{ax+2sap},bp))}(V^{-1}e^{is\mathbf{H}_f(w,v)}\boldsymbol{\psi})_1(x) \\ \vdots \\ e^{i\lambda_m(u(ax,bp)-u(\overline{ax+2sap},bp))}(V^{-1}e^{is\mathbf{H}_f(w,v)}\boldsymbol{\psi})_m(x) \end{pmatrix} \end{aligned}$$

$$= V \begin{pmatrix} e^{i\lambda_1(u(ax,bp)-u(\overline{ax+2sap},bp))}e^{isH_f(w,v)}(V^{-1}\boldsymbol{\psi})_1(x) \\ \vdots \\ e^{i\lambda_m(u(ax,bp)-u(\overline{ax+2sap},bp))}e^{isH_f(w,v)}(V^{-1}\boldsymbol{\psi})_m(x) \end{pmatrix} \quad (4.10)$$

By Theorem 4.1, if $w, v \in \mathbf{B}_{\mathrm{real}}(\mathbf{R}^2)$, $f \in \mathcal{F}_a \cap \mathcal{F}_b$ and $a, b \in \mathcal{N}_d$, we obtain

$$(e^{isH_f(w,v)}\phi)(x) = \int_{\mathbf{R}^d} e^{i[w(ax,(by-bx)/2s)-w(ay,(by-bx)/2s)]+i\Phi_{v,f}(x,y;s)}\Delta_s(x,y)\phi(y)dy,$$

for all $\phi \in L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$ and $s \in \mathbf{R}\setminus\{0\}$. Now we set

$$k(x,y) = e^{i[w(ax,(by-bx)/2s-w(ay,(by-bx)/2s)]+i\Phi_{v,f}(x,y;s)}\Delta_s(x,y). \quad (4.11)$$

It is easy to show that $|k(x,y)|$ is bounded and for all $\xi_1, \xi_2 \in \mathbf{R}$,

$$k(x-\xi_1 a-\xi_2 b, y) = e^{i\xi_1(ay-ax)/2s+i\xi_2(by-bx)/2s}k(x,y), \quad (x,y) \in \mathbf{R}^d \times \mathbf{R}^d.$$

We set

$$F_j(x_1,x_2,x_3) = e^{i\lambda_j(u(x_1,x_3)-u(x_1+2sx_2,x_3)} \quad (j = 1,2,\ldots,m).$$

Since $F_j \in L^\infty(\mathbf{R}^3)$, by Lemma 6.2 of [A-T], we obtain for all $\phi \in L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$ and

$$(F_j(ax,ap,bp)K_0\phi)(x) = \int_{\mathbf{R}^d} F_j\left(ax, \frac{ay-ax}{2s}, \frac{by-bx}{2s}\right)k(x,y)\phi(y)dy,$$

where K_0 is the operator of which kernel is given by $k(x,y)$. Hence

$$\begin{aligned}(e^{i\lambda_j(u(ax,bp)-u(ax+2sap,bp))}&e^{isH_f(w,v)}\phi_j)(x) \\ &= \int_{\mathbf{R}^d} e^{i\lambda_j[u(ax,(by-bx)/2s)-u(ay,(by-bx)/2s)]} \\ &\quad \times e^{i[w(ax,(by-bx)/2s)-w(ay,(by-bx)/2s)]+i\Phi_{v,f}(x,y;s)}\Delta_s(x,y)\phi(y)dy.\end{aligned} \quad (4.12)$$

Since obviously $V^{-1}\boldsymbol{\psi} \in \bigoplus^m L^1(\mathbf{R}^d)\cap L^2(\mathbf{R}^d)$ for $\boldsymbol{\psi} \in \bigoplus^m L^1(\mathbf{R}^d)\cap L^2(\mathbf{R}^d)$, we obtain

$$\begin{aligned}
&e^{i\lambda_j(u(ax,bp)-u(ax+2sap,bp))}e^{isH_f(w,v)}(V^{-1}\boldsymbol{\psi})_j(x)\\
&=\int_{\mathbf{R}^d} e^{i\lambda_j[u(ax,(by-bx)/2s)-u(ay,(by-bx)/2s)]} \\
&\quad\times e^{i[w(ax,(by-bx)/2s)-w(ay,(by-bx)/2s)]+i\Phi_{v,f}(x,y;s)}\Delta_s(x,y)(V^{-1}\boldsymbol{\psi})(y)dy.
\end{aligned}\tag{4.13}$$

By the previous arguments, we obtain the following theorem.

Theorem 4.2 *Let $u, v, w \in \mathbf{B}_{\mathrm{real}}(\mathbf{R}^2)$, $f \in \mathcal{F}_a \cap \mathcal{F}_b$ and $a, b \in \mathcal{N}_d$. Let K be an $m \times m$-Hermite matrix and $V = (v_{ij})$ be a unitary matrix satisfying*

$$K = V \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{pmatrix} V^{-1},$$

where $\{\lambda_j\}_{j=1,\ldots,m}(\lambda_1 \leq \cdots \leq \lambda_m)$ are eigenvalues of K. Then for all $\boldsymbol{\psi} \in \bigoplus^m L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$, and $s \in \mathbf{R} \setminus \{0\}$.

$$\begin{aligned}
&(e^{is\mathbf{H}_f(u,K;w,v)}\boldsymbol{\psi})_j(x)\\
&=\sum_{k=1}^m\sum_{l=1}^m v_{jk}\overline{v_{lk}}\int_{\mathbf{R}^d} e^{i\lambda_k[u(ax,(by-bx)/2s)-u(ay,(by-bx)/2s)]} \\
&\quad\times e^{i[w(ax,(by-bx)/2s)-w(ay,(by-bx)/2s)]+i\Phi_{v,f}(x,y;s)}\Delta_s(x,y)\psi_l(y)dy,
\end{aligned}\tag{4.14}$$

where $(e^{is\mathbf{H}_f(u,K;w,v)}\boldsymbol{\psi})_j$ is the j-th component of $e^{is\mathbf{H}_f(u,K;w,v)}\boldsymbol{\psi}$.

5. Application to the external field problem for a spin-$\frac{1}{2}$ particle

In this section, we apply the operator theory in the preceding sections to the external field problem for a spin-$\frac{1}{2}$ charged particle.

Let $a \in \mathcal{N}_d$, $W \in C^2_{\mathrm{real}}(\mathbf{R})$, $u_\epsilon = u_\epsilon(t)$ (ϵ is a parameter) be a function in $C^1_{\mathrm{real}}(\mathbf{R})$ satisfying the following properties.

(1) $tu_\epsilon \in \mathfrak{B}^1(\mathbf{R})$.

(2) $\sup_{t\in\mathbf{R}} |tu_\epsilon(t)| \leq C$ (C is a constant independent of ϵ).

(3) $\lim_{\epsilon\to 0} u_\epsilon(t) = \frac{1}{t}$, $t \in \mathbf{R} \setminus \{0\}$.

Let K be an $m \times m$ Hermitian matrix.

Let $F = (F_{\mu\nu})_{\mu,\nu=0,\ldots,d-1}$ be a tensor field on $\mathbf{M}^d$. Physically, $F_{\mu\nu}$ is an electromagnetic field. A vector potential $A = (A_0, \ldots, A_{d-1})$ of the electromagnetic field is a vector field on $\mathbf{M}^d$ such that

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{5.1}$$

We assume that the vector potential A is the form

$$A_\mu(x) = Q_\mu(x)W'(ax), \qquad \mu = 0, \dots, d-1, \tag{5.2}$$

where $Q_\mu(x) = f_{\mu\nu}(x^\nu - q^\nu)$ ($q \in \mathbf{M}^d$ is a constant vector) with $f \in M_d^{\mathrm{as}}(\mathbf{R})$.

For A_μ, we define

$$\mathbf{A}_\mu : (\psi_1(x), \dots, \psi_m(x)) \mapsto (A_\mu(x)\psi_1(x), \dots, A_\mu(x)\psi_m(x)) \tag{5.3}$$

on $\bigoplus^m \mathcal{S}(\mathbf{R}^d)$. We define

$$\begin{aligned} A_\mu^\epsilon &= apu_\epsilon(ap)A_\mu \\ &\qquad + \frac{1}{2}a_\mu u_\epsilon(ap)[1 - apu_\epsilon(ap)](ax - aq)^2 W'(ax)^2 \end{aligned} \tag{5.4}$$

and

$$\mathbf{A}_\mu^\epsilon : (\psi_1(x), \dots, \psi_m(x)) \mapsto (A_\mu^\epsilon(x)\psi_1(x), \dots, A_\mu^\epsilon(x)\psi_m(x)). \tag{5.5}$$

For all $\boldsymbol{\psi} \in \bigoplus^m[\bigcap_{\mu=0}^{d-1} D(A_\mu)] \cap D((ap)^{-1}(ax - aq)^2 W'(ax)^2)$, we can see that

$$\lim_{\epsilon \to 0} \mathbf{A}_\mu^\epsilon \boldsymbol{\psi} = \mathbf{A}_\mu \boldsymbol{\psi}, \quad \mu = 0, 1, \dots, d-1.$$

For $W \in C^2_{\mathrm{real}}$, let

$$Y(t) = W(t) + (t - aq)W'(t). \tag{5.6}$$

We take $W(t)$ such that $Y(t)$ is bounded. For $\mathbf{A}_\mu^\epsilon$ and $Y(t)$,

$$\begin{aligned} Y(a\mathbf{x}) : (\psi_1(x), \dots, \psi_m(x)) &\mapsto (Y(ax)\psi_1(x), \dots, Y(ax)\psi_m(x)) \\ Y'(a\mathbf{x}) : (\psi_1(x), \dots, \psi_m(x)) &\mapsto (Y'(ax)\psi_1(x), \dots, Y'(ax)\psi_m(x)) \end{aligned}$$

and

$$\mathbf{B}_\mu^\epsilon := \mathbf{A}_\mu^\epsilon - \frac{a_\mu}{4} K Y'(a\mathbf{x}) u_\epsilon(a\mathbf{p}). \tag{5.7}$$

That is, for all $\boldsymbol{\psi} = (\psi_1(x), \dots, \psi_m(x))$,

$$\begin{aligned} \mathbf{B}_\mu^\epsilon : (\psi_1(x), \dots, \psi_m(x)) &\mapsto (A_\mu^\epsilon(x)\psi_1(x), \dots, A_\mu^\epsilon(x)\psi_m(x)) \\ &\quad - \frac{a_\mu}{4} K(Y'(ax)u_\epsilon(ap)\psi_1(x), \dots, Y'(ax)u_\epsilon(ap)\psi_m(x)). \end{aligned}$$

For $\mathbf{B}^{\epsilon}_{\mu}$, we define

$$\mathbf{\Pi}_{0\mu}(\epsilon) := \mathbf{p}_{\mu} - \mathbf{B}^{\epsilon}_{\mu}, \tag{5.8}$$

$$D(\mathbf{\Pi}_{0\mu}(\epsilon)) = D(\mathbf{p}_{\mu}) \cap D(\mathbf{B}^{\epsilon}_{\mu}).$$

Let

$$\mathfrak{W}_r = \left\{ W \in \mathfrak{B}^r(\mathbf{R}) \mid (t - aq)\frac{d^k W}{dt^k} \in L^{\infty}(\mathbf{R}) \ k = 1, \ldots, r \right\},$$

where $r \in \mathbf{N}_0$.

Lemma 5.1 *Let $W \in \mathfrak{W}_2$. Then $\mathbf{\Pi}_{0\mu}(\epsilon)$ is essentially self-adjoint on $\bigoplus^m C_0^{\infty}(\mathbf{R}^d)$*

Proof. $\mathbf{p}_{\mu}$ essentially self-adjoint on $\bigoplus^m C_0^{\infty}(\mathbf{R}^d)$. Since, for $\epsilon > 0$, $\mathbf{A}^{\epsilon}_{\mu}$ and $(a_{\mu}/4)KY'(a\mathbf{x})u_{\epsilon}(a\mathbf{p})$ is bounded, $\mathbf{p}_{\mu} - \mathbf{B}^{\epsilon}_{\mu}$ is essentially self-adjoint on $\bigoplus^m C_0^{\infty}(\mathbf{R}^d)$. □

For $Y(t)$, we introduce on operator-valued Lorentz transformation on $\bigoplus_{\mu=0}^{d-1} \bigoplus^m L^2(\mathbf{R}^d)$:

$$\mathbf{\Lambda}(\epsilon) := e^{\tilde{f}Y(a\mathbf{x})u_{\epsilon}(a\mathbf{p})} \tag{5.9}$$

and set

$$\tilde{\mathbf{\Pi}}^{\mu}(\epsilon) := \mathbf{\Lambda}(\epsilon)^{\mu}{}_{\nu}\mathbf{\Pi}_0{}^{\nu}(\epsilon). \tag{5.10}$$

We define

$$\mathcal{G}_a := \{ f \in \mathcal{F}_a \mid f^{\mu}{}_{\lambda} f^{\lambda}{}_{\nu} = a^{\mu} a_{\nu}, \ \mu, \nu = 0, \ldots, d-1 \}. \tag{5.11}$$

Lemma 5.2 *Let $W \in \mathfrak{W}_2$, $a \in \mathcal{N}_d$ and $f \in \mathcal{G}_a$. Then $\tilde{\mathbf{\Pi}}^{\mu}(\epsilon)$ is essentially self-adjoint on $\bigoplus^m C_0^{\infty}(\mathbf{R}^d)$*

Proof. We have for all $\boldsymbol{\psi} \in \bigoplus^m C_0^{\infty}(\mathbf{R}^d)$,

$$\tilde{\mathbf{\Pi}}^{\mu}(\epsilon)\boldsymbol{\psi} = \mathbf{P}_{\epsilon}{}^{\mu}\boldsymbol{\psi} + \mathbf{V}_{\epsilon}{}^{\mu}\boldsymbol{\psi},$$

where

$$\mathbf{P}_{\epsilon}{}^{\mu} = \mathbf{p}^{\mu} + f^{\mu}{}_{\nu}\mathbf{p}^{\nu}Y(a\mathbf{x})u_{\epsilon}(a\mathbf{p}) - a\mathbf{p}u_{\epsilon}(a\mathbf{p})W'(a\mathbf{x})\mathbf{Q}^{\mu}$$

$$\mathbf{V}_{\epsilon}{}^{\mu} = \frac{1}{2}a^{\mu}a\mathbf{p}u_{\epsilon}(a\mathbf{p})^2 Y(a\mathbf{x})^2 - a^{\mu}a\mathbf{p}u_{\epsilon}(a\mathbf{p})^2 Y(a\mathbf{x})(a\mathbf{x} - a\mathbf{q})W'(a\mathbf{x})$$

$$+\frac{1}{2}a^\mu u_\epsilon(a\mathbf{p})[a\mathbf{p}u_\epsilon(a\mathbf{p})-1](a\mathbf{x}-a\mathbf{p})^2W'(a\mathbf{x})^2$$
$$+\frac{1}{4}a^\mu KY'(a\mathbf{x})u_\epsilon(a\mathbf{p}).$$

${\mathbf{P}_\epsilon}^\mu$ is symmetric on $\bigoplus^m C_0^\infty(\mathbf{R}^d)$ and we have

$$\|{\mathbf{P}_\epsilon}^\mu\boldsymbol{\psi}\| \le C\|(\mathbf{N}+1)\boldsymbol{\psi}\| \quad (C:\text{constant}), \tag{5.12}$$

$$|({\mathbf{P}_\epsilon}^\mu\boldsymbol{\psi},\mathbf{N}\boldsymbol{\psi})-(\mathbf{N}\boldsymbol{\psi},{\mathbf{P}_\epsilon}^\mu\boldsymbol{\psi})| \le D\|(\mathbf{N}+1)^{1/2}\boldsymbol{\psi}\| \quad (D:\text{constant}) \tag{5.13}$$

for $\boldsymbol{\psi}\in\bigoplus^m C_0^\infty(\mathbf{R}^d)$. Since $\mathbf{N}$ is essentially self-adjoint on $\bigoplus^m C_0^\infty(\mathbf{R}^d)$, by Nelson's commutator theorem, we obtain that ${\mathbf{P}_\epsilon}^\mu$ is essentially self-adjoint on $\bigoplus^m C_0^\infty(\mathbf{R}^d)$. It is easy to show that $u_\epsilon(a\mathbf{p})$, $a\mathbf{p}u_\epsilon(a\mathbf{p})$ and $Y(a\mathbf{x})$ are bounded. Thus ${\mathbf{V}_\epsilon}^\mu$ is bounded and self-adjoint. Hence ${\mathbf{P}_\epsilon}^\mu+{\mathbf{V}_\epsilon}^\mu$ is essentially self-adjoint on $\bigoplus^m C_0^\infty(\mathbf{R}^d)$. □

Let

$$\Omega(t)=\int_{aq}^t (s-aq)^2W'(s)^2ds \quad (t\in\mathbf{R}) \tag{5.14}$$

for $W\in C^1_{\mathrm{real}}(\mathbf{R})$. Put

$$\mathbf{M}_\epsilon := \mathbf{M}\left(\frac{-Y\otimes u_\epsilon}{4},K;\frac{\Omega\otimes u_\epsilon}{2};W\otimes u_\epsilon,\mathbf{L}_f\right), \tag{5.15}$$

$$U(\epsilon;K):=e^{-i\mathbf{M}_\epsilon},$$

where $\mathbf{M}$ is given by (3.34).

Let

$$\mathcal{G}_a^0 := \{f\in\mathcal{G}_a \mid f_{\mu\nu}a_\lambda+f_{\nu\lambda}a_\mu+f_{\lambda\mu}a_\nu=0,\ \mu,\nu,\lambda=0,\dots,d-1\}. \tag{5.16}$$

Theorem 5.3 *Let $W\in\mathfrak{W}_2$, $a\in\mathcal{N}_d$ and $f\in\mathcal{G}_a^0$. Assume that $(t-aq)W'\in L^2(\mathbf{R})$. Then*

$$U(\epsilon;K)\mathbf{p}^\mu U(\epsilon;K)^{-1}=\overline{\tilde{\mathbf{\Pi}}^\mu(\epsilon)}. \tag{5.17}$$

Proof. For all $\psi \in \bigoplus^m \mathcal{S}(\mathbf{R}^d)$,

$$U(\epsilon; K)\mathbf{p}^\mu U(\epsilon; K)^{-1}\psi$$
$$= \mathbf{\Lambda}(\epsilon)^\mu{}_\nu \Big\{ \mathbf{\Lambda}(f, (Y - W) \otimes u_\epsilon)^\nu{}_\lambda \mathbf{p}^\lambda \psi - a^\nu u_\epsilon(a\mathbf{p}) W'(a\mathbf{x}) \mathbf{L}_f \psi$$
$$- \frac{a^\nu u_\epsilon(a\mathbf{p}) \Omega'(a\mathbf{x})}{2} \psi + \frac{a^\nu}{4} K Y'(a\mathbf{x}) u_\epsilon(a\mathbf{p}) \psi \Big\}. \tag{5.18}$$

Using

$$\mathbf{\Lambda}(f, (Y - W) \otimes u_\epsilon)^\nu{}_\lambda \mathbf{p}^\lambda \psi$$
$$= \mathbf{p}^\nu \psi - f^\nu{}_\lambda u_\epsilon(a\mathbf{p})(Y(a\mathbf{x}) - W(a\mathbf{x}))\mathbf{p}^\lambda \psi$$
$$+ \frac{1}{2} a^\nu a\mathbf{p} u_\epsilon(a\mathbf{p})^2 (Y(a\mathbf{x}) - W(a\mathbf{x}))^2 \psi \tag{5.19}$$

and

$$a^\nu \mathbf{L}_f \psi = a\mathbf{p}\mathbf{Q}^\nu \psi - f^\nu{}_\lambda \mathbf{p}^\lambda (a\mathbf{x} - a\mathbf{q})\psi,$$

we obtain

$$U(\epsilon; K)\mathbf{p}^\mu U(\epsilon; K)^{-1}\psi = \tilde{\mathbf{\Pi}}^\mu(\epsilon)\psi$$

for $\psi \in \bigoplus^m \mathcal{S}(\mathbf{R}^d)$. By Lemma 5.2, we get the desired result. □

Now, as an Hermitian matrix K, we take

$$K = f_{\mu\nu}\sigma^{\mu\nu}, \tag{5.20}$$

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu], \tag{5.21}$$

where γ^μ ($\mu = 0, 1, \ldots, d-1$) are $m \times m$-matrices satisfying the following anti-commutation relations,

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} I. \tag{5.22}$$

I is the unit matrix. We denote $f_{\mu\nu}\sigma^{\mu\nu}$ by $f \cdot \sigma$. We define

$$\mathbf{H}(\epsilon; f \cdot \sigma) := U(\epsilon; f \cdot \sigma)\mathbf{H}_0 U(\epsilon; f \cdot \sigma)^{-1} \tag{5.23}$$

By Corollary 3.10, we obtain the following result.

Corollary 5.4 *Let $W \in \mathfrak{W}_3$, $a \in \mathcal{N}_d$ and $f \in \mathcal{G}_a^0$. Assume that $(t - aq)W' \in L^2(\mathbf{R})$ and $u_\epsilon \in \mathfrak{B}^2(\mathbf{R})$. Then $\mathbf{H}(\epsilon; f \cdot \sigma)$ is a self-adjoint extension of $-\mathbf{\Pi}_0(\epsilon)^2 \lceil_{D(\mathbf{N}^2)}$.*

Let $W \in \mathfrak{W}_3$, $a \in \mathcal{N}_d$ and $f \in \mathcal{G}_a^0$. Assume that $(t - aq)W' \in L^2(\mathbf{R})$ and $u_\epsilon \in \mathfrak{B}^2(\mathbf{R})$. By Theorem 3.9,

$$\begin{aligned}\mathbf{\Pi}_0(\epsilon)^2 &= \Big[\mathbf{\Lambda}(\epsilon)\Big\{\mathbf{p} - \mathbf{A}^\epsilon + \frac{a}{4}(f, \sigma)Y'(a\mathbf{x})u_\epsilon(a\mathbf{p})\Big\}\Big]^2 \\ &:= \Big[\mathbf{\Lambda}(\epsilon)^0{}_\nu\Big\{\mathbf{p}^\nu - \mathbf{A}^{\epsilon\nu} + \frac{a^\nu}{4}(f \cdot \sigma)Y'(a\mathbf{x})u_\epsilon(a\mathbf{p})\Big\}\Big]^2 \\ &\quad - \sum_{j=1}^{d-1}\Big[\mathbf{\Lambda}(\epsilon)^j{}_\nu\Big\{\mathbf{p}^\nu - \mathbf{A}^{\epsilon\nu} + \frac{a^\nu}{4}(f \cdot \sigma)Y'(a\mathbf{x})u_\epsilon(a\mathbf{p})\Big\}\Big]^2\end{aligned}$$

on $D(\mathbf{N}^2)$. We can write

$$\begin{aligned}&\Big[\mathbf{\Lambda}(\epsilon)\Big\{\mathbf{p} - \mathbf{A}^\epsilon + \frac{a}{4}(f \cdot \sigma)Y'(a\mathbf{x})u_\epsilon(a\mathbf{p})\Big\}\Big]^2 \\ &\quad = [\mathbf{\Lambda}(\epsilon)(\mathbf{p} - \mathbf{A}^\epsilon)]^2 + \frac{1}{2}(f \cdot \sigma)Y'(a\mathbf{x})a\mathbf{p}u_\epsilon(a\mathbf{p}) \\ &\quad = \hat{\mathbf{\Pi}}(\epsilon)^2 + \frac{1}{2}(f \cdot \sigma)Y'(a\mathbf{x})a\mathbf{p}u_\epsilon(a\mathbf{p}),\end{aligned} \tag{5.24}$$

where

$$\hat{\mathbf{\Pi}}^\mu(\epsilon) = \mathbf{\Lambda}(\epsilon)^\mu{}_\nu(\mathbf{p}^\nu - \mathbf{A}^{\epsilon\nu}), \tag{5.25}$$

$$\begin{aligned}\hat{\mathbf{\Pi}}(\epsilon)^2 &= [\mathbf{\Lambda}(\epsilon)(\mathbf{p} - \mathbf{A}^\epsilon)]^2 \\ &= [\mathbf{\Lambda}(\epsilon)^0{}_\nu(\mathbf{p}^\nu - \mathbf{A}^{\epsilon\nu})]^2 - \sum_{j=1}^{d-1}[\mathbf{\Lambda}(\epsilon)^j{}_\nu(\mathbf{p}^\nu - \mathbf{A}^{\epsilon\nu})]^2.\end{aligned}$$

Moreover,

$$\begin{aligned}&f_{\mu\nu}Y'(a\mathbf{x})a\mathbf{p}u_\epsilon(a\mathbf{p}) \\ &\quad = a\mathbf{p}u_\epsilon(a\mathbf{p})f_{\mu\nu}\{2W'(a\mathbf{x}) + (a\mathbf{x} - a\mathbf{q})W''(a\mathbf{x})\} \\ &\quad = a\mathbf{p}u_\epsilon(a\mathbf{p})\{(f_{\mu\nu} - f_{\nu\mu})W'(a\mathbf{x}) + (a_\nu\mathbf{Q}_\mu - a_\mu\mathbf{Q}_\nu)W''(a\mathbf{x})\}.\end{aligned}$$

Since $(a\mathbf{x} - a\mathbf{q})^2\boldsymbol{\psi} \in \bigoplus^m L^2(\mathbf{R}^d)$ for all $\boldsymbol{\psi} \in D(\mathbf{N}^2)$, we get

$$\begin{aligned}&[\mathbf{p}_\nu, a_\mu(a\mathbf{x} - a\mathbf{q})^2W'(a\mathbf{x})^2] \\ &\qquad = 2a_\mu a_\nu(a\mathbf{x} - a\mathbf{q})W'(a\mathbf{x})^2 + 2a_\mu a_\nu(a\mathbf{x} - a\mathbf{q})^2W''(a\mathbf{x}) \\ &[\mathbf{p}_\mu, a_\nu(a\mathbf{x} - a\mathbf{q})^2W'(a\mathbf{x})^2] \\ &\qquad = 2a_\mu a_\nu(a\mathbf{x} - a\mathbf{q})W'(a\mathbf{x})^2 + 2a_\mu a_\nu(a\mathbf{x} - a\mathbf{q})^2W''(a\mathbf{x}).\end{aligned}$$

Hence

$$[\mathbf{p}_\nu, a_\mu(a\mathbf{x} - a\mathbf{q})^2 W'(a\mathbf{x})^2] = [\mathbf{p}_\mu, a_\nu(a\mathbf{x} - a\mathbf{q})^2 W'(a\mathbf{x})^2] \qquad (5.26)$$

on $D(\mathbf{N}^2)$. We can see that

$$\begin{aligned} [\mathbf{p}_\nu, \mathbf{A}_\mu] &= i f_{\mu\nu} W' + i a_\nu \mathbf{Q}_\mu W''(a\mathbf{x}) \\ [\mathbf{p}_\mu, \mathbf{A}_\nu] &= i f_{\nu\mu} W' + i a_\mu \mathbf{Q}_\nu W''(a\mathbf{x}). \end{aligned}$$

on $D(\mathbf{N}^2)$. Using the above results, we obtain

$$\begin{aligned} &[\mathbf{p}_\nu, \mathbf{A}^\epsilon_\mu] - [\mathbf{p}_\mu, \mathbf{A}^\epsilon_\nu] \qquad (5.27) \\ &\quad = i a \mathbf{p} u_\epsilon(a\mathbf{p})\{(f_{\mu\nu} - f_{\nu\mu}) W'(a\mathbf{x}) + (a_\nu \mathbf{Q}_\mu - a_\mu \mathbf{Q}_\nu) W''(a\mathbf{x})\}. \end{aligned}$$

Thus,

$$\begin{aligned} &\hat{\mathbf{\Pi}}(\epsilon)^2 + \frac{1}{2}(f\cdot\sigma) Y'(a\mathbf{x}) a\mathbf{p} u_\epsilon(a\mathbf{p}) \\ &\quad = \hat{\mathbf{\Pi}}(\epsilon)^2 - \frac{i}{2}\sigma^{\mu\nu}\{[\mathbf{p}_\nu, \mathbf{A}^\epsilon_\mu] - [\mathbf{p}_\mu, \mathbf{A}^\epsilon_\nu]\} \\ &\quad = \gamma^\mu(\mathbf{p}_\mu - \mathbf{A}^\epsilon_\mu)\gamma^\nu(\mathbf{p}_\nu - \mathbf{A}^\epsilon_\nu). \qquad (5.28) \end{aligned}$$

Hence, we can write

$$\mathbf{\Pi}_0(\epsilon)^2 = \gamma^\mu(\mathbf{p}_\mu - \mathbf{A}^\epsilon_\mu)\gamma^\nu(\mathbf{p}_\nu - \mathbf{A}^\epsilon_\nu). \qquad (5.29)$$

Thus we obtain the following result.

Theorem 5.5 *Let $W \in \mathfrak{W}_3$, $a \in \mathcal{N}_d$ and $f \in \mathcal{G}^0_a$. Assumne that $(t - aq)W' \in L^2(\mathbf{R})$ and $u_\epsilon \in \mathfrak{B}^2(\mathbf{R})$. Then the following equation*

$$\mathbf{H}(\epsilon; f\cdot\sigma) + m^2 = \{\gamma^\mu(\mathbf{p}_\mu - \mathbf{A}^\epsilon_\mu) + m\}\{-\gamma^\nu(\mathbf{p}_\nu - \mathbf{A}^\epsilon_\nu\} + m\} \qquad (5.30)$$

holds on $D(\mathbf{N}^2)$. Moreover, $\mathbf{H}(\epsilon; f\cdot\sigma) + m^2$ is a self-adjoint extention of $\{\gamma^\mu(\mathbf{p}_\mu - \mathbf{A}^\epsilon_\mu) + m\}\{-\gamma^\nu(\mathbf{p}_\nu - \mathbf{A}^\epsilon_\nu) + m\}\lceil_{D(\mathbf{N}^2)}$.

We have the following integral-kernel representation of $\mathbf{H}(\epsilon; f\cdot\sigma)$

Theorem 5.6 *Let $W \in C^1_{\mathrm{real}}(\mathbf{R})$ and $a \in \mathcal{N}_d$, $f \in \mathcal{F}_a$. Assume that $f\cdot\sigma$ is an $m\times m$ Hermitian matrix and $\lambda_1, \ldots, \lambda_m$ ($\lambda_1 \le \cdots \le \lambda_m$) are* *eigenvalues of* $f \cdot \sigma$ *and* $V = (v_{ij})$ *is a unitary matrix satisfying*

$$f \cdot \sigma = V \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{pmatrix} V^{-1}.$$

Then, for all $\boldsymbol{\psi} \in \bigoplus^m L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$, *we have*

$$\begin{aligned}(e^{is\mathbf{H}(\epsilon; f\cdot\sigma)}\boldsymbol{\psi})_j(x) \\ = \sum_{k=1}^m \sum_{l=1}^m v_{jk}\overline{v_{lk}} \int_{\mathbf{R}^d} e^{-i\Theta_\epsilon^k(x,y;s)} \Delta_s(x,y)\psi_l(y)dy,\end{aligned} \tag{5.31}$$

$$\begin{aligned}\Theta_\epsilon^k(x,y;s) = &-\frac{\lambda_k}{4} u_\epsilon\left(\frac{ay-ax}{2s}\right)(Y(ax) - Y(ay)) \\ &+ \frac{1}{2} u_\epsilon\left(\frac{ay-ax}{2s}\right)(\Omega(ax) - \Omega(ay)) \\ &- \frac{1}{2s}(y^\mu - q^\mu)\left\{1 - \exp\left[u_\epsilon\left(\frac{ay-ax}{2s}\right)(W(ax) \right.\right. \\ &\left.\left. - W(ay))f\right]\right\}_{\mu\nu}(x^\nu - q^\nu),\end{aligned} \tag{5.32}$$

where $(e^{is\mathbf{H}(\epsilon; f\cdot\sigma)}\boldsymbol{\psi})_j$ *is the* j*-th component of* $e^{is\mathbf{H}(\epsilon; f\cdot\sigma)}\boldsymbol{\psi}$.

Proof. We have only to take

$$u = -\frac{1}{2}\Omega \otimes u_\epsilon, \quad v = -W \otimes u_\epsilon, \quad w = -\frac{1}{4} Y \otimes u_\epsilon$$

as u, v, w in Theorem 4.2. □

From the assumption of u_ϵ,

$$\lim_{\epsilon \to 0} u_\epsilon(t) = u_{-1}(t) = \frac{1}{t} \quad (t \in \mathbf{R} \setminus \{0\}).$$

Let

$$\mathbf{M} = \mathbf{M}\left(\frac{-Y \otimes u_{-1}}{4}, K; \frac{\Omega \otimes u_{-1}}{2}; W \otimes u_{-1}, \mathbf{L}_f\right) \tag{5.33}$$

and

$$U(K) = e^{-i\mathbf{M}}.$$

Lemma 5.7 *Let $W \in \mathfrak{W}_1$. And assume that $(t-aq)W' \in L^2(\mathbf{R})$. Then*

$$\begin{aligned} \text{s-}\lim_{\epsilon\to 0} U(\epsilon;K) &= U(K), \\ \text{s-}\lim_{\epsilon\to 0} U(\epsilon;K)^{-1} &= U(K)^{-1}. \end{aligned}$$

Proof. Using general convergence theorem ([R-S1, Theorems VIII.25 and VIII.21]), we have only to show that $\mathbf{M}_\epsilon$ converges to $\mathbf{M}$ on common core for $\mathbf{M}_\epsilon$ and $\mathbf{M}$.

Let $D = \bigoplus^m [\bigcap_{j,k=0}^\infty D((ap)^{-j}\overline{L_f}^k)]$. Then for all $\psi \in D$,

$$\begin{aligned} \mathbf{M}_\epsilon\psi &= -\frac{1}{4}Y(a\mathbf{x})u_\epsilon(a\mathbf{p})\psi \\ &\quad + \frac{1}{2}\Omega(a\mathbf{x})u_\epsilon(a\mathbf{p})\psi + u_\epsilon(a\mathbf{p})W(a\mathbf{x})\overline{\mathbf{L}_f}\psi, \\ \mathbf{M}\psi &= -\frac{1}{4}Y(a\mathbf{x})(a\mathbf{p})^{-1}\psi \\ &\quad + \frac{1}{2}\Omega(a\mathbf{x})(a\mathbf{p})^{-1}\psi + (a\mathbf{p})^{-1}W(a\mathbf{x})\overline{\mathbf{L}_f}\psi. \end{aligned}$$

By functional calculus, $\lim_{\epsilon\to 0}\mathbf{M}_\epsilon\psi = \mathbf{M}\psi$ for $\psi \in D$. D is a common core for $\mathbf{M}_\epsilon$ and $\mathbf{M}$. Thus by [R-S1, Theorem VIII.25], we can easily show that $\mathbf{M}_\epsilon$ converges to $\mathbf{M}$ in strong resolvent sense. By [R-S1, Theorem VIII.21], this is equivalent to $\lim_{\epsilon\to 0} e^{is\mathbf{M}_\epsilon}\psi = e^{is\mathbf{M}}\psi$ for all $\psi \in L^2(\mathbf{R}^d)$ and $s \in \mathbf{R}\setminus\{0\}$. □

By Lemma 5.7, we can show the following Lemma.

Lemma 5.8 *Let $W \in \mathfrak{W}_1$. Assume that $(t-aq)W' \in L^2(\mathbf{R})$ and $f \in \mathcal{G}_a^0$. Let*

$$\begin{aligned} \mathbf{H}(\epsilon;K) &= U(\epsilon;K)\mathbf{H}_0U(\epsilon;K)^{-1}, \\ \mathbf{H}(K) &= U(K)\mathbf{H}_0U(K)^{-1}. \end{aligned} \tag{5.34}$$

Then $\mathbf{H}(\epsilon;K)$ converges to $\mathbf{H}(K)$ in the strong resolvent sense as $\epsilon \to 0$.

Since $\mathbf{H}(\epsilon;K)$ converges to $\mathbf{H}(K)$ in the strong resolvent sense as $\epsilon \to 0$, $e^{is\mathbf{H}(\epsilon;K)}$ strongly converges to $e^{is\mathbf{H}(K)}$ for $s \in \mathbf{R}\setminus\{0\}$ as $\epsilon \to 0$.

Now we take $f\cdot\sigma$ as an $m\times m$ Hermitian matrix K. By Theorem 5.6, if $W \in C^1_{\mathrm{real}}(\mathbf{R})$, $a \in \mathcal{N}_d$, $f \in \mathcal{F}_a$, then, for all $\psi \in \bigoplus^m L^1(\mathbf{R}^d)\cap L^2(\mathbf{R}^d)$, we have

$$\begin{aligned}&(e^{is\mathbf{H}(\epsilon;f\cdot\sigma)}\boldsymbol{\psi})_j(x)\\&=\sum_{k=1}^{m}\sum_{l=1}^{m}v_{jk}\overline{v_{lk}}\int_{\mathbf{R}^d}e^{-i\Theta_\epsilon^k(x,y;s)}\Delta_s(x,y)\psi_l(y)dy.\end{aligned}\tag{5.35}$$

Since $e^{is\mathbf{H}(\epsilon;f\cdot\sigma)}\boldsymbol{\psi}$ strongly converges to $e^{is\mathbf{H}(f\cdot\sigma)}\boldsymbol{\psi}$ for $s\in\mathbf{R}\setminus\{0\}$, we obtain the following theorem.

Theorem 5.9 *Let $W\in C^1_{\mathrm{real}}(\mathbf{R})$ and $a\in\mathcal{N}_d$, $f\in\mathcal{F}_a$. Assume that $f\cdot\sigma$ is an $m\times m$ Hermitian matrix and $\lambda_1,\dots,\lambda_m$ $(\lambda_1\le\cdots\le\lambda_m)$ are eigenvalues of $f\cdot\sigma$ and $V=(v_{ij})$ is a unitary matrix satisfying*

$$f\cdot\sigma=V\begin{pmatrix}\lambda_1&&0\\&\ddots&\\0&&\lambda_m\end{pmatrix}V^{-1}.$$

Then, for all $\boldsymbol{\psi}\in\bigoplus^m L^1(\mathbf{R}^d)\cap L^2(\mathbf{R}^d)$, we have

$$(e^{is\mathbf{H}(f\cdot\sigma)}\boldsymbol{\psi})_j(x)=\sum_{k=1}^{m}\sum_{l=1}^{m}v_{jk}\overline{v_{lk}}\int_{\mathbf{R}^d}e^{-i\Theta^k(x,y;s)}\Delta_s(x,y)\psi_l(y)dy,\tag{5.36}$$

$$\begin{aligned}&\Theta^k(x,y;s)\\&=\frac{\lambda_k s}{2}\frac{(Y(ay)-Y(ax))}{ay-ax}+s\frac{\Omega(ay)-\Omega(ax)}{ay-ax}\\&\quad-\frac{1}{2s}(y^\mu-q^\mu)\left\{1-\exp\left[-2s\frac{W(ay)-W(ax)}{ay-ax}f\right]\right\}_{\mu\nu}(x^\nu-q^\nu),\end{aligned}\tag{5.37}$$

where $(e^{is\mathbf{H}(f\cdot\sigma)}\boldsymbol{\psi})_j$ is the j-th component of $e^{is\mathbf{H}(f\cdot\sigma)}\boldsymbol{\psi}$.

We can prove the following theorem.

Theorem 5.10 *Let $W\in\mathfrak{W}_3$, $a\in\mathcal{N}_d$ and $f\in\mathcal{G}_a^0$. Assume that $(t-aq)W'\in L^2(\mathbf{R})$. Then $\mathbf{H}(f\cdot\sigma)$ is a self-adjoint extension of $-\gamma^\mu(\mathbf{p}_\mu-\mathbf{A}_\mu)\gamma^\nu(\mathbf{p}_\nu-\mathbf{A}_\nu)\lceil_{D(\mathbf{N}^2)}$. In particular, $\mathbf{H}(f\cdot\sigma)+m^2$ is a self-adjoint extension of $\{\gamma^\mu(\mathbf{p}_\mu-\mathbf{A}_\mu)+m\}\{-\gamma^\nu(\mathbf{p}_\nu-\mathbf{A}_\nu)+m\}\lceil_{D(\mathbf{N}^2)}$.*

Proof. We showed that $\mathbf{H}(\epsilon;f\cdot\sigma)=-\gamma^\mu(\mathbf{p}_\mu-\mathbf{A}^\epsilon_\mu)\gamma^\nu(\mathbf{p}_\nu-\mathbf{A}^\epsilon_\nu)$ on $D(\mathbf{N}^2)$. By Lemma 5.8, $\mathbf{H}(\epsilon;f\cdot\sigma)$ converges to $\mathbf{H}(f\cdot\sigma)$ in the strong resolvent sense as $\epsilon \to 0$. And

$$\lim_{\epsilon\to 0} \mathbf{H}(\epsilon; f\cdot\sigma)\boldsymbol{\psi} = -\gamma^\mu(\mathbf{p}_\mu - \mathbf{A}_\mu)\gamma^\nu(\mathbf{p}_\nu - \mathbf{A}_\nu)\boldsymbol{\psi}$$

on $\boldsymbol{\psi} \in D(\mathbf{N}^2)$.

For $\boldsymbol{\psi} \in D(\mathbf{N}^2)$ and $\boldsymbol{\phi} \in \bigoplus^m L^2(\mathbf{R}^d)$, we have

$$((\mathbf{H}(\epsilon; f\cdot\sigma) - z^*)\boldsymbol{\psi},\ (\mathbf{H}(\epsilon; f\cdot\sigma) - z)^{-1}\boldsymbol{\phi}) = (\boldsymbol{\psi}, \boldsymbol{\phi}) \quad (z \in \mathbf{C}\setminus\mathbf{R}).$$

Since $D(\mathbf{N}^2) \subset D(\mathbf{\Pi}_0(\epsilon)^2) \cap D(\gamma^\mu(\mathbf{p}_\mu - \mathbf{A}_\mu)\gamma^\nu(\mathbf{p}_\nu - \mathbf{A}_\nu))$, the limit $\epsilon \to 0$ gives

$$\begin{aligned}((-\gamma^\mu(\mathbf{p}_\mu - \mathbf{A}_\mu)\gamma^\nu(\mathbf{p}_\nu - \mathbf{A}_\nu) - z^*)\boldsymbol{\psi},\ (\mathbf{H}(f\cdot\sigma) - z)^{-1}\boldsymbol{\phi})\\ = (\boldsymbol{\psi}, \boldsymbol{\phi}).\end{aligned}$$

Thus, for all $\boldsymbol{\eta} \in D(\mathbf{H}(f\cdot\sigma))$,

$$(-\gamma^\mu(\mathbf{p}_\mu - \mathbf{A}_\mu)\gamma^\nu(\mathbf{p}_\nu - \mathbf{A}_\nu)\boldsymbol{\psi}, \boldsymbol{\eta}) = (\boldsymbol{\psi}, \mathbf{H}(f\cdot\sigma)\boldsymbol{\eta}).$$

Hence by self-adjointness of $\mathbf{H}(f\cdot\sigma)$, $\boldsymbol{\psi} \in D(\mathbf{H}(f\cdot\sigma))$ and $-\gamma^\mu(\mathbf{p}_\mu - \mathbf{A}_\mu)\gamma^\nu(\mathbf{p}_\nu - \mathbf{A}_\nu)\boldsymbol{\psi} = \mathbf{H}(f\cdot\sigma)\boldsymbol{\psi}$. So we get $-\gamma^\mu(\mathbf{p}_\mu - \mathbf{A}_\mu)\gamma^\nu(\mathbf{p}_\nu - \mathbf{A}_\nu)\lceil_{D(\mathbf{N}^2)} \subset \mathbf{H}(f\cdot\sigma)$. □

Since $\mathbf{H}(\epsilon; f\cdot\sigma)$ is self-adjoint for $\epsilon > 0$,

$$[\mathbf{H}(\epsilon; f\cdot\sigma) + m^2 \pm i\eta]^{-1}, \quad \eta > 0,$$

are bounded operators. By functional calculus

$$\begin{aligned}&[\mathbf{H}(\epsilon; f\cdot\sigma) + m^2 \pm i\eta]^{-1}\\ &\quad = \lim_{\delta\to 0} \mp i \int_\delta^\infty e^{\pm is(m^2 \pm i\eta)} e^{\pm is\mathbf{H}(\epsilon; f\cdot\sigma)} ds \end{aligned} \tag{5.38}$$

and

$$\begin{aligned}&(\boldsymbol{\phi},\ [\mathbf{H}(\epsilon; f\cdot\sigma) + m^2 \pm i\eta]^{-1}\boldsymbol{\psi})\\ &\quad = \lim_{\delta\to 0} \mp i \int_\delta^\infty e^{\pm is(m^2 \pm i\eta)} (\boldsymbol{\phi}, e^{\pm is\mathbf{H}(\epsilon; f\cdot\sigma)}\boldsymbol{\psi}) ds,\end{aligned} \tag{5.39}$$

where $\boldsymbol{\phi}, \boldsymbol{\psi} \in \bigoplus^m L^2(\mathbf{R}^d)$. We can write

$$\begin{aligned}&(\boldsymbol{\phi}, e^{\pm is\mathbf{H}(\epsilon; f\cdot\sigma)}\boldsymbol{\psi})\\ &= \sum_{k=1}^m \int_{\mathbf{R}^d\times\mathbf{R}^d} e^{-i\Theta_\epsilon^k(x,y;s)} \Delta_s(x,y) \overline{\left(\sum_{j=1}^m \overline{v_{jk}}\phi_j(x)\right)} \left(\sum_{l=1}^m \overline{v_{lk}}\psi_l(y)\right) dy\, dx\end{aligned}$$

$$= \sum_{k=1}^{m} \int_{\mathbf{R}^d \times \mathbf{R}^d} e^{-i\Theta_\epsilon^k(x,y;s)} \Delta_s(x,y) \overline{(V^*\boldsymbol{\phi})_k(x)} (V^*\boldsymbol{\psi})_k(y) dy\, dx,$$

so that

$$\begin{aligned}&(\boldsymbol{\phi},\ [\mathbf{H}(\epsilon; f\cdot\sigma) + m^2 \pm i\eta]^{-1}\boldsymbol{\psi}) \\ &\quad = \lim_{\delta\to 0} \mp i \sum_{k=1}^{m} \int_\delta^\infty \int_{\mathbf{R}^d\times\mathbf{R}^d} e^{\pm is(m^2\pm i\eta)} e^{-i\Theta_\epsilon^k(x,y;s)} \\ &\qquad \times \Delta_s(x,y) \overline{(V^*\boldsymbol{\phi})_k(x)} (V^*\boldsymbol{\psi})_k(y) dy\, dx\, ds.\end{aligned}$$

Theorem 5.11 *Let $W \in C^1_{\mathrm{real}}(\mathbf{R})$ and $a \in \mathcal{N}_d$, $f \in \mathcal{F}_a$. Assume that $f\cdot\sigma$ is an $m\times m$ Hermitian matrix and $\lambda_1,\ldots,\lambda_m$ ($\lambda_1 \le \cdots \le \lambda_m$) are eigenvalues of $f\cdot\sigma$ and $V = (v_{ij})$ is a unitary matrix satisfying*

$$f\cdot\sigma = V \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{pmatrix} V^{-1}.$$

Then, for all $\boldsymbol{\phi},\boldsymbol{\psi} \in \bigoplus^m L^1(\mathbf{R}^d)\cap L^2(\mathbf{R}^d)$, we have

$$\begin{aligned}&(\boldsymbol{\phi},\ [\mathbf{H}(\epsilon; f\cdot\sigma) + m^2 \pm i\eta]^{-1}\boldsymbol{\psi}) \\ &\quad = \lim_{\delta\to 0} \mp i \sum_{k=1}^{m} \int_{\mathbf{R}^d\times\mathbf{R}^d} \left(\int_\delta^\infty e^{\pm is(m^2\pm i\eta)} e^{-i\Theta_\epsilon^k(x,y;s)} \Delta_s(x,y) ds \right) \\ &\qquad \times \overline{(V^*\boldsymbol{\phi})_k(x)} (V^*\boldsymbol{\psi})_k(y) dy\, dx. \end{aligned} \tag{5.40}$$

Corollary 5.12 *Let $W \in C^1_{\mathrm{real}}(\mathbf{R})$ and $a \in \mathcal{N}_d$, $f \in \mathcal{F}_a$. Assume that $f\cdot\sigma$ is an $m\times m$ Hermitian matrix and $\lambda_1,\ldots,\lambda_m$ ($\lambda_1 \le \cdots \le \lambda_m$) are eigenvalues of $f\cdot\sigma$ and $V = (v_{ij})$ is a unitary matrix satisfying*

$$f\cdot\sigma = V \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{pmatrix} V^{-1}.$$

Then, for all $\boldsymbol{\phi},\boldsymbol{\psi} \in \bigoplus^m L^1(\mathbf{R}^d)\cap L^2(\mathbf{R}^d)$, we have

$$\begin{aligned}&(\boldsymbol{\phi},\ [\mathbf{H}(f\cdot\sigma) + m^2 \pm i\eta]^{-1}\boldsymbol{\psi}) \\ &\quad = \lim_{\delta\to 0} \mp i \sum_{k=1}^{m} \int_{\mathbf{R}^d\times\mathbf{R}^d} \left(\int_\delta^\infty e^{\pm is(m^2\pm i\eta)} e^{-i\Theta^k(x,y;s)} \Delta_s(x,y) ds \right)\end{aligned}$$

$$\times \overline{(V^*\boldsymbol{\phi})_k(x)}(V^*\boldsymbol{\psi})_k(y)dy\,dx. \tag{5.41}$$

Let

$$(e^{\pm isH_k^V(f\cdot\sigma)}\psi_k)(x) := \int_{\mathbf{R}^d} e^{-i\Theta^k(x,y;s)}\Delta_s(x,y)\psi_k(y)dy. \tag{5.42}$$

For $\delta > 0$, we can write for $\phi, \psi \in L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$,

$$\begin{aligned}\lim_{\eta\to 0}\int_0^\delta & e^{\pm is(m^2\pm i\eta)}(\phi^*, e^{\pm isH_k^V(f\cdot\sigma)}\psi)_{L^2}ds \\ &= \int_0^\delta e^{\pm ism^2}(\phi^*, e^{\pm isH_k^V(f\cdot\sigma)}\psi)_{L^2}ds,\end{aligned} \tag{5.43}$$

where $(\cdot,\cdot)_{L^2}$ is the inner product of $L^2(\mathbf{R}^d)$.

In particular, if $d \geq 3$, using (4.2),

$$\begin{aligned}\int_\delta^\infty & |e^{\pm is(m^2\pm i\eta)}(\phi^*, e^{\pm isH_k^V(f\cdot\sigma)}\psi)_{L^2}|ds \\ &\leq (\text{Constant independent of } \eta) \times \int_\delta^\infty |s|^{-d/2}ds < \infty.\end{aligned}$$

By the dominated convergence theorem, we have

$$\begin{aligned}\lim_{\eta\to 0}\int_\delta^\infty & e^{\pm is(m^2\pm i\eta)}(\phi^*, e^{\pm isH_k^V(f\cdot\sigma)}\psi)_{L^2}ds \\ &= \int_\delta^\infty e^{\pm ism^2}(\phi^*, e^{\pm isH_k^V(f\cdot\sigma)}\psi)_{L^2}ds.\end{aligned} \tag{5.44}$$

Hence if $d \geq 3$, for all $\phi, \psi \in L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$, we have

$$\begin{aligned}\lim_{\eta\to 0}\int_0^\infty & e^{\pm is(m^2\pm i\eta)}(\phi^*, e^{\pm isH_k^V(f\cdot\sigma)}\psi)_{L^2}ds \\ &= \int_0^\infty e^{\pm ism^2}(\phi^*, e^{\pm isH_k^V(f\cdot\sigma)}\psi)_{L^2}ds.\end{aligned} \tag{5.45}$$

Let

$$\tilde{S}^k_{\pm,1}(\phi,\psi) := \int_0^\infty e^{\pm ism^2}(\phi^*, e^{\pm isH_k^V(f\cdot\sigma)}\psi)_{L^2}ds \tag{5.46}$$

and let

$$\begin{aligned}T^k_{\pm,\delta,1}(\phi,\psi) &:= \int_0^\delta e^{\pm ism^2}(\phi^*, e^{\pm isH_k^V(f\cdot\sigma)}\psi)_{L^2}ds, \\ T^k_{\pm,\delta,2}(\phi,\psi) &:= \int_\delta^\infty e^{\pm ism^2}(\phi^*, e^{\pm isH_k^V(f\cdot\sigma)}\psi)_{L^2}ds.\end{aligned}$$

Then we have

$$\tilde{S}^k_{\pm,1}(\phi,\psi) = T^k_{\pm,\delta,1}(\phi,\psi) + T^k_{\pm,\delta,2}(\phi,\psi). \tag{5.47}$$

It is easy to show that $T^k_{\pm,\delta,1}(\phi,\psi)$ and $T^k_{\pm,\delta,2}(\phi,\psi)$ are jointly continuous bilinear functional on $\mathcal{S}(\mathbf{R}^d)\times\mathcal{S}(\mathbf{R}^d)$, so we can see that $\tilde{S}^k_{\pm,1}(\cdot,\cdot)$ are jointly continuous bilinear functionals on $\mathcal{S}(\mathbf{R}^d)\times\mathcal{S}(\mathbf{R}^d)$. Hence, by the nuclear theorem, there exist unique $S^k_{\pm,1}\in\mathcal{S}'(\mathbf{R}^d\times\mathbf{R}^d)$ respectively, such that, for all $\phi,\psi\in\mathcal{S}(\mathbf{R}^d)$

$$S^k_{\pm,1}(\phi\otimes\psi) = \tilde{S}^k_{\pm,1}(\phi,\psi), \tag{5.48}$$

where $(\phi\otimes\psi)(x,y)=\phi(x)\psi(y)$.

We can see that for $\boldsymbol{\phi},\boldsymbol{\psi}\in\bigoplus^m L^1(\mathbf{R}^d)\cap L^2(\mathbf{R}^d)$

$$\begin{aligned}
&\lim_{\eta\to0}(\boldsymbol{\phi}^*,\ [\mathbf{H}(f\cdot\sigma)+m^2\pm i\eta]^{-1}\boldsymbol{\psi})\\
&\quad=\lim_{\eta\to0}\int_0^\infty e^{\pm is(m^2\pm i\eta)}(\boldsymbol{\phi}^*,e^{\pm is\mathbf{H}(f\cdot\sigma)}\boldsymbol{\psi})ds\\
&\quad=\sum_{k=1}^m((V^*\boldsymbol{\phi}^*)_k,e^{\pm isH_k^V(f\cdot\sigma)}(V^*\boldsymbol{\psi})_k)_{L^2}\\
&\quad=\sum_{k=1}^m S^k_{\pm,1}((V^t\boldsymbol{\phi})_k\otimes(V^*\boldsymbol{\psi})_k),
\end{aligned}\tag{5.49}$$

where $\boldsymbol{\phi}^*(x)=(\overline{\phi_1(x)},\ldots,\overline{\phi_m(x)})$

Let

$$S_{\pm,1}(\boldsymbol{\phi}\otimes\boldsymbol{\psi})=\sum_{k=1}^m S^k_{\pm,1}((V^t\boldsymbol{\phi})_k\otimes(V^*\boldsymbol{\psi})_k). \tag{5.50}$$

$S_{\pm,1}(\boldsymbol{\phi}\otimes\boldsymbol{\psi})$ are tempered distributions on $\mathbf{R}^d\times\mathbf{R}^d$.

Thus we obtain the following theorem.

Theorem 5.13 *Let $d\ge3$. Let $W\in C^1_{\rm real}(\mathbf{R})$ and $a\in\mathcal{N}_d$, $f\in\mathcal{F}_a$. Then there exist unique $S_{\pm,1}\in[\bigoplus^m\mathcal{S}(\mathbf{R}^d\times\mathbf{R}^d)]'$, respectively such that*

$$S_{\pm,1}(\boldsymbol{\phi}\otimes\boldsymbol{\psi})=\lim_{\eta\to0}(\boldsymbol{\phi}^*,\ [\mathbf{H}(f\cdot\sigma)+m^2\pm i\eta]^{-1}\boldsymbol{\psi}).$$

for $\boldsymbol{\phi},\boldsymbol{\psi}\in\bigoplus^m\mathcal{S}(\mathbf{R}^d)$. Moreover

$$S_{\pm,1}(\boldsymbol{\phi}\otimes\boldsymbol{\psi})=\lim_{\delta\to0}\int_\delta^\infty e^{\pm ism^2}(\boldsymbol{\phi}^*,e^{\pm is\mathbf{H}(f\cdot\sigma)}\boldsymbol{\psi})ds.$$

Assume that W is slowly increasing C^∞-function. That is, W satisfies

$$\sup_{x\in\mathbf{R}^d} |(1+|x|^2)^{-l}W(x)| < \infty$$

for some $l \in \mathbf{N}_0$. Then we can see that $S_{\pm,1}((-{}^t\gamma^\mu(-\mathbf{p}_\mu - \mathbf{A}_\mu)+m)\boldsymbol{\phi}\otimes\boldsymbol{\psi})$ (${}^t\gamma^\mu$ is the transposed matrix of γ^μ) are jointly continuous bilinear functionals on $\mathcal{S}(\mathbf{R}^d)\times\mathcal{S}(\mathbf{R}^d)$. By the nuclear theorem, there exist unique $S_\pm \in \bigoplus^m \mathcal{S}'(\mathbf{R}^d\times\mathbf{R}^d)$ respectively, such that, for all $\boldsymbol{\phi},\boldsymbol{\psi}\in\bigoplus^m\mathcal{S}(\mathbf{R}^d)$,

$$S_\pm(\boldsymbol{\phi}\otimes\boldsymbol{\psi}) = S_{\pm,1}((-{}^t\gamma^\mu(-\mathbf{p}_\mu - \mathbf{A}_\mu)+m)\boldsymbol{\phi}\otimes\boldsymbol{\psi})$$

We have

$$S_\pm(({}^t\gamma^\mu(-\mathbf{p}_\mu - \mathbf{A}_\mu)+m)\boldsymbol{\phi}\otimes\boldsymbol{\psi}) = \delta(\boldsymbol{\phi}\otimes\boldsymbol{\psi})$$

for all $\boldsymbol{\phi},\boldsymbol{\psi}\in\bigoplus^m\mathcal{S}(\mathbf{R}^d)$. In this sence, $S_\pm(\boldsymbol{\phi}\otimes\boldsymbol{\psi})$ are Green's functions of a Dirac operator $\gamma^\mu(\mathbf{p}_\mu - \mathbf{A}_\mu)+m$.

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