Hokkaido Mathematical Journal Vol. 26 (1997) p. 493–508

Extremal rings and quasiconformal mappings

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(Received December 12, 1995; Revised August 20, 1996)

Abstract. New functional inequalities are obtained for the capacities of Grötzsch and Teichmüller rings, and for complete elliptic integrals, thus solving two conjectures. These results are applied to refine earlier estimates in quasiconformal Schwarz lemma and Mori's theorem.

Key words: conformal capacity, distortion, modulus, quasiconformal, ring.

1. Introduction

For $n \geq 2$, $s > 1$ and $t > 0$, let $R_{G,n}(s)$ denote the Grötzsch ring in $\mathbf{R}^n$, whose complementary components are the closed unit ball $\overline{B}^n$ and the ray $[s, \infty)$ along the x_1-axis, and let $R_{T,n}(t)$ denote the Teichmüller ring in $\mathbf{R}^n$, whose complementary components are the segment $[-1, 0]$ and the ray $[t, \infty)$ along the x_1-axis. The conformal capacities of $R_{G,n}(s)$ and $R_{T,n}(t)$ are denoted by

$$\gamma_n(s) = \text{cap } R_{G,n}(s), \quad \tau_n(t) = \text{cap } R_{T,n}(t), \tag{1.1}$$

respectively. The modulus $M_n(r)$ of the Grötzsch ring $R_{G,n}(1/r)$, $0 < r < 1$, is defined by

$$M_n(r) = [\omega_{n-1}/\gamma_n(1/r)]^{1/(n-1)}, \tag{1.2}$$

where ω_{n-1} is the $(n-1)$-dimensional measure of the unit sphere S^{n-1} in $\mathbf{R}^n$ [G], [Vä], [Vu1]. The capacities in (1.1) are related [G, §18] by

$$\gamma_n(s) = 2^{n-1}\tau_n(s^2 - 1), \quad s > 1. \tag{1.3}$$

For $K > 0$, define the increasing homeomorphism $\varphi_{K,n}(r)$ from $[0, 1]$ onto $[0, 1]$ by

$$\varphi_{K,n}(r) = 1/\gamma_n^{-1}(K\gamma_n(1/r)) = M_n^{-1}(\alpha M_n(r)) \tag{1.4}$$

for $r \in (0, 1)$, $\varphi_{K,n}(0) = \varphi_{K,n}(1) - 1 = 0$, where $\alpha = K^{1/(1-n)}$. These functions arise in the study of quasiconformal mappings in $\mathbf{R}^n$ [AVV1]–

1991 Mathematics Subject Classification : Primary: 30C62, Secondary: 30C65, 33E05.

[AVV6], [AV1], [G], [LV], [Q1]–[Q4], [Vä], [VV], [Vu1], [Vu2].

As usual, for $n = 2$, we let

$$\mu(r) = M_2(r), \quad \tau(t) = \tau_2(t) \text{ and } \varphi_K(r) = \varphi_{K,2}(r). \tag{1.5}$$

It is well-known that $\mu(r)$ has the explicit expression [LV, p. 60]

$$\mu(r) = \frac{\pi}{2}\frac{\mathcal{K}'(r)}{\mathcal{K}(r)}, \tag{1.6}$$

where

$$\begin{aligned} \mathcal{K} &= \mathcal{K}(r) = \int_0^{\pi/2} (1 - r^2 \sin^2 x)^{-1/2} dx, \\ \mathcal{K}' &= \mathcal{K}'(r) = \mathcal{K}(r'), \end{aligned} \tag{1.7}$$

$r' = \sqrt{1 - r^2}$, are complete elliptic integrals of the first kind [BF], [Bo], [BB].

Let D and D' be domains in $\mathbf{R}^n$, and for $K \geq 1$, $f : D \to D'$ be a K-quasiconformal mapping. The inner dilatation $K_I(f)$ and the outer dilatation $K_O(f)$ of f are defined by [Vä], [Vu1]

$$K_I(f) = \sup \frac{M(f(\Gamma))}{M(\Gamma)}, \quad K_O(f) = \sup \frac{M(\Gamma)}{M(f(\Gamma))}, \tag{1.8}$$

respectively, where $M(\Gamma)$ is the modulus of the curve family Γ. The suprema are taken over all curve families Γ in D such that $M(\Gamma)$ and $M(f(\Gamma))$ are not simultaneously 0 or ∞. Then $1 \leq K_I(f), K_O(f) \leq K$.

Recently, many inequalities were obtained for the above–mentioned special functions and several conjectures were raised [AVV3]–[AVV11], [QVV], [VV]. Among these conjectures, the following appear in [AVV7] and [AVV9]:

(C_1) For each $K \in (1, \infty)$, the function $f(r) = r^{-1/K}\varphi_K(r)$ is convex on $(0, 1)$.

(C_2) $\frac{3}{2}\tau(t) < \tau(2t) + \tau(\sqrt{2t}) < 3\tau(t)$, for all $t \in (0, \infty)$.

On the other hand, some of the function-theoretic applications of the above-mentioned functions rely on the monotoneity and explicit estimates of these functions. An example of such an explicit estimate is [W], [LV], [He]

$$r^{1/K} < \varphi_K(r) < 4^{1-1/K} r^{1/K}, \tag{1.9}$$

for all $K > 1$ and $r \in (0, 1)$, from which the well-known explicit quasicon-

formal Schwarz lemma follows, that is,

$$|f(z)| \le 4^{1-1/K}|z|^{1/K}, \tag{1.10}$$

for each K-quasiconformal mapping f of the unit disk B into itself with $f(0) = 0$ and for $z \in B$. In [QVV, Theorem 1.5], the upper bound in (1.9) was improved to

$$\varphi_K(r) < (1 + r')^{2(1-1/K)} r^{1/K}, \tag{1.11}$$

for $r \in (0,1)$ and $K > 1$, where $r' = \sqrt{1-r^2}$, and hence, the inequality (1.10) was sharpened to

$$|f(z)| \le (1 + \sqrt{1-|z|^2})^{2(1-1/K)}|z|^{1/K}, \tag{1.12}$$

for all $z \in B$.

In this paper we obtain some new functional inequalities for $\mu(r)$ and $\tau_n(t)$, prove conjecture (C_2), disprove conjecture (C_1), and sharpen the upper bound for $\varphi_K(r)$, by studying some properties of complete elliptic integrals $\mathcal{K}(r)$, $\mathcal{K}'(r)$ in (1.7) and those of the second kind

$$\mathcal{E} = \mathcal{E}(r) = \int_0^{\pi/2} (1 - r^2 \sin^2 x)^{1/2} dx, \quad \mathcal{E}' = \mathcal{E}'(r) = \mathcal{E}(r'), \tag{1.13}$$

for $r \in (0,1)$, where $r' = \sqrt{1-r^2}$. These inequalities lead to sharper estimates in the quasiconformal Schwarz lemma and Mori's theorem.

Throughout this paper, r' denotes $\sqrt{1-r^2}$, for $r \in [0,1]$.

We now state some of the main results of this paper.

Theorem 1.14 *Conjecture* (C_1) *is false.*

Theorem 1.15 *The function* $F(t) = [\tau(2t) + \tau(\sqrt{2t})]/\tau(t)$ *is strictly increasing from* $(0,\infty)$ *onto* $(\frac{3}{2}, 3)$. *In particular, for all* $t \in (0,\infty)$,

$$\frac{3}{2}\tau(t) < \tau(2t) + \tau(\sqrt{2t}) < 3\tau(t), \tag{1.16}$$

so Conjecture (C_2) *is true.*

Theorem 1.17 *For* $n \ge 2$, $p \in (0,\infty)$, *let* $C_n(p) = p^{-1/4}$, *if* $n = 2$, *and* $= p^{-1/2}$, *if* $n \ge 3$. *Then*

(1) *For each* $t \in (0,\infty)$,

$$C_n(p) < \frac{\tau_n(pt)}{\tau_n(t)} < 1, \quad \text{if} \quad 1 < p < \infty, \tag{1.18}$$

$$1 < \frac{\tau_n(pt)}{\tau_n(t)} < C_n(p), \quad \text{if} \quad 0 < p < 1. \tag{1.19}$$

(2) *For* $t \in (0,\infty)$ *and* $p \geq 1$,

$$\begin{aligned}\left(1+\frac{1}{p}\right) C_n(p)\tau_n(t) &\leq \tau_n(pt) + \tau_n((pt)^{1/p}) \\ &\leq (1+p^{n-1})\tau_n(t).\end{aligned} \tag{1.20}$$

(3) *For* $p > 1$ *and* $t \geq p^{1/(p-1)}$,

$$\tau_n(pt) + \tau_n((pt)^{1/p}) \geq (1 + C_n(p))\tau_n(t). \tag{1.21}$$

Remark 1.22. For $n = 2$, Theorem 1.17 (1) improves [AVV8, Theorem 1.8 (3) and (4)].

Theorem 1.23 *For* $r \in (0,1)$, *define the function* g *on* $[1,\infty)$ *by*

$$g(K) = \varphi_K(r)[(1+r')^{1+r'}/r]^{1/K}.$$

Then g *is strictly decreasing from* $[1,\infty)$ *onto* $(1,(1+r')^{1+r'}]$. *In particular, for* $r \in (0,1)$ *and* $K > 1$,

$$\varphi_K(r) < (1+r')^{(1+r')(1-1/K)} r^{1/K}. \tag{1.24}$$

Theorem 1.25 *For* $K \geq 1$, *let* f *be a* K*-quasiconformal automorphism of the closed unit disk* $\overline{B}$ *with* $f(0) = 0$.

(a) *For each* $z \in \overline{B}$,

$$|f(z)| \leq (1+u)^{(1+u)(1-1/K)} |z|^{1/K}, \tag{1.26}$$

where $u = \sqrt{1-|z|^2}$.

(b) *For* $z_1, z_2 \in \overline{B}$, *let* $v = (|z_1| + |z_2|)/2$, $w = |z_1 - z_2|/(4v)$ *and* $t = 2w$. *Then*

$$\begin{aligned}|f(z_1) - f(z_2)| &\leq C(z_1,z_2)^{1-1/K} |z_1 - z_2|^{1/K} \\ &\leq 64^{1-1/K} |z_1 - z_2|^{1/K},\end{aligned} \tag{1.27}$$

where $C(z_1, z_2) = 4(1+v')^{1+v'}(1+w')^{1+w'}\sqrt{(1+t')/2}$.

(c) *For* $|z_1| = |z_2| = 1$,

$$\begin{aligned} |f(z_1) - f(z_2)| &\le [8(1+r')^{2(1+r')}(1+s')]^{(1-1/K)/2}|z_1 - z_2|^{1/K} \\ &\le 16^{1-1/K}|z_1 - z_2|^{1/K}, \end{aligned}$$

where $r = |z_1 - z_2|/4$ *and* $s = 2r$.

2. Preliminary Results

To prove the main theorems stated in Section 1, we need some functional inequalities for complete elliptic integrals. In this section, we study some monotone properties of some functions defined in terms of complete elliptic integrals, from which several functional inequalities follow. We apply these results to derive sharp inequalities for the special functions $\mu(r)$, $\tau_n(t)$ and $\varphi_K(r)$.

The following well known relations will be useful in our proofs [BF], [He], [AVV10, Lemma 2.1].

$$\frac{d\mathcal{K}}{dr} = \frac{\mathcal{E} - r'^2\mathcal{K}}{rr'^2}, \quad \frac{d\mathcal{E}}{dr} = \frac{\mathcal{E} - \mathcal{K}}{r}, \tag{2.1}$$

$$\mathcal{K}\mathcal{E}' + \mathcal{K}'\mathcal{E} - \mathcal{K}\mathcal{K}' = \frac{\pi}{2}, \tag{2.2}$$

$$\frac{d\mu}{dr} = -\frac{\pi^2}{4}\frac{1}{rr'^2\mathcal{K}^2(r)}, \tag{2.3}$$

$$\frac{\partial s}{\partial r} = \frac{1}{K}\frac{ss'^2\mathcal{K}^2(s)}{rr'^2\mathcal{K}^2(r)}, \tag{2.4}$$

$$\frac{\partial s}{\partial K} = \frac{2}{\pi K}ss'^2\mathcal{K}(s)\mathcal{K}'(s), \tag{2.5}$$

where $s = \varphi_K(r)$, $0 < r < 1$, $0 < K < \infty$.

Lemma 2.6 *For* $p > 1$, $t \in (0, \infty)$, *let* $x = \sqrt{t^p/(t^p+1)}$ *and* $y = \sqrt{t/(t+1)}$. *Then, as a function of* t, $f(t) = p\mathcal{K}(y)\mathcal{K}'(y) - \mathcal{K}(x)\mathcal{K}'(x)$ *is strictly increasing on* $(0, 1]$ *and decreasing on* $[1, \infty)$, *with*

$$f((0,1]) = f([1,\infty)) = ((p-1)\pi\log 2,\ (p-1)\mathcal{K}^2(1/\sqrt{2})].$$

Proof. Clearly, $f(1/t) = f(t)$. Hence, it is enough to prove the result only for $t \in (0, 1]$, so that $0 < x \le y \le 1/\sqrt{2}$, with equalities if and only if $t = 1$.

Next, since [Bo]

$$\mathcal{K}(r) = \log(4/r') + O(r'^2 \log r') \tag{2.7}$$

as r tends to 1, it follows that $f(0^+) = (p-1)\pi \log 2$. Clearly, $f(1) = (p-1)\mathcal{K}^2(1/\sqrt{2})$.

Now, differentiation gives

$$\frac{dx}{dt} = \frac{pxx'^2}{2t}, \quad \frac{dy}{dt} = \frac{yy'^2}{2t}$$

so that

$$tf'(t) = p[h(y) - h(x)]$$

by (2.2), where $h(r) = \mathcal{K}'(r)[\mathcal{E}(r) - r'^2\mathcal{K}(r)]$. Hence, $f'(t) > 0$ for $t \in (0, 1)$ by [AVV10, p. 539], yielding the result. □

Corollary 2.8 (a) *For $p > 1$, $t \in (0, \infty)$, let $x = \sqrt{t^p/(t^p+1)}$ and $y = \sqrt{t/(t+1)}$. Then $f_1(t) = \mu(y)/\mu(x)$ is strictly increasing from $(0, \infty)$ onto $(1/p, p)$. In particular,*

$$\frac{1}{p}\mu(x) < \mu(y) < p\mu(x). \tag{2.9}$$

(b) *For each $p > 1$, the function $f_2(t) = \tau(t)/\tau(t^p)$ is strictly increasing from $(0, \infty)$ onto $(1/p, p)$. In particular, for all $t \in (0, \infty)$,*

$$\frac{1}{p}\tau(t) < \tau(t^p) < p\tau(t). \tag{2.10}$$

Proof. (a) By l'Hospital's Rule, $f_1(0^+) = 1/p$ and $f_1(\infty) = p$.

Next, by differentiation,

$$\frac{4}{\pi} t\mathcal{K}'^2(x)\mathcal{K}^2(y) f_1'(t) = f(t),$$

where f is as in Lemma 2.6, and the result follows.

(b) Put $x = \sqrt{t^p/(t^p+1)}$ and $y = \sqrt{t/(t+1)}$. Then $f_2(t) = f_1(t)$ by (1.2) and (1.3). Now the result follows from (a). □

Corollary 2.11 *For $r \in (0,1)$, $K \in [1,\infty)$, $p \in [1,\infty)$ and $c = 2/p$,*

$$\frac{\varphi_K^c(r)}{\varphi_K^c(r) + \varphi_{1/K}^c(r')} \le \varphi_K^2\left(\sqrt{\frac{r^c}{r^c + r'^c}}\right). \tag{2.12}$$

The inequality is reversed if $0 < K \le 1$, with equality iff $p = 1$ or $K = 1$.

Proof. By Corollary 2.8 (a), as a function of K, $\mu(\sqrt{s^c/(s^c + s'^c)})/\mu(s)$, where $s = \varphi_K(r)$, is strictly increasing on $[1,\infty)$, and hence,

$$\mu\left(\sqrt{\frac{s^c}{s^c + s'^c}}\right)\Big/\mu(s) \ge \mu\left(\sqrt{\frac{r^c}{r^c + r'^c}}\right)\Big/\mu(r),$$

from which (2.12) follows.

The remaining conclusions are clear. □

Lemma 2.13 *For $p > 1$ and $t \in (0,\infty)$, let $x = \sqrt{t/(t+1)}$ and $y = \sqrt{pt/(pt+1)}$. Then, as a function of t, $f(t) \equiv \mathcal{K}(y)\mathcal{K}'(y) - \mathcal{K}(x)\mathcal{K}'(x)$ is strictly increasing from $(0,\infty)$ onto $(-\frac{\pi}{4}\log p, \frac{\pi}{4}\log p)$.*

Proof. Clearly, $p(x/x')^2 = (y/y')^2$, $\frac{dx}{dt} = \frac{xx'^2}{2t}$ and $\frac{dy}{dt} = \frac{yy'^2}{2t}$. Hence, differentiation and (2.2) give

$$tf'(t) = h(y) - h(x) > 0 \tag{2.14}$$

by [AVV10, p. 539], where $h(r) = \mathcal{K}'(r)[\mathcal{E}(r) - r'^2\mathcal{K}(r)]$. This yields the monotoneity of f.

By (2.7), $f(0^+) = -f(\infty) = -\frac{\pi}{4}\log p$. □

Corollary 2.15 (a) *For $p > 1$, $t \in (0,\infty)$, let $x = \sqrt{t/(t+1)}$ and $y = \sqrt{pt/(pt+1)}$. Then $g_1(t) = \mu(y)/\mu(x)$ is strictly decreasing on $(0, 1/\sqrt{p}]$ and increasing on $[1/\sqrt{p}, \infty)$ with $g_1(0^+) = g_1(\infty) = 1$. In particular, for $p > 1$ and $t \in (0,\infty)$, $u^2 = 1/(1+\sqrt{p})$,*

$$p^{-1/4} < \left(\frac{\mathcal{K}(u)}{\mathcal{K}'(u)}\right)^2 \le \frac{\mu(y)}{\mu(x)} < 1, \tag{2.16}$$

with equality iff $t = 1/\sqrt{p}$.

(b) *For each $p > 1$, the function $g_2(t) = \tau(pt)/\tau(t)$ is strictly decreasing on $(0, 1/\sqrt{p}]$ and increasing on $[1/\sqrt{p}, \infty)$ with $g_2(0^+) = g_2(\infty) = 1$.*

In particular, for $p > 1$ *and* $t \in (0, \infty)$, $u^2 = 1/(1+\sqrt{p})$,

$$p^{-1/4} < \left(\frac{\mathcal{K}(u)}{\mathcal{K}'(u)}\right)^2 \le \frac{\tau(pt)}{\tau(t)} < 1, \tag{2.17}$$

with equality iff $t = 1/\sqrt{p}$.

Proof. (a) The first inequality in (2.16) follows from [AVV10, Theorem 2.2 (3)]. Next, since $g_1(t) = g_1(1/(pt))$ by the relation [LV]

$$\mu(r)\mu(r') = \pi^2/4, \tag{2.18}$$

we need only prove the result (a) for $t \in (0, 1/\sqrt{p})$.

Differentiation and (2.2) give

$$\frac{4}{\pi} t(\mathcal{K}'(x)\mathcal{K}(y))^2 g_1'(t) = f(t) < f(1/\sqrt{p}) = 0,$$

where f is as in Lemma 2.13, yielding the monotoneity of g_1.

By l'Hospital's Rule, $g_1(0^+) = g_1(\infty) = 1$. The second and third inequalities in (2.16) follow.

The result (b) follows from (a) and the relations (1.2) and (1.3). □

In [QVV, Lemma 2.1], it was shown that

$$\frac{2}{\pi} r'^2 \mathcal{K}(r)\mathcal{K}'(r) + \log r < 2\log(1+r'), \tag{2.19}$$

for $r \in (0,1)$, from which the inequality (1.12) was established. For further improvement of the inequality (1.12), we need to improve the estimate (2.19).

Lemma 2.20 *Let* $m(r) = \frac{2}{\pi} r'^2 \mathcal{K}(r)\mathcal{K}'(r)$. *Then there exists a unique* $r_0 \in (\sin 55^\circ, \sin 56^\circ)$ *such that the function* $f(r) = m(r) + \log r - (1+r')\log(1+r')$ *is strictly decreasing on* $(0, r_0]$ *and increasing on* $[r_0, 1)$. *In particular, for* $r \in (0,1)$,

$$-0.196 < f(r_0) \le f(r) < 0, \tag{2.21}$$

with equality iff $r = r_0$.

Proof. By differentiation,

$$\frac{r'}{r} f'(r) = f_1(r) = 1 + \log(1+r') - \frac{4}{\pi}\mathcal{K}'(r)\cdot r' \frac{\mathcal{K}(r) - \mathcal{E}(r)}{r^2}. \tag{2.22}$$

Let $g_1(r) = \mathcal{K}(r) - \mathcal{E}(r)$ and $g_2(r) = r^2/r'$. Then

$$\frac{g_1'(r)}{g_2'(r)} = \frac{r'}{1+r'^2}\mathcal{E}(r),$$

which is strictly decreasing on $(0,1)$. Hence, the function $f_2(r) = r'[\mathcal{K}(r) - \mathcal{E}(r)]/r^2$ is strictly decreasing from $(0,1)$ onto $(0,\pi/4)$ by Monotone l'Hospital's Rule [AVV5, Lemma 2.2]. Hence, for $r \in [a,b] \subset (0,1)$,

$$f_1(r) \le f_3(a,b) = 1 + \log(1+a') - \frac{4}{\pi}b'\mathcal{K}'(b)\frac{\mathcal{K}(b) - \mathcal{E}(b)}{b^2}.$$

Since $f_3(0, \sin 42^\circ) = -0.057127\ldots$,

$$f_1(r) < 0 \quad \text{for} \quad r \in (0, \sin 42^\circ]. \tag{2.23}$$

Next, let $f_4(r) = r^2 f_1(r)$. Then, $f_4(r) = r^2 + 2r' + r^2\log(1+r') - \frac{4}{\pi}r'\mathcal{K}(r)\mathcal{E}'(r)$, by (2.2). By differentiation,

$$\begin{aligned}\frac{r'}{r}f_4'(r) &= 3r' - 3 + 2r'\log(1+r') + \frac{4}{\pi}\mathcal{K}(r)\mathcal{E}'(r)\\ &\quad - \frac{4}{\pi}\frac{1}{r^2}[\mathcal{E}(r)\mathcal{E}'(r) - \mathcal{K}(r)\mathcal{E}'(r) + r^2\mathcal{K}(r)\mathcal{K}'(r)].\end{aligned}$$

Clearly, $3r' - 3 + 2r'\log(1+r')$ is decreasing on $(0,1)$. Hence, by [QV, Theorem 2.1 (7)], it follows that

$$\begin{aligned}\frac{r'}{r}f_4'(r) &\ge f_5(a,b) = 3(b'-1) + 2b'\log(1+b')\\ &\quad + \frac{4}{\pi a^2}\{(1+a^2)\mathcal{K}(a)\mathcal{E}'(a) - \mathcal{E}(a)\mathcal{E}'(a) - a^2\mathcal{K}(a)\mathcal{K}'(a)\},\end{aligned}$$

for $r \in [a,b] \subset (0,1)$. By computation, we have

$$\begin{aligned} f_5(\sin 73^\circ, 1) &= 0.102028\ldots, \quad & f_5(\sin 63^\circ, \sin 73^\circ) &= 0.052752,\\ f_5(\sin 55^\circ, \sin 63^\circ) &= 0.0364292, & f_5(\sin 48^\circ, \sin 55^\circ) &= 0.018148,\end{aligned}$$

and

$$f_5(\sin 42^\circ, \sin 48^\circ) = 0.001202.$$

Hence, $f_4'(r) > 0$ for $r \in [\sin 42^\circ, 1)$, and f_4 is strictly increasing on $[\sin 42^\circ, 1)$. Since $f_4(\sin 55^\circ) = -0.005950 < 0$ and $f_4(\sin 56^\circ) = 0.002438 >$

0, f_4 has a zero $r_0 \in (\sin 55^\circ, \sin 56^\circ)$ such that

$$f_4(r) \begin{cases} < 0, & \text{if } 0 < r < r_0, \\ > 0, & \text{if } r_0 < r < 1. \end{cases} \tag{2.24}$$

It follows from (2.23) and (2.24) that, on $(0,1)$, f_1 has a unique zero r_0 such that $f_1(r) < 0$ for $r \in (0, r_0)$ and $f_1(r) > 0$ for $r \in (r_0, 1)$. This yields the piecewise monotoneity of f, so that the second and third inequalities in (2.21) follow.

Finally, since $m(r) + \log r$ is decreasing on $(0,1)$ [W], [AVV10, Lemma 4.2 (1)],

$$\begin{aligned} f(r_0) &> m(\sin 56^\circ) + \log \sin 56^\circ - (1 + \cos 55^\circ) \log(1 + \cos 55^\circ) \\ &= -0.195968 > -0.196. \end{aligned}$$

This yields the first inequality in (2.21). □

Remark 2.25. By Lemma 2.20, is is natural to ask whether the function $f(r) = m(r) + \log r - (1 + r') \log(1 + r')$ is convex on $(0,1)$. The answer to this question, however, is negative. As a matter of fact, differentiation gives

$$\begin{aligned} r'^3 f''(r) = F(r) &= 1 + \log(1 + r') - \frac{r^2 r'}{1 + r'} \\ &\quad + \frac{4}{\pi} r' \mathcal{K}' \left\{ \left(\frac{\mathcal{E}'}{\mathcal{K}'} + 1 - 2r^2 \right) \frac{\mathcal{K} - \mathcal{E}}{r^2} - \mathcal{E} \right\}. \end{aligned}$$

Clearly, $F(0) = -\infty$ and $F(1) = 1$. Hence, there exist r_1 and r_2 with $0 < r_1 < r_2 < 1$ such that f is concave on $(0, r_1)$ and convex on $(r_2, 1)$.

3. Proofs of Main Results

In this section, making use of the lemmas established in Section 2, we prove the theorems stated in Section 1.

3.1. *Proof of Theorem* 1.14. Let $s = \varphi_K(r)$. Then, by (2.4),

$$K f'(r) = \frac{1}{r} [g(r, K) - 1] f(r), \tag{3.2}$$

$$\begin{aligned} \frac{K r^2 f''(r)}{f(r)} &= \frac{[g(r,K) - 1]^2}{K} + r \frac{\partial g}{\partial r} - g(r,K) + 1 \\ &\equiv f_1(r, K), \end{aligned} \tag{3.3}$$

where

$$g(r,K) = \{s'\mathcal{K}(s)/(r'\mathcal{K}(r))\}^2,$$
$$r\frac{\partial g}{\partial r} = \frac{2g(r,K)}{r'^2\mathcal{K}(r)\mathcal{K}'(r)}[\mathcal{K}(r)\mathcal{E}'(r) - \mathcal{K}(s)\mathcal{E}'(s)].$$

By l'Hospital's Rule, we have

$$\lim_{K\to 1}\frac{\mathcal{K}(r)\mathcal{E}'(r) - \mathcal{K}(s)\mathcal{E}'(s)}{K-1}$$
$$= -\frac{2}{\pi}\mathcal{K}(r)\mathcal{K}'(r)\{\mathcal{E}(r)\mathcal{E}'(r) - \mathcal{K}(r)\mathcal{E}'(r) + r^2\mathcal{K}(r)\mathcal{K}'(r)\}, \tag{3.4}$$

$$\lim_{K\to 1}\frac{1-g(r,K)}{K-1} = \frac{1}{r'^2\mathcal{K}^2(r)}\lim_{K\to 1}\frac{r'^2\mathcal{K}^2(r) - s'^2\mathcal{K}^2(s)}{K-1}$$
$$= \frac{4}{\pi}\mathcal{K}'(r)[\mathcal{K}(r) - \mathcal{E}(r)], \tag{3.5}$$

and

$$\lim_{K\to 1}\frac{[1-g(r,K)]^2}{(K-1)} = 0. \tag{3.6}$$

Let $f_2(r,K) = \frac{\pi}{4}r'^2 f_1(r,K)/(K-1)$ for $K \in (1,\infty)$ and $f_2(r,1) = \lim_{K\to 1} f_2(r,K)$. then, by (3.4)–(3.6), it follows that

$$f_2(r,1) = r'^2\mathcal{K}'(r)[\mathcal{K}(r) - \mathcal{E}(r)] - \mathcal{E}'(r)[\mathcal{E}(r) - r'^2\mathcal{K}(r)]$$
$$- r^2\mathcal{K}(r)[\mathcal{K}'(r) - \mathcal{E}'(r)],$$

and hence, $f_2(1,1) = -\pi/2$. Hence f cannot be convex on $(0,1)$, when K is close to 1.

On the other hand, it is clear that, when $K = 2$, $f(r) = 2/(1+r)$ is convex on $(0,1)$. □

3.7. *Proof of Theorem* 1.15. Set $x^2 = 2t/(1+2t)$, $r^2 = t/(1+t)$ and $y^2 = \sqrt{2t}/(1+\sqrt{2t})$, with $x, r, y > 0$. Then $r = x/\sqrt{2-x^2}$, $y^2 = x/(x+x')$,

$$\frac{dr}{dx} = \frac{rr'^2}{xx'^2}, \quad \frac{dy}{dx} = \frac{yy'^2}{2xx'^2} \tag{3.8}$$

and

$$F(t) = F_1(x) = \frac{\mu(x)+\mu(y)}{\mu(r)}$$

by (1, 2), (1.3). It is sufficient to prove that F_1 is strictly increasing from $(0, 1)$ onto $(3/2, 3)$.

By differentiation,

$$\frac{4}{\pi}\frac{x(x'\mathcal{K}'(r)\mathcal{K}(x)\mathcal{K}(y))^2}{\mathcal{K}^2(x)+2\mathcal{K}^2(y)}F_1'(x) = F_2(x) \tag{3.9}$$

by virtue of (2.3) and (3.8), where

$$F_2(x) = 2\mathcal{K}(x)\mathcal{K}(y)\frac{\mathcal{K}'(x)\mathcal{K}(y)+\mathcal{K}(x)\mathcal{K}'(y)}{\mathcal{K}^2(x)+2\mathcal{K}^2(y)} - \mathcal{K}(r)\mathcal{K}'(r).$$

Next, by Lemma 2.6,

$$2\mathcal{K}(y)\mathcal{K}'(y) > \pi\log 2 + \mathcal{K}(x)\mathcal{K}'(x),$$

so that

$$F_2(x) > F_3(x) + \frac{\mathcal{K}^2(x)}{\mathcal{K}^2(x)+2\mathcal{K}^2(y)}\pi\log 2, \tag{3.10}$$

where $F_3(x) = \mathcal{K}(x)\mathcal{K}'(x) - \mathcal{K}(r)\mathcal{K}'(r)$.

It is easy to verify that $y < \sqrt{x}$. Hence

$$\frac{\mathcal{K}(y)}{\mathcal{K}(x)} < \frac{\mathcal{K}(\sqrt{x}\,)}{\mathcal{K}(x)} < \sqrt[4]{1+x} < \sqrt[4]{2}, \tag{3.11}$$

by [QV, Theorem 1.5] or [AVV11, Theorem 3.11]. On the other hand,

$$F_3(x) > -\frac{\pi}{4}\log 2, \quad \text{for} \quad 0 < x < 1, \tag{3.12}$$

by Lemma 2.13.

From (3.11) and (3.12), it follows from (3.10) that

$$\begin{aligned} F_2(x) &> -\frac{\pi}{4}\log 2 + \frac{\pi\log 2}{1+2[\mathcal{K}(y)/\mathcal{K}(x)]^2} \\ &> -\frac{\pi}{4}\log 2 + \frac{\pi\log 2}{1+2\sqrt{2}} = \frac{(3-2\sqrt{2})\pi}{4(1+2\sqrt{2})}\log 2 > 0, \end{aligned}$$

for $x \in (0, 1)$. Hence, $F_1'(x) > 0$ for $x \in (0, 1)$ by (3.9), yielding the montoneity of F_1.

Finally, we have the limiting values:

$$F(0^+) = F_1(0^+) = \lim_{x\to 0}\frac{\log(4/x)+\log(4/y)}{\log(4/r)} = \frac{3}{2},$$

$$F(\infty) = F_1(1^-) = \lim_{x\to 1^-} \frac{\mathcal{K}(r)}{\mathcal{K}(y)}\left[1+\frac{\mathcal{K}(y)}{\mathcal{K}(x)}\right]$$
$$= \lim_{x\to 1^-} \frac{\log(4/r')}{\log(4/y')}\left[1+\frac{\log(4/y')}{\log(4/x')}\right] = 3.$$

This completes the proof. □

3.13. *Proof of Theorem* 1.17. (1) We need only prove the lower bound in (1.18). For $n \geq 3$, this was proved in [AVV8, Theorem 1.8 (4)], while for $n = 2$, it follows from (2.17).

(2) From the second inequality in [AVV8, Theorem 1.8 (1)] and the monotoneity of $\tau_n(t)$, it follows that

$$\tau_n(pt) + \tau_n((pt)^{1/p}) \leq (1+p^{n-1})\tau_n(pt) \leq (1+p^{n-1})\tau_n(t).$$

Next, it follows from the first inequality in [AVV8, Theorem 1.8 (1)] and (1.18) that

$$\tau_n(pt) + \tau_n((pt)^{1/p}) \geq \left(1+\frac{1}{p}\right)\tau_n(pt) \geq \left(1+\frac{1}{p}\right)C_n(p)\tau_n(t).$$

For (3), we note that $t \geq (pt)^{1/p}$ for $t \geq p^{1/(p-1)}$. Hence, by (1.18),

$$\tau_n(pt) + \tau_n((pt)^{1/p}) \geq \tau_n(pt) + \tau_n(t) \geq (1+C_n(p))\tau_n(t).$$

□

3.14. *Proof of Theorem* 1.23. Let $s = \varphi_K(r)$. Making use of (2.5), we get

$$\frac{K^2 g'(K)}{g(K)} = \frac{2}{\pi} s'^2 \mathcal{K}^2(s)\frac{\mathcal{K}'(r)}{\mathcal{K}(r)} + \log r - (1+r')\log(1+r'), \quad (3.15)$$

by logarithmic differentiation. Since $s'^2\mathcal{K}^2(s)$ is strictly decreasing in K on $[1,\infty)$ [AVV10, Theorem 2.2 (3)], it follows from (3.15) and Lemma 2.20 that

$$K^2 g'(K)/g(K) < m(r) + \log r - (1+r')\log(1+r') < 0.$$

□

3.16. *Proof of Theorem* 1.25. The result (a) follows from the inequality (1.24) and the quasiconformal Schwarz lemma, namely [LV],

$$|f(z)| \leq \varphi_K(|z|). \quad (3.17)$$

The result (b) follows from [AVV6, Lemma 4.6] and (1.24), while part (c) follows from part (b). □

Remark 3.18. The inequality (1.24) enables one to improve some known distortion results for quasiconformal mappings. Theorem 1.25 is only one such example.

Conjecture 3.19 Our computational work supports the validity of the following conjecture:

For each $p > 1$, the function $f(t) = [\tau(pt) + \tau((pt)^{1/p})]/\tau(t)$ is strictly increasing from $(0, \infty)$ onto $(1 + 1/p, p + 1)$.

Acknowledgments This work began during the first author's visit to the Mathematics Department, University of Helsinki with support from CIMO and the Finnish Ministry of Education and completed at the University of Auckland with support from its Department of Mathematics. The authors thank Professor Matti Vuorinen at Helsinki and Professor David Gauld at Auckland for their hospitality and helpful discussions. The authors are grateful to the referee for many helpful suggestions.

References

[AV1] Anderson G.D. and Vamanamurthy M.K., *Hölder continuity of quasiconformal mappings of the unit ball.* Proc. Amer. Math. Soc. **104**(1) (1988), 227–230.

[AV2] Anderson G.D. and Vamanamurthy M.K., *Rotation of plane quasiconformal mappings.* Tôhoku Math. J. **23** (1971), 606–620.

[AVV1] Anderson G.D., Vamanamurthy M.K. and Vuorinen M., *Dimension-free quasiconformal distortion in n-space.* Trans. Amer. Math. Soc. **297** (1986), 687–706.

[AVV2] Anderson G.D., Vamanamurthy M.K. and Vuorinen M., *Sharp distortion theorems for quasiconformal mappings.* Trans. Amer. Math. **305** (1988), 95–111.

[AVV3] Anderson G.D., Vamanamurthy M.K. and Vuorinen M., *Distortion functions for plane quasiconformal mappings.* Israel J. Math. **62** (1988), 1–16.

[AVV4] Anderson G.D., Vamanamurthy M.K. and Vuorinen M., *Special functions of quasiconformal theory.* Exposition. Math. **7** (1989), 97–138.

[AVV5] Anderson G.D., Vamanamurthy M.K. and Vuorinen M., *Inequalities for quasiconformal mappings in space.* Pacific J. Math. **1** (1993), 1–18.

[AVV6] Anderson G.D., Vamanamurthy M.K. and Vuorinen M., *Inequalities for plane quasiconformal mappings.* in The Mathematical Heritage of Wilhelm Magnus-Groups, Geometry and Special Functions, ed. by W. Abikoff, J.S. Birman, and K. Kuiken, Contempary Math., Vol. 169, Amer. Math. Soc., Providence, RI, 1994, pp. 1–27.

[AVV7] Anderson G.D., Vamanamurthy M.K. and Vuorinen M., *Hypergeometric functions and elliptic integrals.* in Current Topics in Analytic Function Theory, World Scientific Publ. Co. Singapore–London, 1992, pp. 48–85.

[AVV8] Anderson G.D., Vamanamurthy M.K. and Vuorinen M., *Functional inequalities for the Teichmüller ring capacity.* Israel J. Math. **88** (1994), 367–380.

[AVV9] Anderson G.D., Vamanamurthy M.K. and Vuorinen M., *Conformal Invariants, Inequalities and Quasiconformal Mappings.* Wiley, 1997.

[AVV10] Anderson G.D., Vamanamurthy M.K. and Vuorinen M., *Functional inequalities for complete elliptic integrals and their ratios.* SIAM J. Math. Anal. **21** (1990), 536–549.

[AVV11] Anderson G.D., Vamanamurthy M.K. and Vuorinen M., *Functional inequalities for hypergeometric functions and complete elliptic integrals.* SIAM J. Math. Anal. **23**(2) (1992), 512–524.

[BB] Borwein J.M. and Borwein P.B., *Pi and the AGM.* John Wiley and Sons, New York, 1987.

[BF] Byrd P.F. and Friedman M.D., *Handbook of Elliptic Integrals for Engineers and Physicists.* Grundlehren Math. Wiss. **57**, Springer-Verlag, Berlin, Göttingen, Heidelberg and New York, 1954.

[Bo] Bowman F., *Introduction to Elliptic Functions with Applications.* Dover, New York, 1961.

[G] Gehring F.W., *Symmetrization of rings in space.* Trans. Amer. Math. Soc. **101** (1961), 499–519.

[He] He C.-Q., *Distortion estimates of quasiconformal mappings.* Sci. Sinica Ser. A **27** (1984), 225–232.

[LV] Lehto O. and Virtanen K.I., *Quasiconformal Mappings in the Plane.* 2nd Ed., Die Grundlehren der math. Wiss., Band **126** Springer-Verlag, Berlin, Heidelberg and New York, 1973.

[Q1] Qiu S.-L., *Two better values of the coefficient of Hölder continuity of K-quasiconformal mappings in space.* Hunan Ann. Math. **8**(1-2) (1988), 41–46.

[Q2] Qiu S.-L., *Distortion properties of K-quasiconformal maps and a better estimate of Mori's constant.* Acta Math. Sinica **4** (1992), 492–504.

[Q3] Qiu S.-L., *Straightening symmetrization of rings with its applications to K-quasiconformal mappings in space.* J. Hunan Univ. **19**(5) (1992), 51–56.

[Q4] Qiu S.-L., *Covering theorems for quasiconformal mappings in n-space.* J. Hangzhou Inst. of Elect. Eng. **11**(1) (1991), 1–13.

[QV] Qiu S.-L. and Vamanamurthy M.K., *Sharp estimates for complete elliptic integrals.* SIAM J. Math. Anal. **27** (1996), 823–834.

[QVV] Qiu S.-L., Vamanamurthy M.K. and Vuorinen M., *Bounds for quasiconformal distortion functions.* J. Math. Anal. Appl. **205** (1997), 43–64.

[Vä] Väisälä J., *Lectures on n-Dimensional Quasiconformal Mappings.* Lecture Notes in Mathematics, vol. 229, Springer-Verlag, Berlin, Heidelberg and New York, 1971.

[VV] Vamanamurthy M.K. and Vuorinen M., *Functional inequalities, Jacobi products and quasiconformal maps.* Illinois J. of Math. **38**(3) (1994), 394–419.

[Vu1] Vuorinen M., *Conformal Geometry and Quasiregular Mappings.* Lecture Notes in Mathematics, vol. 1319, Springer-Verlag, Berlin, Heidelberg and New York, 1988.

[Vu2] Vuorinen M., *Conformal invariants and quasiregular mappings.* J. Analyse Math. **45** (1985), 69–115.

[W] Wang C.-F., *On the precision of Mori's theorem in Q-mapping.* Science Record **4** (1960), 329–333.

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