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Indefinite equi-centroaffinely homogeneous surfaces with vanishing Pick-Invariant in $\mathbf{R}^4$

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Abstract. A nondegenerate equi-centroaffine surface in $\mathbf{R}^4$ is called homogeneous if for any two points p and q on the surface there exists an equi-centroaffine transformation in $\mathbf{R}^4$ which takes the surface to itself and takes p to q. In this paper we classify the indefinite equi-centroaffinely homogeneous surfaces with vanishing Pick-Invariant in $\mathbf{R}^4$ up to centroaffine transformations.

Key words: equi-centroaffine metric, homogeneous surface, Pick-Invariant, centroaffine normalization.

0. Introduction

Affine homogeneous submanifolds (hypersurfaces) in $\mathbf{R}^{n+1}$ form an interesting class of submanifolds (hypersurfaces) in affine differential geometry. They are orbits of subgroups of the affine transformation group in $\mathbf{R}^{n+1}$.

H. Guggenheimer [1], K. Nomizu and T. Sasaki [10] classified the equiaffinely homogeneous surfaces in $\mathbf{R}^3$. H.L. Liu and C.P. Wang [7] classified the centroaffinely homogeneous surfaces in $\mathbf{R}^3$. Recently, C.P. Wang [14] gave a classification of the flat equiaffinely homogeneous surfaces for the equiaffine metric (or so called Burstin-Mayer metric) in $\mathbf{R}^4$. In high codimensional, the class of the homogeneous submanifolds is very large and it is difficult to determine all of them. Therefore, we classify them under certain conditions. In this paper, we give all of the indefinite equi-centroaffinely homogeneous surfaces with vanishing Pick-Invariant in $\mathbf{R}^4$. For the definite surfaces, the situation is much easier, see [4].

In the equiaffine geometry of hypersurfaces in $\mathbf{R}^{n+1}$, the condition of vanishing Pick-Invariant locally characterizes locally strongly convex hyperquadrics. The definitions of the equi-centroaffine metric and the equi-centroaffine normalizations in this paper are motivated from the definitions

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of the equiaffine metric and the affine normal. For the high codimensional, it is difficult to determine all of the (hyper)surfaces with vanishing Pick-Invariant, even if for the definite (hyper)surfaces. In this paper, we give 15 equi-centroaffinely homogeneous surfaces in $\mathbf{R}^4$ with vanishing Pick-Invariant. Many of them contains one or two parameter(s).

The metric we use here is centroaffine metric. It is different with the Burstin-Mayer equiaffine metric. The Laplacian of the Burstin-Mayer metric acting on the position vector field of the surface gives a vector field tangent to the surface. But the Laplacian of the centroaffine metric acting on the position vector field of the surface gives a vector field transversal to the surface. Therefore, we use it and the position vector field of the surface as the centroaffine normalizations. With the Burstin-Mayer equiaffine metric, we can not get such normalization. So, the equi-centroaffine surfaces given here and the equiaffine surfaces given in [14] (or [13]) are different.

1. Equi-centroaffine surfaces in $\mathbf{R}^4$

Let $x : \mathbf{M} \to \mathbf{R}^4$ be an oriented immersed surface such that $x(p) \notin \mathrm{d}x(T_pM)$ for all $p \in \mathbf{M}$ and $x(\mathbf{M})$ is not contained in a hyperplane of $\mathbf{R}^4$. We consider the oriented basis of TM. For any local basis $\sigma = \{E_1, E_2\}$ of TM with dual basis $\{\theta_1, \theta_2\}$ we define

$$\begin{aligned} G^\sigma &= [E_1(x), E_2(x), x, \mathrm{d}^2x] \\ &= \sum_{i,j=1}^{2} [E_1(x), E_2(x), x, E_iE_j(x)]\theta_i \otimes \theta_j, \end{aligned} \tag{1.1}$$

where [ ] is the standard determinant in $\mathbf{R}^4$. G^σ is a symmetric 2-form. We assume that G^σ is nondegenerate. If τ is another local basis of TM, we have

$$|\det G^\tau|^{-\frac{1}{4}}\, G^\tau = |\det G^\sigma|^{-\frac{1}{4}}\, G^\sigma, \tag{1.2}$$

where | | denotes the absolute value of the real number. Therefore

$$G = |\det G^\sigma|^{-\frac{1}{4}}\, G^\sigma \tag{1.3}$$

is independent of the choice of the basis σ and thus a globally defined symmetric 2-form. From (1.1) we know that G is an invariant under equi-centroaffine transformations. G is called the equi-centroaffine metric of $x : \mathbf{M} \to \mathbf{R}^4$. Let Δ denote the Laplacian of G. $\{x, \Delta x\}$ is called the equi-centroaffine normalization of $x : \mathbf{M} \to \mathbf{R}^4$. We call x a nondegenerate equi-centroaffine surface if G^σ is nondegenerate. x is called definite or indefinite if G^σ is definite or indefinite, respectively. In this paper we consider indefinite surfaces in $\mathbf{R}^4$.

For the equi-centroaffine surface x, let $\nabla = \{\Gamma_{ij}^k\}$ and $\widehat{\nabla} = \{\widehat{\Gamma}_{ij}^k\}$ be the induced connection and the Levi-Civita connection of the equi-centroaffine metric G. We define the Fubini-Pick form C by

$$C_{ij}^k := \Gamma_{ij}^k - \widehat{\Gamma}_{ij}^k, \quad i, j, k = 1, 2. \tag{1.4}$$

We know that C_{ij}^k is symmetric for i and j. The Pick-Invariant is defined by

$$J = \frac{1}{2} G^{ij} C_{ik}^l C_{lj}^k. \tag{1.5}$$

Let $x : \mathbf{M} \to \mathbf{R}^4$ be an equi-centroaffine surface with the indefinite equi-centroaffine metric G. We choose the asymptotic local basis $\sigma = \{E_1, E_2\}$ of TM such that

$$G = 2e^w(\mathrm{d}u\mathrm{d}v) = e^w(\mathrm{d}u \otimes \mathrm{d}v + \mathrm{d}v \otimes \mathrm{d}u). \tag{1.6}$$

Then $\{E_1(x), E_2(x), x, \Delta x\}$ (or $\{x_u, x_v, x, x_{uv}\}$) forms a local moving frame for $\mathbf{R}^4$ on $\mathbf{M}$, where Δ is the Laplacian of G. By (1.6) we have $[x_u, x_v, x, x_{uu}] = [x_u, x_v, x, x_{vv}] = 0$. Without loss of generality, we may assume that $[x_u, x_v, x, x_{uv}] > 0$. Then

$$\begin{aligned} G^\sigma &= [E_1(x), E_2(x), x, \mathrm{d}^2x] = [x_u, x_v, x, \mathrm{d}^2x] \\ &= [x_u, x_v, x, x_{uv}](\mathrm{d}u \otimes \mathrm{d}v + \mathrm{d}v \otimes \mathrm{d}u). \end{aligned} \tag{1.7}$$

From (1.3), (1.6) and (1.7) we obtain

$$[x_u, x_v, x, x_{uv}] = e^{2w}. \tag{1.8}$$

We assume that

$$\begin{cases} x_{uu} = \mu_1 x_u + e^{-w}\varphi x_v + \psi x \\ x_{vv} = e^{-w}\lambda x_u + \mu_2 x_v + \theta x. \end{cases}$$

The relation (1.8) yields $\mu_1 = w_u$ and $\mu_2 = w_v$. Hence

$$\begin{cases} x_{uu} = w_u x_u + e^{-w}\varphi x_v + \psi x \\ x_{vv} = e^{-w}\lambda x_u + w_v x_v + \theta x. \end{cases} \tag{1.9}$$

Proposition 1.1 *Let* Φ, Ψ, Λ *and* Θ *be the forms on* $\mathbf{M}$ *defined by*

$$\Phi = \varphi \mathrm{d}u^3 = e^w \frac{[x_u, x_{uu}, x, x_{uv}]}{[x_u, x_v, x, x_{uv}]} \mathrm{d}u^3, \tag{1.10}$$

$$\Psi = \psi \mathrm{d}u^2 = \frac{[x_u, x_v, x_{uu}, x_{uv}]}{[x_u, x_v, x, x_{uv}]} \mathrm{d}u^2, \tag{1.11}$$

$$\Lambda = \lambda \mathrm{d}v^3 = e^w \frac{[x_{vv}, x_v, x, x_{uv}]}{[x_u, x_v, x, x_{uv}]} \mathrm{d}v^3, \tag{1.12}$$

$$\Theta = \theta \mathrm{d}v^2 = \frac{[x_u, x_v, x_{vv}, x_{uv}]}{[x_u, x_v, x, x_{uv}]} \mathrm{d}v^2. \tag{1.13}$$

Then $\{\Phi, \Psi, \Lambda, \Theta\}$ *are globally defined on* $\mathbf{M}$ *and equi-centroaffine invariants.*

Proposition 1.2 (i) *If two indefinite equi-centroaffine surfaces* $x, \tilde{x} : \mathbf{M} \to \mathbf{R}^4$ *are centroaffinely equivalent, then there exists a constant* α *such that*

$$G = \alpha \tilde{G}, \quad \Phi = \alpha \tilde{\Phi}, \quad \Psi = \tilde{\Psi}, \quad \Lambda = \alpha \tilde{\Lambda}, \quad \Theta = \tilde{\Theta}. \tag{1.14}$$

(ii) *Conversely, if* $x, \tilde{x} : \mathbf{M} \to \mathbf{R}^4$ *are indefinite equi-centroaffine surfaces such that there exists a constant* α *so that* (1.14) *holds, then* x *and* $\tilde{x}$ *are centroaffinely equivalent.*

In the following, we derive the integrability conditions for the equi-centroaffine surface $x : \mathbf{M} \to \mathbf{R}^4$. From (1.9) we have

$$\begin{cases} x_{uu} = w_u x_u + e^{-w} \varphi x_v + \psi x \\ x_{vv} = e^{-w} \lambda x_u + w_v x_v + \theta x \\ x_{uvu} = (w_{uv} + e^{-2w} \lambda \varphi) x_u + (e^{-w} \varphi_v + \psi) x_v \\ \qquad\qquad + (e^{-w} \varphi \theta + \psi_v) x + w_u x_{uv} \\ x_{uvv} = (e^{-w} \lambda_u + \theta) x_u + (w_{uv} + e^{-2w} \lambda \varphi) x_v \\ \qquad\qquad + (e^{-w} \lambda \psi + \theta_u) x + w_v x_{uv}. \end{cases} \tag{1.15}$$

Hence $x_{uvuv} = x_{uvvu}$ yields

$$\begin{aligned} &(w_{uv} + e^{-2w} \lambda \varphi)_v + e^{-2w} \lambda \varphi_v \\ &\qquad = (e^{-w} \lambda_u + \theta)_u + \theta_u + w_v (w_{uv} + e^{-2w} \lambda \varphi), \end{aligned}$$

$$\begin{aligned}&(e^{-w}\varphi_v + \psi)_v + \psi_v + w_u(w_{uv} + e^{-2w}\lambda\varphi)\\ &\qquad = e^{-2w}\varphi\lambda_u + (w_{uv} + e^{-2w}\lambda\varphi)_u,\\ &e^{-w}\theta\varphi_v + (e^{-w}\varphi\theta + \psi_v)_v + w_u(e^{-w}\lambda\psi + \theta_u)\\ &\qquad = e^{-w}\psi\lambda_u + (e^{-w}\lambda\psi + \theta_u)_u + w_v(e^{-w}\varphi\theta + \psi_v),\end{aligned}$$

that is,

$$(e^{-w}w_{uv} + e^{-3w}\lambda\varphi)_v + e^{-3w}\lambda\varphi_v - e^{-w}(e^{-w}\lambda_u)_u - 2e^{-w}\theta_u = 0, \quad (1.16)$$

$$(e^{-w}w_{uv} + e^{-3w}\lambda\varphi)_u + e^{-3w}\varphi\lambda_u - e^{-w}(e^{-w}\varphi_v)_v - 2e^{-w}\psi_v = 0, \quad (1.17)$$

$$\begin{aligned}&(e^{-2w}\varphi\theta + e^{-w}\psi_v)_v + e^{-2w}\theta\varphi_v\\ &\qquad = (e^{-2w}\lambda\psi + e^{-w}\theta_u)_u + e^{-2w}\psi\lambda_u. \end{aligned} \quad (1.18)$$

Let κ be the Gauss curvature of x, then from (1.6) we have

$$\kappa = -e^{-w}w_{uv}. \quad (1.19)$$

2. Equi-centroaffinely homogeneous surfaces with $J = 0$ in $\mathbf{R}^4$

An equi-centroaffine surface in $\mathbf{R}^4$ is called equi-centroaffinely homogeneous, if for any two points p and q on the surface there exists an equi-centroaffine transformation in $\mathbf{R}^4$ which takes the surface to itself and takes p to q. Therefore, an equi-centroaffinely homogeneous surface is an orbit of an equi-centroaffine group which acts on a fixed point in $\mathbf{R}^4$. In this section, we prove the following theorem.

Theorem 2.1 *Let $x : \mathbf{M} \to \mathbf{R}^4$ be a nondegenerate indefinite equi-centroaffinely homogeneous surface with vanishing Pick-Invariant in $\mathbf{R}^4$. Assume that $x(\mathbf{M})$ is not contained in a hyperplane of $\mathbf{R}^4$. Then x is centroaffinely equivalent to one (or a part) of the following surfaces in $\mathbf{R}^4$:*

(i)
$$\begin{cases} x_1x_2x_3x_4 = 1\\ x_1^{\alpha}x_3 = x_2^{\alpha}x_4,\end{cases}$$

where $\alpha \neq 0$ is constant;

(ii)
$$\begin{cases} x_4 = \frac{1}{2}x_3(\log x_1 - \log x_2), \quad x_1 > 0,\ x_2 > 0\\ x_1x_2x_3^2 = 1;\end{cases}$$

(iii) $$\begin{cases} x_1x_2(x_3^2+x_4^2)=1 \\ x_1 = x_2 \exp\left(\dfrac{2}{\alpha}\arctan\dfrac{x_4}{x_3}\right), \end{cases}$$

where $\alpha \neq 0$ *is constant*;

(iv) $$\begin{cases} x_1x_4 = x_2x_3 \\ x_2x_4 = 1; \end{cases}$$

(v) $$\begin{cases} (x_1^2+x_2^2)(x_3^2+x_4^2)=1 \\ \arctan\dfrac{x_2}{x_1} = \alpha\arctan\dfrac{x_4}{x_3}, \end{cases}$$

where $\alpha \neq 0$ *is constant*;

(vi) $$\begin{cases} x_1 = x_2\arctan\dfrac{x_4}{x_3} \\ x_2^2(x_3^2+x_4^2)=1; \end{cases}$$

(vii) $$\begin{cases} x_1x_4 - x_2x_3 = 2\log x_2 \\ x_2x_4 = 1; \end{cases}$$

(viii) $$\begin{cases} x_1x_4 - x_2x_3 = \arcsin x_4 \\ x_3^2+x_4^2 = 1; \end{cases}$$

(ix) $$\begin{cases} x_1x_4 = x_2x_3 \\ x_2^2+x_4^2 = 1; \end{cases}$$

(x) $$\begin{cases} x_1^2+x_2^2+x_3^2+x_4^2 = 1 \\ x_1x_4 = x_2x_3; \end{cases}$$

(xi) $$\begin{cases} (x_1^2+x_2^2)(x_3^2+x_4^2)=1 \\ \log(x_1^2+x_2^2) = \alpha\arcsin(x_2x_3+x_1x_4), \end{cases}$$

where α *is constant*;

(xii) $$\begin{cases} x_3x_4^\alpha = 1 \\ x_1x_3 - x_2x_4 = \beta, \end{cases}$$

where α and β are constants;

$$\text{(xiii)} \qquad \begin{cases} x_3 = x_4 \log x_4, \quad x_4 > 0 \\ x_1x_4 - x_2x_3 = \alpha, \end{cases}$$

where α is constant;

$$\text{(xiv)} \qquad \begin{cases} x_1x_4 - x_2x_3 = \alpha \\ x_3^2 + x_4^2 = \exp\left[2\beta \arctan \dfrac{x_3}{x_4}\right], \end{cases}$$

where α and β are constants;

$$\text{(xv)} \qquad \begin{cases} x_4(x_2x_3 - x_1x_4) = 2\alpha x_3^3 \\ x_4(3x_1x_4 - x_2x_3)^3 = 8x_3^3, \end{cases}$$

where α is constant.

Remark 2.1. The generating groups of the surfaces (i)–(xi) in Theorem 2.1 are abelian groups; the generating groups of the surfaces (xii)–(xv) in Theorem 2.1 are no abelian groups. See the following examples.

Example 2.1. The surface defined by

$$x = (e^{u+v}, e^{u-v}, e^{-u-\alpha v}, e^{-u+\alpha v}), \quad u, v \in \mathbf{R},$$

for any $\alpha \in \mathbf{R}$, $\alpha > 0$, is a homogeneous equi-centroaffine surface in $\mathbf{R}^4$. Its generating group is

$$\left\{ \left(\begin{matrix} e^{u+v} & 0 & 0 & 0 \\ 0 & e^{u-v} & 0 & 0 \\ 0 & 0 & e^{-u-\alpha v} & 0 \\ 0 & 0 & 0 & e^{-u+\alpha v} \end{matrix} \right) \middle| \, u, v \in \mathbf{R} \right\},$$

which acts on the point ${}^t(1, 1, 1, 1)$, and

$$G = (8\alpha)^{-\frac{1}{2}} 16\alpha du dv.$$

The surface is equi-centroaffinely equivalent to the surface (i) in Theorem 2.1.

Example 2.2. The surface defined by

$$x = (e^{u+v}, e^{u-v}, e^{-u}, ve^{-u}), \quad u, v \in \mathbf{R},$$

is a homogeneous equi-centroaffine surface in $\mathbf{R}^4$. Its generating group is

$$\left\{ \begin{pmatrix} e^{u+v} & 0 & 0 & 0 \\ 0 & e^{u-v} & 0 & 0 \\ 0 & 0 & e^{-u} & 0 \\ 0 & 0 & ve^{-u} & e^{-u} \end{pmatrix} \middle| u, v \in \mathbf{R} \right\},$$

which acts on the point ${}^t(1,1,1,0)$, and

$$G = (4)^{-\frac{1}{2}} 8 \mathrm{d}u \mathrm{d}v.$$

The surface is equi-centroaffinely equivalent to the surface (ii) in Theorem 2.1.

Example 2.3. The surface defined by

$$x = (e^{u+v}, e^{u-v}, e^{-u}\cos(\alpha v), e^{-u}\sin(\alpha v)), \quad u, v \in \mathbf{R},$$

for any $\alpha \in \mathbf{R}$, $\alpha > 0$, is a homogeneous equi-centroaffine surface in $\mathbf{R}^4$. Its generating group is

$$\left\{ \begin{pmatrix} \begin{pmatrix} e^{u+v} & 0 \\ 0 & e^{u-v} \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & e^{-u} \begin{pmatrix} \cos(\alpha v) & -\sin(\alpha v) \\ \sin(\alpha v) & \cos(\alpha v) \end{pmatrix} \end{pmatrix} \middle| u, v \in \mathbf{R} \right\},$$

which acts on the point ${}^t(1,1,1,0)$, and

$$G = (4\alpha)^{-\frac{1}{2}} 8\alpha \mathrm{d}u \mathrm{d}v.$$

The surface is equi-centroaffinely equivalent to the surface (iii) in Theorem 2.1.

Example 2.4. The surface defined by

$$x = (ue^v, e^v, ue^{-v}, e^{-v}), \quad u, v \in \mathbf{R},$$

is a homogeneous equi-centroaffine surface in $\mathbf{R}^4$. Its generating group is

$$\left\{ \begin{pmatrix} e^v & ue^v & 0 & 0 \\ 0 & e^v & 0 & 0 \\ 0 & 0 & e^{-v} & ue^{-v} \\ 0 & 0 & 0 & e^{-v} \end{pmatrix} \middle| u, v \in \mathbf{R} \right\},$$

which acts on the point ${}^t(0,1,0,1)$, and

$$G = (2)^{-\frac{1}{2}} 4 du dv.$$

The surface is equi-centroaffinely equivalent to the surface (iv) in Theorem 2.1.

Example 2.5. The surface defined by

$$x = (e^u \cos(\alpha v), e^u \sin(\alpha v), e^{-u} \cos v, e^{-u} \sin v), \quad u, v \in \mathbf{R},$$

for any $\alpha \in \mathbf{R}$, $\alpha > 0$, is a homogeneous equi-centroaffine surface in $\mathbf{R}^4$. Its generating group is

$$\left\{ \begin{pmatrix} e^u \begin{pmatrix} \cos(\alpha v) & -\sin(\alpha v) \\ \sin(\alpha v) & \cos(\alpha v) \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & e^{-u} \begin{pmatrix} \cos v & -\sin v \\ \sin v & \cos v \end{pmatrix} \end{pmatrix} \middle| \, u, v \in \mathbf{R} \right\},$$

which acts on the point ${}^t(1,0,1,0)$, and

$$G = -(2\alpha)^{-\frac{1}{2}} 4\alpha du dv.$$

The surface is equi-centroaffinely equivalent to the surface (v) in Theorem 2.1.

Example 2.6. The surface defined by

$$x = (ve^u, e^u, e^{-u} \cos v, e^{-u} \sin v), \quad u, v \in \mathbf{R},$$

is a homogeneous equi-centroaffine surface in $\mathbf{R}^4$. Its generating group is

$$\left\{ \begin{pmatrix} \begin{pmatrix} e^u & ve^u \\ 0 & e^u \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & e^{-u} \begin{pmatrix} \cos v & -\sin v \\ \sin v & \cos v \end{pmatrix} \end{pmatrix} \middle| \, u, v \in \mathbf{R} \right\},$$

which acts on the point ${}^t(0,1,1,0)$, and

$$G = (2)^{-\frac{1}{2}} 4 du dv.$$

The surface is equi-centroaffinely equivalent to the surface (vi) in Theorem 2.1.

Example 2.7. The surface defined by

$$x = ((u+v)e^v, e^v, (u-v)e^{-v}, e^{-v}), \quad u, v \in \mathbf{R},$$

is a homogeneous equi-centroaffine surface in $\mathbf{R}^4$. Its generating group is

$$\left\{ \begin{pmatrix} e^v & (u+v)e^v & 0 & 0 \\ 0 & e^v & 0 & 0 \\ 0 & 0 & e^{-v} & (u-v)e^{-v} \\ 0 & 0 & 0 & e^{-v} \end{pmatrix} \middle| u, v \in \mathbf{R} \right\},$$

which acts on the point ${}^t(0,1,0,1)$, and

$$G = (2)^{-\frac{1}{2}} 4 \mathrm{d}u \mathrm{d}v.$$

The surface is equi-centroaffinely equivalent to the surface (vii) in Theorem 2.1.

Example 2.8. The surface defined by

$$x = (u\cos v + v\sin v, u\sin v - v\cos v, \cos v, \sin v), \quad u, v \in \mathbf{R},$$

is a homogeneous equi-centroaffine surface in $\mathbf{R}^4$. Its generating group is

$$\left\{ \begin{pmatrix} \begin{pmatrix} \cos v & -\sin v \\ \sin v & \cos v \end{pmatrix} & \begin{pmatrix} u & v \\ -v & u \end{pmatrix} \begin{pmatrix} \cos v & -\sin v \\ \sin v & \cos v \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} \cos v & -\sin v \\ \sin v & \cos v \end{pmatrix} \end{pmatrix} \middle| u, v \in \mathbf{R} \right\},$$

which acts on the point ${}^t(0,0,1,0)$, and

$$G = -\left(\frac{1}{2}\right)^{-\frac{1}{2}} \mathrm{d}u \mathrm{d}v.$$

The surface is equi-centroaffinely equivalent to the surface (viii) in Theorem 2.1.

Example 2.9. The surface defined by

$$x = (u\cos v, \cos v, u\sin v, \sin v), \quad u \in \mathbf{R}, v \in [0, 2\pi],$$

is a homogeneous equi-centroaffine surface in $\mathbf{R}^4$. Its generating group is

$$\left\{ \begin{pmatrix} \cos v \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} & -\sin v \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \\ \sin v \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} & \cos v \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \end{pmatrix} \middle| u \in \mathbf{R}, v \in [0, 2\pi] \right\},$$

which acts on the point ${}^t(0,1,0,0)$, and

$$G = \left(\frac{1}{2}\right)^{-\frac{1}{2}} dudv.$$

The surface is equi-centroaffinely equivalent to the surface (ix) in Theorem 2.1.

Example 2.10. The surface defined by

$$x = (\cos u \cos v, \cos u \sin v, \sin u \cos v, \sin u \sin v), \quad u, v \in [0, 2\pi],$$

is a homogeneous equi-centroaffine surface in $\mathbf{R}^4$. Its generating group is

$$\left\{ \begin{pmatrix} \cos u \begin{pmatrix} \cos v & -\sin v \\ \sin v & \cos v \end{pmatrix} & -\sin u \begin{pmatrix} \cos v & -\sin v \\ \sin v & \cos v \end{pmatrix} \\ \sin u \begin{pmatrix} \cos v & -\sin v \\ \sin v & \cos v \end{pmatrix} & \cos u \begin{pmatrix} \cos v & -\sin v \\ \sin v & \cos v \end{pmatrix} \end{pmatrix} \middle| u, v \in [0, 2\pi] \right\},$$

which acts on the point ${}^t(1,0,0,0)$, and

$$G = -\left(\frac{1}{2}\right)^{-\frac{1}{2}} dudv.$$

The surface is equi-centroaffinely equivalent to the surface (x) in Theorem 2.1.

Example 2.11. The surface defined by

$$\begin{aligned} x \;=\; & (e^{\alpha v}\cos(u+v), e^{\alpha v}\sin(u+v), e^{-\alpha v}\cos(-u+v), \\ & \qquad e^{-\alpha v}\sin(-u+v)), \quad u, v \in \mathbf{R}, \end{aligned}$$

for any $\alpha \in \mathbf{R}$, is a homogeneous equi-centroaffine surface in $\mathbf{R}^4$. Its generating group is

$$\left\{ \begin{pmatrix} e^{\alpha v} \begin{pmatrix} \cos(u+v) & -\sin(u+v) \\ \sin(u+v) & \cos(u+v) \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & e^{-\alpha v} \begin{pmatrix} \cos(-u+v) & -\sin(-u+v) \\ \sin(-u+v) & \cos(-u+v) \end{pmatrix} \end{pmatrix} \right\},$$

$u, v \in \mathbf{R}$, which acts on the point ${}^t(1,0,1,0)$, and

$$G = -[2(\alpha^2+1)]^{-\frac{1}{2}}4(\alpha^2+1)\mathrm{d}u\mathrm{d}v.$$

The surface is equi-centroaffinely equivalent to the surface (xi) in Theorem 2.1.

Example 2.12. The surface defined by

$$x = (vu + \alpha u^k, vu^{-k} + \beta u^{-1}, u^{-k}, u), \quad u, v \in \mathbf{R},\ u > 0,$$

for any α, β, $k \in \mathbf{R}$, $k \neq 0, -1$, is a homogeneous equi-centroaffine surface in $\mathbf{R}^4$. Its generating group is

$$\left\{ \left. \begin{pmatrix} u^k & 0 & 0 & vu \\ 0 & u^{-1} & vu^{-k} & 0 \\ 0 & 0 & u^{-k} & 0 \\ 0 & 0 & 0 & u \end{pmatrix} \right| u, v \in \mathbf{R}, u > 0 \right\},$$

which acts on the point ${}^t(\alpha, \beta, 1, 1)$, and

$$\begin{aligned} G = & \left[\frac{1}{4}(1+k)^4 u^{-4k}\right]^{-\frac{1}{4}} [-(1+k)^2 u^{-2k}\mathrm{d}u\mathrm{d}v \\ & - 2(k-1)(k+1)(k\alpha+\beta)u^{-k-2}\mathrm{d}u^2]. \end{aligned}$$

The surface is equi-centroaffinely equivalent to the surface (xii) in Theorem 2.1.

Example 2.13. The surface defined by

$$\begin{aligned} x = & (vu \log u + \alpha(1+\log u)u^{-1}, vu + \alpha u^{-1}, u\log u, u), \\ & u, v \in \mathbf{R},\ u > 0, \end{aligned}$$

for any $\alpha \in \mathbf{R}$, is a homogeneous equi-centroaffine surface in $\mathbf{R}^4$. Its generating group is

$$\left\{ \left. \begin{pmatrix} u^{-1}\begin{pmatrix} 1 & \log u \\ 0 & 1 \end{pmatrix} & vu \begin{pmatrix} 1 & \log u \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & u\begin{pmatrix} 1 & \log u \\ 0 & 1 \end{pmatrix} \end{pmatrix} \right| u, v \in \mathbf{R}, u > 0 \right\},$$

which acts on the point ${}^t(\alpha, \alpha, 0, 1)$, and

$$G = \left[\frac{1}{4}(u^4)\right]^{-\frac{1}{4}} u^2\mathrm{d}u\mathrm{d}v.$$

The surface is equi-centroaffinely equivalent to the surface (xiii) in Theorem 2.1.

Example 2.14. The surface defined by

$$\begin{aligned} x \;=\; & (e^{-ku}(\alpha\cos u + \beta\sin u) + ve^{ku}\sin u, e^{-ku}(\beta\cos u - \alpha\sin u) \\ & + ve^{ku}\cos u, e^{ku}\sin u, e^{ku}\cos u), \quad u, v \in \mathbf{R}, \end{aligned}$$

for any α, β, $k \in \mathbf{R}$, is a homogeneous equi-centroaffine surface in $\mathbf{R}^4$. Its generating group is

$$\left\{ \left(\begin{array}{cc} e^{-ku}\begin{pmatrix} \cos u & \sin u \\ -\sin u & \cos u \end{pmatrix} & ve^{ku}\begin{pmatrix} \cos u & \sin u \\ -\sin u & \cos u \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & e^{ku}\begin{pmatrix} \cos u & \sin u \\ -\sin u & \cos u \end{pmatrix} \end{array} \right) \middle| \; u, v \in \mathbf{R} \right\},$$

which acts on the point ${}^t(\alpha, \beta, 0, 1)$, and

$$G = \left[\frac{1}{4}e^{8ku}\right]^{-\frac{1}{4}} [e^{4ku}\mathrm{d}u\mathrm{d}v + 4ke^{2ku}(k\alpha - \beta)\mathrm{d}u^2].$$

The surface is equi-centroaffinely equivalent to the surface (xiv) in Theorem 2.1.

Example 2.15. The surface defined by

$$x = (vu^{-\frac{1}{2}} + \alpha v^3 u^{\frac{3}{2}}, u^{-\frac{1}{2}} + 3\alpha v^2 u^{\frac{3}{2}}, vu^{\frac{3}{2}}, u^{\frac{3}{2}}), \quad u, v \in \mathbf{R}, \; u > 0,$$

for any $\alpha \in \mathbf{R}$, is a homogeneous equi-centroaffine surface in $\mathbf{R}^4$. Its generating group is

$$\left\{ \begin{pmatrix} u^{-\frac{3}{2}} & vu^{-\frac{1}{2}} & 3\alpha v^2 u^{\frac{1}{2}} & \alpha v^3 u^{\frac{3}{2}} \\ 0 & u^{-\frac{1}{2}} & 6\alpha v u^{\frac{1}{2}} & 3\alpha v^2 u^{\frac{3}{2}} \\ 0 & 0 & u^{\frac{1}{2}} & vu^{\frac{3}{2}} \\ 0 & 0 & 0 & u^{\frac{3}{2}} \end{pmatrix} \middle| \; u, v \in \mathbf{R}, u > 0 \right\},$$

which acts on the point ${}^t(0, 1, 0, 1)$, and

$$G = (4)^{-\frac{1}{4}}(-4)\mathrm{d}u\mathrm{d}v.$$

The surface is equi-centroaffinely equivalent to the surface (xv) in Theorem 2.1.

For the proof of the Theorem 2.1, we need the following Lemma.

Lemma *Let $x : \mathbf{M} \to \mathbf{R}^4$ be a nondegenerate indefinite equi-centroaffinely homogeneous surface with vanishing Pick-Invariant. Then x is flat.*

Proof. Let $x : \mathbf{M} \to \mathbf{R}^4$ be a nondegenerate indefinite equi-centroaffinely homogeneous surface with vanishing Pick-Invariant. Since x is equi-centroaffinely homogeneous, $\psi \neq 0$ means that $\psi(p) \neq 0$ for all points $p \in \mathbf{M}$. $\psi^2 + \theta^2 = 0$ implies that x lies in a hyperplane of $\mathbf{R}^4$, so we may assume that $\psi^2 + \theta^2 \neq 0$. From the homogeneity of x, we know that any equi-centroaffinely invariant function on the surface is constant. Therefore, the Gauss curvature κ of x is constant.

By a direct computation we have

$$C^1_{11} = C^1_{12} = C^2_{12} = C^2_{22} = 0,$$
$$C^2_{11} = e^{-w}\varphi, \quad C^1_{22} = e^{-w}\lambda.$$

Then the Pick-Invariant of x is given by

$$J = e^{-3w}\lambda\varphi.$$

Hence $J \equiv 0$ if and only if $\lambda\varphi \equiv 0$. Setting $(u, v) \to (v, u)$, we may assume that $\lambda = 0$. We consider the following cases, respectively.

(1) Let $\lambda = \varphi = 0$. By (1.16) and (1.17) we have $\theta_u = 0$ and $\psi_v = 0$. When $\theta \neq 0$, $\theta^{-3}(\theta_v - 2w_v\theta)^2 =$ constant and $\theta_u = 0$ yield $w_{uv} = 0$; when $\psi \neq 0$, $\psi^{-3}(\psi_u - 2w_u\psi)^2 =$ constant and $\psi_v = 0$ yield $w_{uv} = 0$. Therefore $\kappa = -e^{-w}w_{uv} = 0$.

(2) Let $\lambda = 0$, $\varphi \neq 0$. By (1.16) we have $\theta_u = 0$. If $\theta \neq 0$, $\theta^{-3}(\theta_v - 2w_v\theta)^2 =$ constant and $\theta_u = 0$ yield $w_{uv} = 0$. When $\theta = 0$, we have $\psi \neq 0$. By (1.18) we get $(e^{-w}\psi_v)_v = 0$. $\psi^{-1}(e^{-w}\psi_v)^2 =$ constant and $(e^{-w}\psi_v)_v = 0$ yield $\psi_v = 0$. $\psi \neq 0$, $\psi_v = 0$ and $\psi^{-3}(\psi_u - 2w_u\psi)^2 =$ constant yield $w_{uv} = 0$. Therefore $\kappa = -e^{-w}w_{uv} = 0$. x is flat. □

This completes the proof of the Lemma.

The proof of Theorem 2.1 Let $x : \mathbf{M} \to \mathbf{R}^4$ be an equi-centroaffinely homogeneous surface with vanishing Pick-Invariant in $\mathbf{R}^4$. From Lemma, x is flat. By the flatness of the surface, we can choose a local basis such that $w \equiv 0$. Since $\psi^2 + \theta^2 = 0$ implies that x lies in a hyperplane of $\mathbf{R}^4$, Setting $(u, v) \to (v, u)$, we may choose that $\psi \neq 0$. Therefore we have

$$\frac{{\psi_v}^2}{\psi} = \text{constant}; \qquad \frac{\psi_u^2}{\psi^3} = \text{constant}. \tag{2.1}$$

Let $\epsilon = \pm 1$, $\epsilon\psi > 0$. Then we can put

$$\psi_u = b(\epsilon\psi)^{\frac{3}{2}}, \qquad \psi_v = a(\epsilon\psi)^{\frac{1}{2}}. \tag{2.2}$$

From (2.2) we have

$$\begin{aligned} \psi\psi_{uv} &= \frac{3}{2}b(\epsilon\psi)^{\frac{1}{2}}\epsilon\psi_v\psi = \frac{3}{2}\psi_u\psi_v, \\ \psi\psi_{vu} &= \frac{1}{2}a(\epsilon\psi)^{-\frac{1}{2}}\epsilon\psi\psi_u = \frac{1}{2}\psi_u\psi_v, \end{aligned}$$

that is

$$\psi_u\psi_v = 0. \tag{2.3}$$

Therefore, we obtain

(A) If $\psi_u = \psi_v = 0$, ψ is constant; $e^{-2w}\psi\theta =$ constant yields θ is constant; $\varphi^2\psi^{-3} =$ constant yields that φ is constant; $e^{-6w}\psi^3\lambda^2 =$ constant yields that λ is constant.

(B) If $\psi_v = 0$, $\psi_u \neq 0$, from (2.1) we get

$$\psi(u) = \epsilon\frac{4}{b^2}u^{-2}, \quad u \neq 0. \tag{2.4}$$

(C) If $\psi_v \neq 0$, $\psi_u = 0$, from (2.1) we get

$$\psi(v) = \epsilon\frac{a^2}{4}v^2, \quad v \neq 0. \tag{2.5}$$

We assume

$$\varphi^2\psi^{-3} = \epsilon c^2; \quad \lambda^2\psi^3 = \epsilon d^2; \quad \theta\psi = e, \tag{2.6}$$

where c, d and e are constants. Since $J = e^{-3w}\lambda\varphi = 0$, we have $cd = 0$. Thus (1.16), (1.17) and (1.18) become

$$\begin{cases} -\lambda_{uu} - 2\theta_u = 0 \\ -\varphi_{vv} - 2\psi_v = 0 \\ \varphi\theta_v + \psi_{vv} + 2\theta\varphi_v = \lambda\psi_u + \theta_{uu} + 2\psi\lambda_u. \end{cases} \tag{2.7}$$

From (2.2), (2.6) and (2.7) we get

$$\begin{cases} 4be - 3b^2d = 0 \\ 4a + 3a^2c = 0 \\ 4ace + \epsilon a^2 + 4bd - \epsilon b^2 e = 0. \end{cases} \tag{2.8}$$

Therefore, for the case (B), we have: (I) $a = c = d = e = 0$, $b \neq 0$; (II) $a = d = e = 0$, $b \neq 0$, $c \neq 0$; (III) $a = c = 0$, $b \neq 0$, $d \neq 0$, $e \neq 0$, $\epsilon = 1$, $3b^2 = 16$, $e \pm \sqrt{3}d = 0$. For the case (C), we have: $b = d = 0$, $a \neq 0$, $c \neq 0$, $e \neq 0$, $3ac = -4$, $16e = 3\epsilon a^2$. We solve the equations (1.15) in the separated cases.

Case (A): $\psi_u = \psi_v = 0$.
In this case, φ, ψ, λ and θ are constants satisfying $\lambda\varphi = 0$. Then (1.16), (1.17) and (1.18) are identically satisfied. From (1.15) we have

$$\begin{cases} x_{uu} = \varphi x_v + \psi x \\ x_{vv} = \lambda x_u + \theta x \\ x_{uvu} = \lambda\varphi x_u + \psi x_v + \varphi\theta x \\ x_{uvv} = \theta x_u + \lambda\varphi x_v + \lambda\psi x. \end{cases} \tag{2.9}$$

Since the case $\lambda = 0$ is equivalent to the case $\varphi = 0$ by setting $(u, v) \to (v, u)$, so we may assume that $\varphi = 0$. Setting $(u, v) \to (-u, v)$, we may also assume that $\lambda \geq 0$. From (2.9) we have

$$\begin{cases} x_{uu} = \psi x \\ x_{vv} = \lambda x_u + \theta x. \end{cases}$$

(i) If $\psi > 0$, setting $(\sqrt{\psi}u, v) \to (u, v)$, we may assume that $\psi = 1$. From $x_{uu} = x$ we get

$$x = f(v)e^u + g(v)e^{-u}.$$

By $x_{vv} = \lambda x_u + \theta x$ we obtain

$$f''(v) = (\lambda + \theta)f(v), \quad g''(v) = (\theta - \lambda)g(v).$$

(1) If $\theta > 0$, setting $(u, \sqrt{\theta}v) \to (u, v)$, we may assume that $\theta = 1$. Then

$$f''(v) = (1 + \lambda)f(v), \quad g''(v) = (1 - \lambda)g(v).$$

(a) When $\lambda \in [0, 1)$, we have

$$f(v) = c_1 e^{\sqrt{1+\lambda}v} + c_2 e^{-\sqrt{1+\lambda}v}, \quad g(v) = c_3 e^{\sqrt{1-\lambda}v} + c_4 e^{-\sqrt{1-\lambda}v},$$

where $c_i \in \mathbf{R}^4$, $i = 1, 2, 3, 4$ and $[c_1, c_2, c_3, c_4] \neq 0$. Then

$$x = (c_1, c_2, c_3, c_4)\ {}^t(e^{u+\sqrt{1+\lambda}v}, e^{u-\sqrt{1+\lambda}v}, e^{-u+\sqrt{1-\lambda}v}, e^{-u-\sqrt{1-\lambda}v}). \tag{2.10}$$

The surface is centroaffinely equivalent to Example 2.1.

(b) When $\lambda = 1$, we have

$$f(v) = c_1 e^{\sqrt{2}v} + c_2 e^{-\sqrt{2}v}, \quad g(v) = c_3 v + c_4,$$

where $c_i \in \mathbf{R}^4$, $i = 1, 2, 3, 4$ and $[c_1, c_2, c_3, c_4] \neq 0$. Then

$$x = (c_1, c_2, c_3, c_4)\ {}^t(e^{u+\sqrt{2}v}, e^{u-\sqrt{2}v}, ve^{-u}, e^{-u}). \tag{2.11}$$

The surface is centroaffinely equivalent to Example 2.2 .

(c) When $\lambda \in (1, +\infty)$, we have

$$\begin{aligned} f(v) &= c_1 e^{\sqrt{1+\lambda}v} + c_2 e^{-\sqrt{1+\lambda}v}, \\ g(v) &= c_3 \cos(\sqrt{\lambda - 1}v) + c_4 \sin(\sqrt{\lambda - 1}v), \end{aligned}$$

where $c_i \in \mathbf{R}^4$, $i = 1, 2, 3, 4$ and $[c_1, c_2, c_3, c_4] \neq 0$. Then

$$\begin{aligned} x = &(c_1, c_2, c_3, c_4)\ {}^t(e^{u+\sqrt{1+\lambda}v}, e^{u-\sqrt{1+\lambda}v}, \\ &e^{-u}\cos(\sqrt{\lambda - 1}v), e^{-u}\sin(\sqrt{\lambda - 1}v)). \end{aligned} \tag{2.12}$$

The surface is centroaffinely equivalent to Example 2.3.

(2) If $\theta = 0$, then

$$f''(v) = \lambda f(v), \quad g''(v) = -\lambda g(v).$$

(a) When $\lambda = 0$, we have

$$f(v) = c_1 v + c_2, \quad g(v) = c_3 v + c_4,$$

where $c_i \in \mathbf{R}^4$, $i = 1, 2, 3, 4$ and $[c_1, c_2, c_3, c_4] \neq 0$. Then

$$x = (c_1, c_2, c_3, c_4)\ {}^t(ve^u, e^u, ve^{-u}, e^{-u}). \tag{2.13}$$

The surface is centroaffinely equivalent to Example 2.4.

(b) When $\lambda \in (0, +\infty)$, we have

$$f(v) = c_1 e^{\sqrt{\lambda}v} + c_2 e^{-\sqrt{\lambda}v}, \quad g(v) = c_3 \cos(\sqrt{\lambda}v) + c_4 \sin(\sqrt{\lambda}v),$$

where $c_i \in \mathbf{R}^4$, $i = 1, 2, 3, 4$ and $[c_1, c_2, c_3, c_4] \neq 0$. Then

$$x = (c_1, c_2, c_3, c_4)\ {}^t(e^{u+\sqrt{\lambda}v}, e^{u-\sqrt{\lambda}v}, e^{-u}\cos(\sqrt{\lambda}v), e^{-u}\sin(\sqrt{\lambda}v)). \tag{2.14}$$

The surface is centroaffinely equivalent to Example 2.3.

(3) If $\theta < 0$, setting $(u, \sqrt{-\theta}v) \to (u, v)$, we may assume that $\theta = -1$.

Then

$$f''(v) = (\lambda - 1)f(v), \quad g''(v) = -(\lambda + 1)g(v).$$

(a) When $\lambda \in [0, 1)$, we have

$$\begin{aligned} f(v) &= c_1 \cos(\sqrt{1-\lambda}v) + c_2 \sin(\sqrt{1-\lambda}v), \\ g(v) &= c_3 \cos(\sqrt{1+\lambda}v) + c_4 \sin(\sqrt{1+\lambda}v), \end{aligned}$$

where $c_i \in \mathbf{R}^4$, $i = 1, 2, 3, 4$ and $[c_1, c_2, c_3, c_4] \neq 0$. Then by a centroaffine transformation

$$\begin{aligned} x = {}^t(&e^u \cos(\sqrt{1-\lambda}v), e^u \sin(\sqrt{1-\lambda}v), \\ &e^{-u} \cos(\sqrt{1+\lambda}v), e^{-u} \sin(\sqrt{1+\lambda}v)). \end{aligned} \tag{2.15}$$

The surface is centroaffinely equivalent to Example 2.5.

(b) When $\lambda = 1$, we have

$$f(v) = c_1 v + c_2, \quad g(v) = c_3 \cos(\sqrt{2}v) + c_4 \sin(\sqrt{2}v),$$

where $c_i \in \mathbf{R}^4$, $i = 1, 2, 3, 4$ and $[c_1, c_2, c_3, c_4] \neq 0$. Then

$$x = (c_1, c_2, c_3, c_4)\ {}^t(ve^u, e^u, e^{-u} \cos(\sqrt{2}v), e^{-u} \sin(\sqrt{2}v)). \tag{2.16}$$

The surface is centroaffinely equivalent to Example 2.6.

(c) When $\lambda \in (1, +\infty)$, we have

$$\begin{aligned} f(v) &= c_1 e^{\sqrt{\lambda-1}v} + c_2 e^{-\sqrt{\lambda-1}v}, \\ g(v) &= c_3 \cos(\sqrt{\lambda+1}v) + c_4 \sin(\sqrt{\lambda+1}v), \end{aligned}$$

where $c_i \in \mathbf{R}^4$, $i = 1, 2, 3, 4$ and $[c_1, c_2, c_3, c_4] \neq 0$. Then

$$\begin{aligned} x = (c_1, c_2, c_3, c_4)\ {}^t(&e^{u+\sqrt{\lambda-1}v}, e^{u-\sqrt{\lambda-1}v}, \\ &e^{-u} \cos(\sqrt{\lambda+1}v), e^{-u} \sin(\sqrt{\lambda+1}v)). \end{aligned} \tag{2.17}$$

The surface is centroaffinely equivalent to Example 2.3.

(ii) If $\psi = 0$, $x_{uu} = 0$ yields

$$x = f(v)u + g(v).$$

By $x_{vv} = \lambda x_u + \theta x$, we obtain

$$f''(v) = \theta f(v), \quad g''(v) = \lambda f(v) + \theta g(v).$$

In this case, $\theta \neq 0$. Setting $(u, \sqrt{|\theta|}v) \to (u, v)$, we may assume that $\theta = \pm 1$.
(1) When $\theta = 1$, we have

$$f(v) = c_1 e^v + c_2 e^{-v}, \quad g(v) = c_3 e^v + c_4 e^{-v} + c_1 \frac{\lambda}{2} v e^v + c_2 \frac{-\lambda}{2} v e^{-v},$$

where $c_i \in \mathbf{R}^4$, $i = 1, 2, 3, 4$ and $[c_1, c_2, c_3, c_4] \neq 0$. Then

$$x = (c_1, c_2, c_3, c_4)\ {}^t\!\left(ue^v + \frac{\lambda}{2} v e^v, ue^{-v} - \frac{\lambda}{2} v e^{-v}, e^v, e^{-v}\right). \tag{2.18}$$

When $\lambda = 0$ the surface is centroaffinely equivalent to Example 2.4 and when $\lambda > 0$ the surface is centroaffinely equivalent to Example 2.7.
(2) When $\theta = -1$, we have

$$\begin{aligned} f(v) &= c_1 \cos v + c_2 \sin v, \\ g(v) &= c_3 \cos v + c_4 \sin v + c_1 \frac{\lambda}{2} v \sin v + c_2 \frac{-\lambda}{2} v \cos v, \end{aligned}$$

where $c_i \in \mathbf{R}^4$, $i = 1, 2, 3, 4$ and $[c_1, c_2, c_3, c_4] \neq 0$. Then

$$\begin{aligned} x = (c_1, c_2, c_3, c_4)\ {}^t\Big(& u \cos v + \frac{\lambda}{2} v \sin v, u \sin v \\ & - \frac{\lambda}{2} v \cos v, \cos v, \sin v\Big). \end{aligned} \tag{2.19}$$

When $\lambda = 0$ the surface is centroaffinely equivalent to Example 2.9 and when $\lambda > 0$ the surface is centroaffinely equivalent to Example 2.8.
(iii) If $\psi < 0$, setting $(\sqrt{-\psi}u, v) \to (u, v)$, we may assume that $\psi = -1$. From $x_{uu} = -x$, we get

$$x = f(v) \cos u + g(v) \sin u.$$

By $x_{vv} = \lambda x_u + \theta x$, we obtain

$$f''(v) = \lambda g(v) + \theta f(v), \quad g''(v) = -\lambda f(v) + \theta g(v).$$

(1) If $\lambda = 0$, we have

$$f''(v) = \theta f(v), \quad g''(v) = \theta g(v).$$

(a) When $\theta > 0$, setting $(u, \sqrt{\theta}v) \to (u, v)$, we may assume that $\theta = 1$. Hence

$$f(v) = c_1 e^v + c_2 e^{-v}, \quad g(v) = c_3 e^v + c_4 e^{-v},$$

where $c_i \in \mathbf{R}^4$, $i = 1,2,3,4$ and $[c_1, c_2, c_3, c_4] \neq 0$. Then

$$x = (c_1, c_2, c_3, c_4)\ {}^t(e^v \cos u, e^{-v} \cos u, e^v \sin u, e^{-v} \sin u). \tag{2.20}$$

The surface is centroaffinely equivalent to Example 2.5.

(b) When $\theta = 0$, we have

$$f(v) = c_1 v + c_2, \quad g(v) = c_3 v + c_4,$$

where $c_i \in \mathbf{R}^4$, $i = 1,2,3,4$ and $[c_1, c_2, c_3, c_4] \neq 0$. Then

$$x = (c_1, c_2, c_3, c_4)\ {}^t(v \cos u, \cos u, v \sin u, \sin u). \tag{2.21}$$

The surface is centroaffinely equivalent to Example 2.9.

(c) When $\theta < 0$, setting $(u, \sqrt{-\theta} v) \to (u, v)$, we may assume that $\theta = -1$. Hence

$$f(v) = c_1 \cos v + c_2 \sin v, \quad g(v) = c_3 \cos v + c_4 \sin v,$$

where $c_i \in \mathbf{R}^4$, $i = 1,2,3,4$ and $[c_1, c_2, c_3, c_4] \neq 0$. Then

$$x = (c_1, c_2, c_3, c_4)\ {}^t(\cos u \cos v, \cos u \sin v, \sin u \cos v, \sin u \sin v). \tag{2.22}$$

The surface is centroaffinely equivalent to Example 2.10.

(2) If $\lambda > 0$, let $\alpha^2 - \beta^2 = \theta$ and $2\alpha\beta = \lambda$, we have

$$\begin{aligned} f(v) &= c_1 e^{\alpha v} \cos(\beta v) + c_2 e^{\alpha v} \sin(\beta v) \\ &\qquad + c_3 e^{-\alpha v} \cos(\beta v) + c_4 e^{-\alpha v} \sin(\beta v), \\ g(v) &= -c_1 e^{\alpha v} \sin(\beta v) + c_2 e^{\alpha v} \cos(\beta v) \\ &\qquad + c_3 e^{-\alpha v} \sin(\beta v) - c_4 e^{-\alpha v} \cos(\beta v), \end{aligned}$$

where $c_i \in \mathbf{R}^4$, $i = 1,2,3,4$ and $[c_1, c_2, c_3, c_4] \neq 0$. Then by a centroaffine transformation

$$\begin{aligned} x = {}^t(&e^{\alpha v} \cos(u + \beta v), e^{\alpha v} \sin(u + \beta v), \\ &e^{-\alpha v} \cos(-u + \beta v), e^{-\alpha v} \sin(-u + \beta v)). \end{aligned} \tag{2.23}$$

The surface is centroaffinely equivalent to Example 2.11.

Case (B): $\psi_u \neq 0$, $\psi_v = 0$.

(I) In this case, we have $\varphi = \lambda = \theta = 0$, $\psi(u) = \epsilon \frac{4}{b^2} u^{-2}$. Thus (1.15) becomes

$$\begin{cases} x_{uu} = \epsilon \dfrac{4}{b^2} u^{-2} x \\ x_{vv} = 0. \end{cases} \tag{2.24}$$

$x_{vv} = 0$ yields

$$x = f(u)v + g(u).$$

From $x_{uu} = \epsilon \frac{4}{b^2} u^{-2} x$ we get

$$\begin{cases} f''(u) = \epsilon \dfrac{4}{b^2} u^{-2} f(u) \\ g''(u) = \epsilon \dfrac{4}{b^2} u^{-2} g(u). \end{cases}$$

(1) If $\epsilon = 1$, we have

$$\begin{cases} f(u) = c_1 u^{\frac{1+\sqrt{1+\frac{16}{b^2}}}{2}} + c_2 u^{\frac{1-\sqrt{1+\frac{16}{b^2}}}{2}} \\ g(u) = c_3 u^{\frac{1+\sqrt{1+\frac{16}{b^2}}}{2}} + c_4 u^{\frac{1-\sqrt{1+\frac{16}{b^2}}}{2}}, \end{cases}$$

where $c_i \in \mathbf{R}^4$, $i = 1, 2, 3, 4$ and $[c_1, c_2, c_3, c_4] \neq 0$. Then

$$x = (c_1, c_2, c_3, c_4)\ {}^t\!\left(vu^{\frac{1+\sqrt{1+\frac{16}{b^2}}}{2}}, vu^{\frac{1-\sqrt{1+\frac{16}{b^2}}}{2}}, u^{\frac{1+\sqrt{1+\frac{16}{b^2}}}{2}}, u^{\frac{1-\sqrt{1+\frac{16}{b^2}}}{2}}\right). \tag{2.25}$$

The surface is centroaffinely equivalent to Example 2.12.

(2) If $\epsilon = -1$, we have

$$\begin{cases} f''(u) = -\dfrac{4}{b^2} u^{-2} f(u) \\ g''(u) = -\dfrac{4}{b^2} u^{-2} g(u). \end{cases}$$

(i) When $b^2 > 4^2$,

$$\begin{cases} f(u) = c_1 u^{\frac{1+\sqrt{1-\frac{16}{b^2}}}{2}} + c_2 u^{\frac{1-\sqrt{1-\frac{16}{b^2}}}{2}} \\ g(u) = c_3 u^{\frac{1+\sqrt{1-\frac{16}{b^2}}}{2}} + c_4 u^{\frac{1-\sqrt{1-\frac{16}{b^2}}}{2}}, \end{cases}$$

where $c_i \in \mathbf{R}^4$, $i = 1, 2, 3, 4$ and $[c_1, c_2, c_3, c_4] \neq 0$. Then

$$x = (c_1, c_2, c_3, c_4)\ {}^t(vu^{\frac{1+\sqrt{1-\frac{16}{b^2}}}{2}}, vu^{\frac{1-\sqrt{1-\frac{16}{b^2}}}{2}}, u^{\frac{1+\sqrt{1-\frac{16}{b^2}}}{2}}, u^{\frac{1-\sqrt{1-\frac{16}{b^2}}}{2}}). \tag{2.26}$$

The surface is centroaffinely equivalent to Example 2.12.
(ii) When $b^2 = 4^2$,

$$\begin{cases} f(u) = c_1\sqrt{u}\log u + c_2\sqrt{u} \\ g(u) = c_3\sqrt{u}\log u + c_4\sqrt{u}, \end{cases}$$

where $c_i \in \mathbf{R}^4$, $i = 1, 2, 3, 4$ and $[c_1, c_2, c_3, c_4] \neq 0$. Then

$$x = (c_1, c_2, c_3, c_4)\ {}^t(v\sqrt{u}\log u, v\sqrt{u}, \sqrt{u}\log u, \sqrt{u}). \tag{2.27}$$

The surface is centroaffinely equivalent to Example 2.13.
(iii) When $b^2 < 4^2$,

$$\begin{cases} f(u) = c_1\sqrt{u}\cos\dfrac{\sqrt{\frac{16}{b^2}-1}}{2}\log u + c_2\sqrt{u}\sin\dfrac{\sqrt{\frac{16}{b^2}-1}}{2}\log u \\ g(u) = c_3\sqrt{u}\cos\dfrac{\sqrt{\frac{16}{b^2}-1}}{2}\log u + c_4\sqrt{u}\sin\dfrac{\sqrt{\frac{16}{b^2}-1}}{2}\log u, \end{cases}$$

where $c_i \in \mathbf{R}^4$, $i = 1, 2, 3, 4$ and $[c_1, c_2, c_3, c_4] \neq 0$. Then by a centroaffine transformation

$$\begin{aligned} x = {}^t\Big(& v\sqrt{u}\cos\frac{\sqrt{\frac{16}{b^2}-1}}{2}\log u, v\sqrt{u}\sin\frac{\sqrt{\frac{16}{b^2}-1}}{2}\log u, \\ & \sqrt{u}\cos\frac{\sqrt{\frac{16}{b^2}-1}}{2}\log u, \sqrt{u}\sin\frac{\sqrt{\frac{16}{b^2}-1}}{2}\log u\Big). \end{aligned} \tag{2.28}$$

The surface is centroaffinely equivalent to Example 2.14.
(II) In this case, we have $\lambda = \theta = 0$, $\psi(u) = \epsilon\frac{4}{b^2}u^{-2}$, $\varphi(u) = \frac{2^3c}{b^3}u^{-3}$. We write $\frac{2^3c}{b^3}$ also by c. Thus (1.15) becomes

$$\begin{cases} x_{uu} = cu^{-3}x_v + \epsilon\dfrac{4}{b^2}u^{-2}x \\ x_{vv} = 0. \end{cases}$$

$x_{vv} = 0$ yields

$$x = f(u)v + g(u).$$

From $x_{uu} = cu^{-3}x_v + \epsilon\frac{4}{b^2}u^{-2}x$, we get

$$\begin{cases} f''(u) = \epsilon\dfrac{4}{b^2}u^{-2}f(u) \\ g''(u) = cu^{-3}f(u) + \epsilon\dfrac{4}{b^2}u^{-2}g(u). \end{cases}$$

(1) If $\epsilon = 1$, we have

$$\begin{cases} f(u) = c_1 u^{\frac{1+\sqrt{1+\frac{16}{b^2}}}{2}} + c_2 u^{\frac{1-\sqrt{1+\frac{16}{b^2}}}{2}} \\ g(u) = c_3 u^{\frac{1+\sqrt{1+\frac{16}{b^2}}}{2}} + c_4 u^{\frac{1-\sqrt{1+\frac{16}{b^2}}}{2}} \\ \qquad\qquad + c_1 \alpha u^{-\frac{1-\sqrt{1+\frac{16}{b^2}}}{2}} + c_2 \beta u^{-\frac{1+\sqrt{1+\frac{16}{b^2}}}{2}}, \end{cases}$$

where $c_i \in \mathbf{R}^4$, $i = 1, 2, 3, 4$, $[c_1, c_2, c_3, c_4] \neq 0$,

$$\alpha = \frac{c}{\left(\frac{1-\sqrt{1+\frac{16}{b^2}}}{2}\right)\left(\frac{1-\sqrt{1+\frac{16}{b^2}}}{2} + 1\right) - \frac{4}{b^2}},$$

$$\beta = \frac{c}{\left(\frac{1+\sqrt{1+\frac{16}{b^2}}}{2}\right)\left(\frac{1+\sqrt{1+\frac{16}{b^2}}}{2} + 1\right) - \frac{4}{b^2}}.$$

Then by a centroaffine transformation

$$x = {}^t\Big(vu^{\frac{1+\sqrt{1+\frac{16}{b^2}}}{2}} + \alpha u^{-\frac{1-\sqrt{1+\frac{16}{b^2}}}{2}}, vu^{\frac{1-\sqrt{1+\frac{16}{b^2}}}{2}} q$$
$$+ \beta u^{-\frac{1+\sqrt{1+\frac{16}{b^2}}}{2}}, u^{\frac{1+\sqrt{1+\frac{16}{b^2}}}{2}}, u^{\frac{1-\sqrt{1+\frac{16}{b^2}}}{2}}\Big). \tag{2.29}$$

The surface is centroaffinely equivalent to Example 2.12.

(2) If $\epsilon = -1$, we have

$$\begin{cases} f''(u) = -\dfrac{4}{b^2}u^{-2}f(u) \\ g''(u) = cu^{-3}f(u) - \dfrac{4}{b^2}u^{-2}g(u). \end{cases}$$

(i) When $b^2 > 4^2$,

$$\begin{cases} f(u) = c_1 u^{\frac{1+\sqrt{1-\frac{16}{b^2}}}{2}} + c_2 u^{\frac{1-\sqrt{1-\frac{16}{b^2}}}{2}} \\ g(u) = c_3 u^{\frac{1+\sqrt{1-\frac{16}{b^2}}}{2}} + c_4 u^{\frac{1-\sqrt{1-\frac{16}{b^2}}}{2}} \\ \qquad\qquad + c_1 \alpha u^{-\frac{1-\sqrt{1-\frac{16}{b^2}}}{2}} + c_2 \beta u^{-\frac{1+\sqrt{1-\frac{16}{b^2}}}{2}}, \end{cases}$$

where $c_i \in \mathbf{R}^4$, $i = 1, 2, 3, 4$, $[c_1, c_2, c_3, c_4] \neq 0$,

$$\alpha = \frac{c}{\left(\frac{1-\sqrt{1-\frac{16}{b^2}}}{2}\right)\left(\frac{1-\sqrt{1-\frac{16}{b^2}}}{2}+1\right)+\frac{4}{b^2}},$$

$$\beta = \frac{c}{\left(\frac{1+\sqrt{1-\frac{16}{b^2}}}{2}\right)\left(\frac{1+\sqrt{1-\frac{16}{b^2}}}{2}+1\right)+\frac{4}{b^2}}.$$

Then by a centroaffine transformation

$$\begin{aligned} x = {}^t\Bigg(& vu^{\frac{1+\sqrt{1-\frac{16}{b^2}}}{2}} + \alpha u^{-\frac{1-\sqrt{1-\frac{16}{b^2}}}{2}}, vu^{\frac{1-\sqrt{1-\frac{16}{b^2}}}{2}} \\ & + \beta u^{-\frac{1+\sqrt{1-\frac{16}{b^2}}}{2}}, u^{\frac{1+\sqrt{1-\frac{16}{b^2}}}{2}}, u^{\frac{1-\sqrt{1-\frac{16}{b^2}}}{2}}\Bigg). \end{aligned} \tag{2.30}$$

The surface is centroaffinely equivalent to Example 2.12.

(ii) When $b^2 = 4^2$,

$$\begin{cases} f(u) = c_1\sqrt{u}\log u + c_2\sqrt{u} \\ g(u) = c_3\sqrt{u}\log u + c_4\sqrt{u} + c_1 c\frac{1}{\sqrt{u}}(2+\log u) + c_2 c\frac{1}{\sqrt{u}}, \end{cases}$$

where $c_i \in \mathbf{R}^4$, $i = 1, 2, 3, 4$ and $[c_1, c_2, c_3, c_4] \neq 0$. Then

$$\begin{aligned} x = (c_1, c_2, c_3, c_4)\, {}^t\Big(& v\sqrt{u}\log u + c\frac{2+\log u}{\sqrt{u}}, v\sqrt{u} \\ & + c\frac{1}{\sqrt{u}}, \sqrt{u}\log u, \sqrt{u}\Big). \end{aligned} \tag{2.31}$$

The surface is centroaffinely equivalent to Example 2.13.

(iii) When $b^2 < 4^2$,

$$\begin{cases} f(u) &= c_1\sqrt{u}\cos\frac{\sqrt{\frac{16}{b^2}-1}}{2}\log u + c_2\sqrt{u}\sin\frac{\sqrt{\frac{16}{b^2}-1}}{2}\log u \\ g(u) &= c_3\sqrt{u}\cos\frac{\sqrt{\frac{16}{b^2}-1}}{2}\log u + c_4\sqrt{u}\sin\frac{\sqrt{\frac{16}{b^2}-1}}{2}\log u \\ &\quad + c_1\frac{1}{\sqrt{u}}\Big(\beta\cos\frac{\sqrt{\frac{16}{b^2}-1}}{2}\log u - \alpha\sin\frac{\sqrt{\frac{16}{b^2}-1}}{2}\log u\Big) \\ &\quad + c_2\frac{1}{\sqrt{u}}\Big(\alpha\cos\frac{\sqrt{\frac{16}{b^2}-1}}{2}\log u + \beta\sin\frac{\sqrt{\frac{16}{b^2}-1}}{2}\log u\Big), \end{cases}$$

where $c_i \in \mathbf{R}^4$, $i = 1,2,3,4$, $[c_1, c_2, c_3, c_4] \neq 0$, $\alpha = \frac{b^2c}{16}\sqrt{\frac{16}{b^2}-1}$, $\beta = \frac{b^2c}{16}$. Then by a centroaffine transformation

$$x = {}^t\Big(v\sqrt{u}\cos\frac{\sqrt{\frac{16}{b^2}-1}}{2}\log u + \frac{\beta\cos\frac{\sqrt{\frac{16}{b^2}-1}}{2}\log u - \alpha\sin\frac{\sqrt{\frac{16}{b^2}-1}}{2}\log u}{\sqrt{u}},$$
$$v\sqrt{u}\sin\frac{\sqrt{\frac{16}{b^2}-1}}{2}\log u + \frac{\alpha\cos\frac{\sqrt{\frac{16}{b^2}-1}}{2}\log u + \beta\sin\frac{\sqrt{\frac{16}{b^2}-1}}{2}\log u}{\sqrt{u}},$$
$$\sqrt{u}\cos\frac{\sqrt{\frac{16}{b^2}-1}}{2}\log u, \sqrt{u}\sin\frac{\sqrt{\frac{16}{b^2}-1}}{2}\log u\Big). \tag{2.32}$$

The surface is centroaffinely equivalent to Example 2.14.

(III) In this case, we have $\varphi = 0$, $\psi(u) = \frac{3}{4}u^{-2}$, $\lambda(u) = (\frac{4}{3})^{\frac{3}{2}}du^3$, $\theta(u) = \frac{4}{3}eu^2$, $e \pm \sqrt{3}d = 0$. Thus (1.15) becomes

$$\begin{cases} x_{uu} = \frac{3}{4}u^{-2}x \\ x_{vv} = \left(\frac{4}{3}\right)^{\frac{3}{2}} du^3 x_u + \frac{4}{3}eu^2 x, \end{cases}$$

that is

$$\begin{cases} x_{uu} = \frac{3}{4}u^{-2}x \\ x_{vv} = \pm\frac{8}{9}eu^3 x_u + \frac{4}{3}eu^2 x. \end{cases} \tag{2.33}$$

$x_{uu} = \frac{3}{4}u^{-2}x$ yields

$$x = f(v)u^{\frac{3}{2}} + g(v)u^{-\frac{1}{2}}.$$

From $x_{vv} = (\frac{4}{3})^{\frac{3}{2}}du^3x_u + \frac{4}{3}eu^2x$ and $f(v) \not\equiv 0$ we get

$$\begin{cases} f''(v) = \dfrac{4}{3}\left(e - \dfrac{d}{\sqrt{3}}\right)g(v) \\ g''(v) = 0. \end{cases}$$

Therefore

$$\begin{cases} f(v) = c_1\left(\dfrac{1}{6}kv^3\right) + c_2\left(\dfrac{1}{2}kv^2\right) + c_3v + c_4 \\ g(v) = c_1v + c_2, \end{cases}$$

where $c_i \in \mathbf{R}^4$, $i = 1, 2, 3, 4$, $[c_1, c_2, c_3, c_4] \neq 0$, $k = \frac{4}{3}(e - \frac{1}{\sqrt{3}}d)$. Then

$$x = (c_1, c_2, c_3, c_4)\ {}^t(vu^{-\frac{1}{2}} + \frac{1}{6}kv^3u^{\frac{3}{2}}, u^{-\frac{1}{2}} + \frac{1}{2}kv^2u^{\frac{3}{2}}, vu^{\frac{3}{2}}, u^{\frac{3}{2}}). \quad (2.34)$$

The surface is centroaffinely equivalent to Example 2.15.

Case (C): $\psi_u = 0,\ \psi_v \neq 0.$
In this case, we have $\lambda = 0$, $\psi(v) = \epsilon\frac{a^2}{4}v^2$, $\varphi(v) = \frac{a^3c}{8}v^3$, $\theta(v) = \epsilon\frac{4e}{a^2}v^{-2}$, $3ac = -4$, $16e = 3\epsilon a^2$. Thus (1.15) becomes

$$\begin{cases} x_{uu} = \dfrac{a^3c}{8}v^3x_v + \epsilon\dfrac{a^2}{4}v^2x \\ x_{vv} = \epsilon\dfrac{4e}{a^2}v^{-2}x, \end{cases}$$

that is

$$\begin{cases} x_{uu} = -\dfrac{8\epsilon}{9}ev^3x_v + \dfrac{4}{3}ev^2x \\ x_{vv} = \dfrac{3}{4}v^{-2}x. \end{cases} \quad (2.35)$$

Setting $(u, v) \to (v, u)$, the equation (2.35) becomes the equation (2.33). This completes the proof of Theorem 2.1 .

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