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Currents invariant by a Kleinian group

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Abstract. The goal of this paper is to give, under some hypotheses, a characterization of currents and distributions invariant by a group of diffeomorphisms of a manifold M and especially in the case of a Kleinian group Γ acting on the n-sphere $\mathbf{S}^n$.

Key words: current, distribution, Kleinian group, Poincaré exponent, bigraded cohomology.

0. Introduction

Let $p \in \mathbf{N}$ and $\Omega^p(M)$ be the space of differential forms of degree p with compact support in M equipped with its usual C^∞-topology. An element T of the (topological) dual $\mathcal{C}_p(M)$ of $\Omega^p(M)$ is called a *current of degree* p and a *distribution* when $p = 0$. An element $T \in \mathcal{C}_p(M)$ is said to be *invariant* (or γ-*invariant*) under the action of a diffeomorphism $\gamma : M \longrightarrow M$ if it satisfies $\langle T, \gamma^*\varphi\rangle = \langle T, \varphi\rangle$ for every $\varphi \in \Omega^p(M)$ or if it vanishes on the space $K^p = \{\varphi - \gamma^*\varphi \ : \ \varphi \in \Omega^p(M)\}$. So the space $\mathcal{C}_p^\Gamma(M)$ (where Γ is the cyclic group generated by γ) of invariant currents on M is canonically isomorphic to the (topological) dual of the quotient $\Omega^p(M)/K^p$. More generally if Γ is a group of diffeomorphisms of M we say that $T \in \mathcal{C}_p(M)$ is Γ-*invariant* if it is invariant by every element $\gamma \in \Gamma$.

In [Ha], Haefliger characterized foliations with minimal leaves in terms of currents invariant by pseudogroups. Thus if the foliation is a suspension with holonomy group Γ, then the interest is focused upon Γ-invariant currents. The case of a Fuchsian group was studied in [HL]: let Γ be a subgroup of the diffeomorphism group Diff($\mathbf{S}^1$) of the circle $\mathbf{S}^1$ whose elements are restriction of elements of a Fuchsian group G of diffeomorphisms of the unit disc $\mathbf{D}$. Suppose that the quotient Riemannian surface $S = G \setminus \mathbf{D}$ is of finite volume, of genus g and with k punctures. Then it was proved in [HL] that *the space of* Γ*-invariant distributions on the circle* $\mathbf{S}^1$ *which vanish on constant functions is isomorphic to the space of harmonic forms on* S *having at most poles of order one at the punctures* x_i*. Its dimension*

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is $\max(2g, 2g+2k-2)$.

Other results in higher dimension can be found in [Ga]. Invariant currents by a locally free action of the affine group GA on a compact 3-manifold with a solvable fundamental group were completely characterized in [Ek].

In this paper we study currents, especially distributions, invariant by Kleinian groups. Distribution is a concept generalizing that of measure. It is well known, easy to prove, that nonelementary Kleinian groups do not admit invariant measure. So a natural question is: Does there exist an invariant distribution? We shall show in Proposition 3.1 that Kleinian group of certain kind admits an invariant distribution.

First of all let Γ be the cyclic group generated by a loxodromic transformation $\gamma : \mathbf{S}^n \longrightarrow \mathbf{S}^n$ and $D = \mathbf{S}^n - \{a_+, a_-\}$ where a_+ and a_- are respectively the repeller and the attractor of γ. The group Γ acts on D properly discontinuously and the quotient $\Gamma \setminus D$ is analytically diffeomorphic to $\mathbf{S}^1 \times \mathbf{S}^{n-1}$. We have the following exact sequence

$$0 \longrightarrow \mathcal{C}_0^\Gamma(\mathbf{S}^n, \{a_+, a_-\}) \xrightarrow{\ i\ } \mathcal{C}_0^\Gamma(\mathbf{S}^n) \xrightarrow{L_0} \mathcal{C}_0^\Gamma(D)$$

where $\mathcal{C}_0^\Gamma(\mathbf{S}^n, \{a_+, a_-\})$ denotes the space of Γ-invariant distributions on $\mathbf{S}^n$ with support contained in $\{a_+, a_-\}$ and L_0 is the *localization map* i.e. L_0 associates to every distribution on $\mathbf{S}^n$ its restriction to D. The question is if L_0 is surjective or not.

In §3, Image(L_0) is shown to be a codimension one subspace of $\mathcal{C}_0^\Gamma(D)$. This determines completely the space $\mathcal{C}_0^\Gamma(\mathbf{S}^n)$. In §4 we construct a cross section of the localization map L_0.

Now we consider the problem in further generality. Let Γ be a Kleinian group acting on $\mathbf{S}^n$ and let $D_\Gamma = \mathbf{S}^n - \Lambda_\Gamma$ be the domain of discontinuity of Γ and consider the exact sequence for p-currents

$$0 \longrightarrow \mathcal{C}_p^\Gamma(\mathbf{S}^n, \Lambda_\Gamma) \xrightarrow{\ i\ } \mathcal{C}_p^\Gamma(\mathbf{S}^n) \xrightarrow{L_p} \mathcal{C}_p^\Gamma(D_\Gamma).$$

Here Λ_Γ is the *limit set* of Γ. For $p = 0$, it is very difficult to determine Image(L_p) in general. But for $p > \delta$ (where δ is the *critical exponent* of Γ), we show in §2 that L_p is surjective. Using this for certain groups, we show that for $p = 0$, Image(L_0) is a subspace of $\mathcal{C}_0^\Gamma(D_\Gamma)$ of codimension ≤ 1.

Also if Γ acts on D_Γ freely and properly discontinuously, we show that $\mathcal{C}_p^\Gamma(D_\Gamma)$ is isomorphic to $\mathcal{C}_p(\Gamma \setminus D_\Gamma)$. This is carried out in §1 in complete generality. This result also can be derived from Haefliger's paper [Ha] where he has studied currents invariant by a pseudo-group. However we shall give a slightly different proof, since some concepts there play a crucial role in later developments.

In Section 5 we study weakly invariant distributions i.e. distributions with invariance lack localized in the limit set Λ_Γ. In §6 we use the preceding results for computing the first bigraded cohomology group of the foliation obtained by suspending a diffeomorphism group Γ.

Unless otherwise stated all the objects considered are assumed to be of class C^∞.

1. Covering space

Let M, X be C^∞-manifolds, Γ a discrete group and $\Gamma \longrightarrow M \xrightarrow{\pi} X$ a regular covering. The aim of this § is to show that, for every $p \in \mathbf{N}$, the space $\mathcal{C}_p^\Gamma(M)$ of Γ-invariant p-currents is canonically isomorphic to the space $\mathcal{C}_p(X)$ of the usual p-currents on the quotient manifold $X = \Gamma \setminus M$.

1.1. Preliminary

Let $\mathbf{j} = (j_1, \ldots, j_p) \in \mathbf{N}^p$ be a multi-index such that $1 \leq j_1 < \cdots < j_p \leq n$. Choose a local chart $\{U, (x_1, \ldots, x_n)\}$ of M. Then every element $\omega \in \Omega^p(M)$ has a local expression

$$\omega = \sum_{\mathbf{j}} \omega_{\mathbf{j}} dx_{j_1} \wedge \cdots \wedge dx_{j_p}$$

where $\omega_{\mathbf{j}}$ are C^∞ functions on U. Let $(U_i)_{i \in I}$ be a locally finite cover of M by charts U_i. We define the k-norm $||\omega||_k$ of ω by

$$||\omega||_k = \max_{i \in I} \left\{ \max_{|\mathbf{s}| \leq k} \left(\sum_{\mathbf{j}} \sup_{x \in U_i} \left| \frac{\partial^{|\mathbf{s}|} \omega_{\mathbf{j}}}{\partial x_1^{s_1} \cdots \partial x_n^{s_n}} (x) \right| \right) \right\}$$

where $\mathbf{s} = (s_1, \ldots, s_n) \in \mathbf{N}^n$ and $|\mathbf{s}| = s_1 + \cdots + s_n$. This number exists because ω has a compact support.

The next Lemma will be useful mainly in a later §. Endow $\Omega^p(M)$ with the usual C^∞-topology. That is, $\omega_n \longrightarrow \omega$ if and only if supp(ω_n) is contained in a fixed compact subset and all the derivatives of ω_n converge to the corresponding derivatives of ω uniformly on this subset.

Lemma 1.2 *A linear form $T : \Omega^p(M) \longrightarrow \mathbf{C}$ is continuous if and only if for every compact set $A \subset M$ there exists a positive constant C, an integer*

$k \in \mathbf{N}$ such that

$$|\langle T, \omega\rangle| \leq C||\omega||_k$$

for every $\omega \in \Omega^p(M)$ with support contained in A.

The proof of this lemma is obvious.

Now let $\overline{\Omega}^p(M)$ be the space of all $\mathbf{C}$-valued p-forms on M (not necessarily compactly supported) and $\overline{\Omega}^p_\Gamma(M)$ the subspace of $\overline{\Omega}^p(M)$ whose elements ω are Γ-invariant and such that the quotient $\Gamma \setminus \mathrm{supp}(\omega)$ is compact in X. Then we have obviously the following:

Proposition 1.3 *$\pi^* : \Omega^p(X) \longrightarrow \overline{\Omega}^p(M)$ is a bijection onto $\overline{\Omega}^p_\Gamma(M)$.*

Lemma 1.4 *There exists a positive C^∞-function $f : M \longrightarrow \mathbf{R}$ such that*

i) *for every compact $B \subset X$, $\mathrm{supp}(f) \cap \pi^{-1}(B)$ is compact; or equivalently for every compact $A \subset M$, $\mathrm{supp}(f) \cap \gamma A \neq \emptyset$ for but finitely many $\gamma \in \Gamma$.*

ii) $\sum_{\gamma\in\Gamma} f \circ \gamma = 1$.

Proof. Let $(U_i)_{i\in I}$ be a locally finite cover of X by relatively compact open sets U_i which are evenly covered by π. Let V_i any lift of U_i; then the family $(V_i)_{i\in I}$ is locally finite but it is not a covering of M. Let $g_i : M \longrightarrow \mathbf{R}_+$ be a C^∞-function such that

$$g_i > 0 \text{ on } V_i \text{ and } g_i = 0 \text{ outside a neighbourhood of } V_i.$$

Clearly the function $g = \sum_{i\in I} g_i$ satisfies i). Hence for every compact $A \subset M$ we have

$$\mathrm{supp}(g \circ \gamma) \cap A \neq \emptyset \text{ for but finitely many } \gamma \in \Gamma.$$

Thus

$$\sum_{\gamma\in\Gamma} g \circ \gamma$$

is a well defined positive C^∞-function. Put

$$f = \frac{g}{\sum_{\gamma\in\Gamma} g \circ \gamma}.$$

It is clear that f satisfies the conditions of Lemma 1.4. □

Given $\omega \in \Omega^p(M)$, let

$$\overline{\omega} = \sum_{\gamma \in \Gamma} \gamma^* \omega \in \overline{\Omega}^p(M).$$

It is easy to show that $\overline{\omega}$ is Γ-invariant and that $\Gamma \setminus \mathrm{supp}(\overline{\omega}) = \pi(\mathrm{supp}(\omega))$ is compact. That is $\overline{\omega} \in \overline{\Omega}^p_\Gamma(M)$. By 1.3 one can define a map

$$\pi_! : \Omega^p(M) \longrightarrow \Omega^p(X)$$

by the condition

$$\pi^*(\pi_!(\omega)) = \sum_{\gamma \in \Gamma} \gamma^* \omega.$$

Lemma 1.5 *The map $\pi_!$ is linear, continuous and surjective.*

Proof. The fact that $\pi_!$ is linear and continuous is obvious. We shall prove that it is surjective. Let $\eta \in \Omega^p(X)$ and put $\omega = f \cdot \pi^*\eta$. Then $\mathrm{supp}(\omega) = \mathrm{supp}(f) \cap \pi^{-1}(\mathrm{supp}(\eta))$ is compact. Also

$$\begin{aligned}\pi^*(\pi_!(\omega)) &= \sum_{\gamma \in \Gamma} (f \circ \gamma) \cdot \gamma^* \pi^* \eta \\ &= \sum_{\gamma \in \Gamma} (f \circ \gamma) \cdot \pi^* \eta \\ &= \pi^* \eta\end{aligned}$$

That is $\pi_!(\omega) = \eta$. □

Let $p \in \mathbf{N}$; in the introduction we have defined K^p to be the linear subspace of $\Omega^p(M)$

$$K^p = \left\{ \sum_{i=1}^{n} (\gamma_i^* \omega_i - \omega_i) \mid \gamma_i \in \Gamma,\ \omega_i \in \Omega^p(M) \right\}.$$

Then we have the following:

Proposition 1.6 *The sequence*

$$0 \longrightarrow K^p \longrightarrow \Omega^p(M) \xrightarrow{\pi_!} \Omega^p(X) \longrightarrow 0$$

is exact for every $p \in \mathbf{N}$.

Proof. The inclusion $K^p \subset \mathrm{Ker}(\pi_!)$ is clear; all that need proof is $\mathrm{Ker}(\pi_!) \subset K^p$. The proof of this fact was communicated to us by G. Hector.

Choose an arbitrary element $\omega \in \mathrm{Ker}(\pi_!)$. Define $O(\omega)$ to be the set of the points $x \in X$ such that ω vanishes all over $\pi^{-1}(x)$. Let U and V be connected open subsets of X such that $\overline{U} \subset V$ and V is evenly covered by π. Then we will have the following: □

Lemma 1.7 *For any ω, there exists $\omega_1 \in \mathrm{Ker}(\pi_!)$ such that $\omega_1 \equiv \omega$ mod K^p and $O(\omega) \cup U \subset O(\omega_1)$.*

This Lemma is sufficient for the proof of Proposition 1.6. For, one can choose finite families $\{U_i\}$ and $\{V_i\}$ $(i = 1, \ldots, k)$ of open subsets of X covering $\pi(\mathrm{supp}(\omega))$ such that $\overline{U_i} \subset V_i$ and V_i is evenly covered by π. But then using 1.7 successively, we will get a sequence of p-forms

$$\omega \equiv \omega_1 \equiv \omega_2 \equiv \cdots \equiv \omega_k = 0 \bmod K^p,$$

showing Proposition 1.6.

Proof of 1.7 Let g be a nonnegative valued C^∞-function on X such that $g = 1$ on U and $g = 0$ outside V, and $\overline{g} = g \circ \pi$. Let $\overline{U}$ (resp. $\overline{V}$) be a connected component of $\pi^{-1}(U)$ (resp. $\pi^{-1}(V)$) $(\overline{U} \subset \overline{V})$ and let γ_j $(0 \leq j \leq l)$ be the elements of Γ such that $\gamma_j(\overline{V}) \cap \mathrm{supp}(\omega) \neq \emptyset$. Let η_j be the restriction of $\overline{g}\omega$ to $\gamma_j(\overline{V})$. Then we have

$$\omega = \sum_{j=0}^{l} \eta_j + (1 - \overline{g})\omega.$$

Of course each term above is a C^∞-form. Now define

$$\omega_1 = \sum_{j=0}^{l} \gamma_j^* \eta_j + (1 - \overline{g})\omega.$$

Notice that $\omega_1 \equiv \omega$ mod K^p. Also it follows immediately that $O(\omega) \subset O(\omega_1)$.

Let us show finally that $U \subset O(\omega_1)$. Let x be an arbitrary point of U. Then $(1 - \overline{g})\omega$ clearly vanishes on $\pi^{-1}(x)$. Also since $\mathrm{supp}(\gamma_j^* \eta_j) \subset \overline{V}$, we have that ω_1 vanishes on $\pi^{-1}(x)$ except at one point in $\pi^{-1}(x) \cap \overline{V}$. But actually ω_1 also vanishes there since $\omega_1 \in \mathrm{Ker}(\pi_!)$. Therefore we have $x \in O(\omega_1)$. □

Since $\mathcal{C}_p^\Gamma(M)$ is canonically isomorphic to the dual space of the quotient $\Omega^p(M)/K^p$, from Proposition 1.6 we get easily the following:

Theorem 1.8 *The space $\mathcal{C}_p^\Gamma(M)$ of Γ-invariant p-currents on M is canonically isomorphic to the space $\mathcal{C}_p(X)$ of p-currents on X. The isomorphism is given by the transpose of $\pi_!$.*

2. Kleinian groups

Let $\mathbf{S}^n$ and $\mathbf{D}^{n+1}$ denote respectively the unit sphere and the unit disc of the Euclidean space $\mathbf{R}^{n+1}$:

$$\mathbf{S}^n = \{x \in \mathbf{R}^{n+1} \mid |x| = 1\} \text{ and } \mathbf{D}^{n+1} = \{x \in \mathbf{R}^{n+1} \mid |x| < 1\}.$$

We denote by

$$dm^2 = \frac{\sum_{i=1}^{n+1} dx_i^2}{(1-|x|^2)^2}$$

the Lobatchevski metric on $\mathbf{D}^{n+1}$. Let $\mathrm{Iso}^+(\mathbf{D}^{n+1})$ and $\mathrm{Conf}^+(\mathbf{S}^n)$ be respectively the group of orientation preserving isometries of $\mathbf{D}^{n+1}$ and the group of the Möbius (or conformal) transformations of $\mathbf{S}^n$. It is well known that

$$\mathrm{Conf}^+(\mathbf{S}^n) = \mathrm{Iso}^+(\mathbf{D}^{n+1}) = \mathrm{SO}(n+1,1)_0.$$

If Γ is a discrete subgroup of $\mathrm{Conf}^+(\mathbf{S}^n)$ the set

$$\Lambda_\Gamma = \overline{\Gamma \cdot a} \cap \mathbf{S}^n$$

is independent of the choice of the point $a \in \mathbf{D}^{n+1}$. It is called the *limit set* of Γ. Its complement $D_\Gamma = \mathbf{S}^n - \Lambda_\Gamma$ is called the *domain of discontinuity* of Γ. Now for fixed $z \in \mathbf{D}^{n+1}$ and $s > 0$

$$\Phi_s(z) = \sum_{\gamma\in\Gamma} |\gamma'(z)|^s$$

(where γ' is the derivative of γ) is called the *absolute Poincaré series* of Γ. If it converges for one point $z \in \mathbf{D}^{n+1}$, it converges for all and uniformly on compact subsets. The number

$$\delta(\Gamma) = \inf\{s > 0 \;:\; \Phi_s(z) \text{ converges for} z \in \mathbf{D}^{n+1}\}$$

is called the *critical exponent* of Γ.

As before we put

$$\mathcal{C}_p^\Gamma(\mathbf{S}^n) = \{\Gamma\text{-invariant } p\text{-currents on } \mathbf{S}^n\}$$

$$\mathcal{C}_p^\Gamma(\mathbf{S}^n, \Lambda_\Gamma) = \{T \in \mathcal{C}_p^\Gamma(\mathbf{S}^n) \mid \mathrm{supp}(T) \subset \Lambda_\Gamma\}.$$

Then there is an exact sequence

$$0 \longrightarrow \mathcal{C}_p^\Gamma(\mathbf{S}^n, \Lambda_\Gamma) \longrightarrow \mathcal{C}_p^\Gamma(\mathbf{S}^n) \xrightarrow{L_p} \mathcal{C}_p^\Gamma(D_\Gamma)$$

where L_p is the localization map.

Problem 2.1 When L_p is surjective?

We have the following

Theorem 2.2 *If $\Gamma \setminus D_\Gamma$ is compact and if $p > \delta(\Gamma)$, then L_p is surjective.*

Let $T \in \mathcal{C}_p^\Gamma(D_\Gamma)$ and define $T^* \in \mathcal{C}_p^\Gamma(\mathbf{S}^n)$ by the following formula: $f \in \mathcal{C}^\infty(D_\Gamma)$ is chosen as in Lemma 1.4 which is of compact support this time, since $\Gamma \setminus D_\Gamma$ is compact; for $\omega \in \Omega^p(M)$, let

$$\langle T^*, \omega\rangle = \sum_{\gamma\in\Gamma} \langle T, (f \circ \gamma^{-1}) \cdot \omega\rangle. \tag{1}$$

Recall that

$$\sum_{\gamma\in\Gamma} f \circ \gamma^{-1} = 1 \quad \text{on} \quad D_\Gamma.$$

To give a meaning to the expression (1), we need estimate $|\langle T, (f\circ\gamma^{-1})\cdot \omega\rangle|$.

Now since T is Γ-invariant we have

$$\begin{aligned} |\langle T, (f \circ \gamma^{-1}) \cdot \omega\rangle| &= |\langle T, f \cdot \gamma^*\omega\rangle| \\ &\leq C||f \cdot \gamma^*\omega||_k \\ &\leq \text{constant } ||\gamma^*\omega||_k \end{aligned}$$

where C is the positive constant chosen in Lemma 1.2 for the compact set $A = \mathrm{supp}(f)$.

Now let us make a simple observation for a Fuchsian group of the first kind. We consider

$$\mathbf{S}^2 = U_+ \cup \mathbf{S}^1 \cup U_-$$

where U_+ and U_- are respectively the upper disc and the lower disc. The group Γ acts on $\mathbf{S}^2$ leaving U_+, $\mathbf{S}^1$ and U_- invariant and $\Gamma \setminus U_+$ and $\Gamma \setminus U_-$ are homeomorphic to a closed Riemann surface of genus $g \geq 2$.

Now Γ has a $4g$-gon as a fundamental domain and the action of each $\gamma \in \Gamma$ looks like Fig. 1.

Imagine $\gamma \in \Gamma$ very far away from $e \in \Gamma$. Then the action of γ, restricted to some compact region, say $\underline{D}$, becomes very much like "minute contraction". For a 0-current (i.e. a distribution), this does not mean $||\gamma^*(\omega)||_k$ small (ω is a function and $||\omega \circ \gamma||_0$ is not small). But if we consider p-current (for p large), the sum $\sum_{\gamma \in \Gamma} ||\gamma^*(\omega)||_k$ actually converges on compact region which we are going to show.

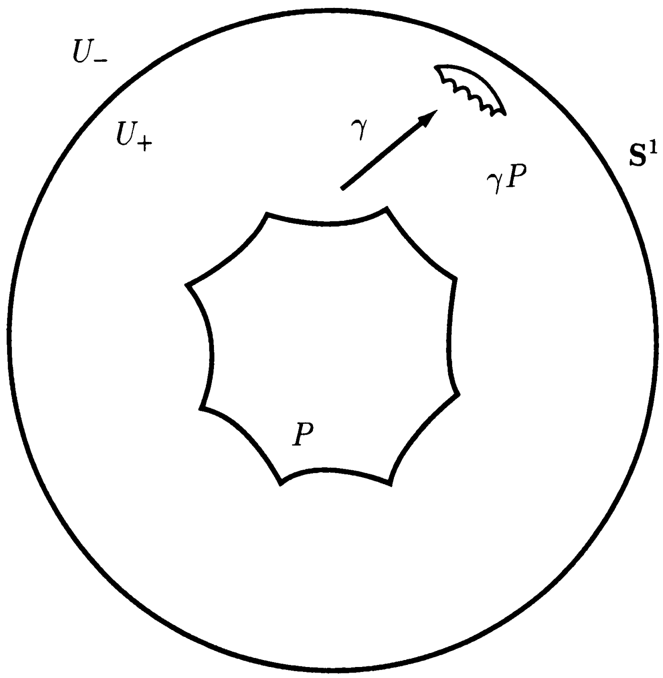

Fig. 1.

$1°$-*k-norm on* $\Omega^p(M)$.

We always consider $\mathbf{S}^n$ to be the unit sphere in $\mathbf{R}^{n+1}$. A Möbius transformation $\in \mathrm{Conf}^+(\mathbf{S}^n)$ is an even-time composite of inversions at n-dimensional spheres orthogonal to $\mathbf{S}^n$. Therefore it acts on $\mathbf{R}^{n+1} \cup \{\infty\}$.

Let V_ε be an ε-neighbourhood of $\mathbf{S}^n$ and let $\pi : V_\varepsilon \longrightarrow \mathbf{S}^n$ be the radial projection.

Given $\omega \in \Omega^p(M)$, we identify ω with $\pi^*(\omega) \in \Omega^p(V_\varepsilon)$ and write it down using coordinates of $\mathbf{R}^{n+1}$. Thus

$$\omega = \sum_{\mathbf{j}} \alpha_{\mathbf{j}}(x_1, \dots, x_{n+1}) dx_{j_1} \wedge \cdots \wedge dx_{j_p}$$

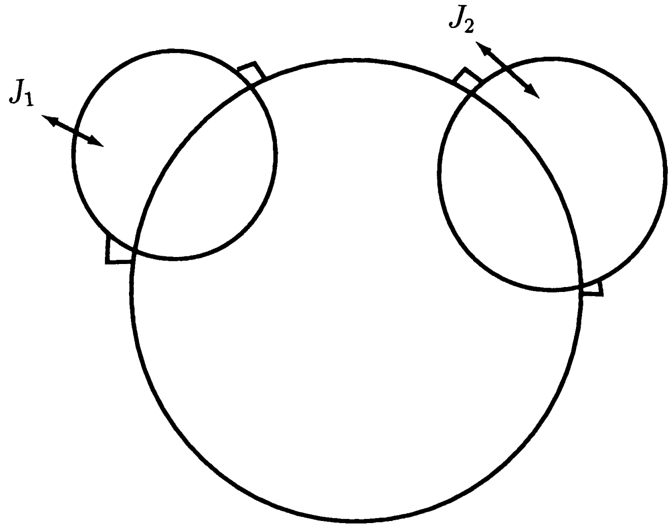

Fig. 2.

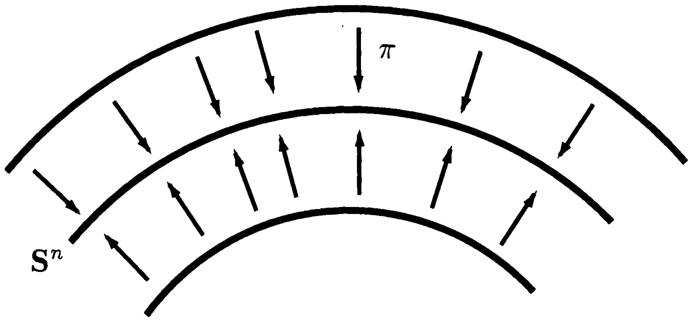

Fig. 3.

where, as in §1, $\mathbf{j} = (j_1, \dots j_p) \in \{1, \dots, n+1\}^p$. Define the k-norm of ω by

$$||\omega||_k = \sum_{\mathbf{j}} ||\alpha_{\mathbf{j}}||_k$$

where

$$||\alpha_{\mathbf{j}}||_k = \max_{|\mathbf{s}| \le k} \left\{ \sup_{x \in V_\varepsilon} \left| \frac{\partial^{|\mathbf{s}|} \alpha_{\mathbf{j}}}{\partial x_1^{s_1} \cdots \partial x_n^{s_n}}(x) \right| \right\}$$

where $\mathbf{s} = (s_1, \dots, s_n)$ and $|\mathbf{s}| = s_1 + \cdots + s_n$. This k-norm is of course equivalent to the usual k-norm defined by using coordinates of $\mathbf{S}^n$.

2°-*Möbius transformation.*

For $\gamma \in \mathrm{Conf}^+(\mathbf{S}^n)$ and $x \in \mathbf{R}^{n+1}$, $D_x\gamma$ (the matrix derivative of γ) is a conformal matrix. Denote by $|D_x\gamma|$ its norm. Now for γ such that $\gamma(0) \neq 0$

$$I(\gamma) = \left\{x \in \mathbf{R}^{n+1} \mid |D_x\gamma| = 1\right\}$$

is an n-sphere perpendicular to $\mathbf{S}^n$ called the *isometric sphere* of γ. It is very small if γ is very far away from e. Suppose $\gamma(0) \neq 0$. Then it is known that such γ decomposes as

$$\gamma = J_\theta \circ J_{I(\gamma)} \circ P$$

where

$P \in \mathrm{SO}(n+1)$; P keeps $I(\gamma)$ invariant

$J_{I(\gamma)}$ is the inversion at $I(\gamma)$

J_θ is the inversion at a plane θ passing through 0.

For details see [Ma]. The transformations J_θ and P does not affect the derivatives of γ. Thus we need only study the derivatives of $J_{I(\gamma)}$.

3°-*Inversion.*

For the estimate of the derivative of $J_{I(\gamma)}$, we shall change the coordinates and consider the following simple situation. Fix $\lambda > 0$ sufficiently small. Then

$$x \in \mathbf{R}^{n+1} \longrightarrow h_\lambda(x) = \frac{\lambda^2}{|x|^2}x \in \mathbf{R}^{n+1}$$

is the inversion at $|x| = \lambda$. Let us estimate k-th derivative at the region $A = \{x \mid |x| \geq a\}$ (where $a > 0$ is fixed. We are considering the situation $\lambda << a$). Now each coordinate of $h_\lambda(x)$ is a rational function

$$\lambda^2\frac{g(x)}{f(x)} \quad f,\ g \text{ homogeneous with } \deg(g) < \deg(f).$$

This property does not change if we take derivatives. That is, we have the

Lemma 2.3 *There exists a positive constant $C = C(a,k)$ such that any i-derivative $(1 \leq i \leq k)$ of the coordinates of h_λ at $x \in \{|x| \geq a\}$ is smaller than $\lambda^2 C$ in norm.*

Let A be a compact set in D_Γ. For $\gamma \in \Gamma$ denote by $||\gamma||_{1,k}^A$ the supremum of any the i-th derivative $(1 \leq i \leq k)$ of the coordinates of γ on A.

Note that in the definition of $||\omega||_k$, we considered the 0-th derivative also. But with $||\gamma||_{1,k}^A$ we do not take the 0-derivative into account.

Corollary 2.4 *There exists a positive constant $C = C(a, k)$ such that*

$$||\gamma||_{1,k}^A \leq \lambda(\gamma)^2 C$$

where $\lambda(\gamma)$ is the radius of the isometric sphere of γ.

Proof. There exists $a > 0$ such that except for finite number of $\gamma \in \Gamma$, the center of the isometric sphere of γ is at least a-apart from A. Now Corollary 2.4 follows from the decomposition $\gamma = J_\theta \circ J_{I(\gamma)} \circ P$ and Lemma 2.3.

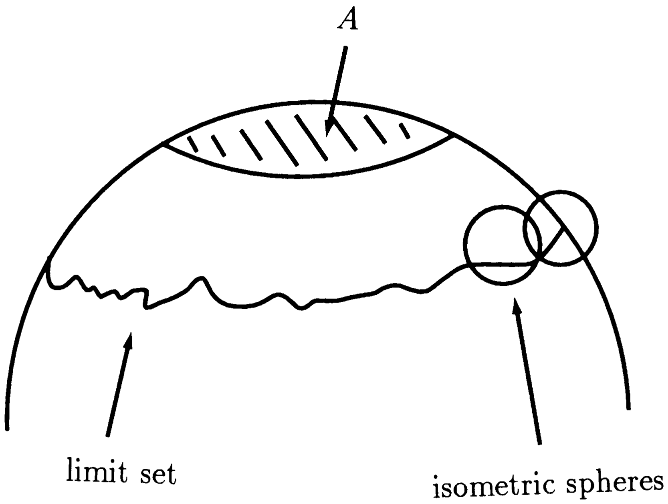

Fig. 4.

Now as before let

$$\omega = \sum_{\mathbf{j}} \alpha_{\mathbf{j}}(x_1, \ldots, x_{n+1}) dx_{j_1} \wedge \cdots \wedge dx_{j_p} \in \Omega^p(\mathbf{S}^n).$$

Let us estimate $||\gamma^*\omega||_k^A$ for $\gamma \in \Gamma$ (A is compact in D_Γ). Let

$$D_x\gamma = \begin{pmatrix} a_{11} & \cdots & a_{1,n+1} \\ \vdots & \ddots & \vdots \\ a_{n+1,1} & \cdots & a_{n+1,n+1} \end{pmatrix}.$$

Then we obtain

$$\gamma^*\omega = \sum_{\mathbf{i}} \left(\sum_{\mathbf{j}} (a_{i_1,j_1} \cdot \cdots \cdot a_{i_p,j_p}) \alpha_{\mathbf{j}} \circ \gamma dx_{j_1} \wedge \cdots \wedge dx_{j_p} \right)$$

and

$$||\gamma^*\omega||_k^A \leq \text{constant} \sum_{\mathbf{j}} \left\{ ||\alpha_{\mathbf{j}} \circ \gamma||_k^A \left(||\gamma||_{1,k}^A \right)^p \right\}$$

because for $\gamma, \sigma \in \Gamma$ we have (easy to show)

$$||\gamma \cdot \sigma||_{1,k}^A \leq C||\gamma||_{1,k}^A \cdot ||\sigma||_{1,k}^A.$$

Now by the Leibnitz rule we have

$$||\alpha_{\mathbf{j}} \circ \gamma||_k^A \leq ||\alpha_{\mathbf{j}}||_k^{\gamma(A)} \cdot Q(||\gamma||_{1,k}^A)$$

where Q is a polynomial with positive coefficients and with leading term 1. This is because we consider 0-th derivative in $||\alpha_{\mathbf{j}} \circ \gamma||_k^A$. By Corollary 2.4 we have $Q \leq$ constant. Thus we get the following:

Lemma 2.5 *We have*

$$||\gamma^*\omega||_k^A \leq C||\omega||_k \cdot \lambda(\gamma)^{2p}.$$

It is easy to show, except for a finite number of $\gamma \in \Gamma$, that we have

$$\frac{1}{2}\lambda(\gamma)^2 \leq |\gamma'(0)| \leq \lambda(\gamma)^2.$$

End of the proof of Theorem 2.2.

Let $\omega \in \Omega^p(\mathbf{S}^n)$ and $T \in \mathcal{C}_p^\Gamma(D_\Gamma)$. Define $\langle T^*, \omega \rangle$ by

$$\begin{aligned} \langle T^*, \omega \rangle &= \sum_{\gamma \in \Gamma} \langle T, f \circ \gamma^{-1} \cdot \omega \rangle \\ &= \sum_{\gamma \in \Gamma} \langle T, f \cdot \gamma^* \omega \rangle. \end{aligned}$$

Then on $A = \text{supp}(f)$ we have

$$\begin{aligned} |\langle T, f \cdot \gamma^*\omega \rangle| &\leq \text{constant}||\gamma^*\omega||_k^A \\ &\leq \text{constant}||\omega||_k \lambda(\gamma)^{2p}. \end{aligned}$$

Now for $z \in \mathbf{D}^{n+1}$, we have

$$||D_z\gamma|| = \frac{\lambda(\gamma)^2}{|z - b(\gamma)|^2}$$

where $b(\gamma)$ is the center of the isometric sphere (see [Ma] p. 189).

Since $|z - b(\gamma)|^2 >$ *constant* for any $\gamma \in \Gamma$, we have

$$\begin{aligned}\sum_{\gamma\in\Gamma} |\langle T, f\cdot\gamma^*(\omega)\rangle| &\leq \text{constant}||\omega||_k \sum_{\gamma\in\Gamma} \lambda(\gamma)^{2p} \\ &\leq \text{constant}||\omega||_k \sum_{\gamma\in\Gamma} ||D_z\gamma||^p \\ &\leq \text{constant}||\omega||_k\end{aligned}$$

if $p > \delta(\Gamma)$ (the critical exponent of Γ). Thus T^* defines a p-current on $\mathbf{S}^n$. It is clear that T^* is Γ-invariant and that $L_p(T^*) = T$. □

Remark 2.6 According to Sullivan [Su], if Γ is convex-cocompact, then we have $\delta(\Gamma) = d_H(\Lambda_\Gamma)$ where d_H denotes the Hausdorff dimension.

3. Invariant distributions

Assume that (1) $\delta(\Gamma) < 1$, (2) Γ acts on D_Γ freely and (3) $\Gamma \setminus D_\Gamma$ is compact and connected. The localization map $L_1 : \mathcal{C}_1^\Gamma(\mathbf{S}^n) \longrightarrow \mathcal{C}_1^\Gamma(D_\Gamma)$ is surjective by Theorem 2.2. Consider the following diagram.

$$\begin{array}{ccccc} \mathcal{C}_1^\Gamma(\mathbf{S}^n) & \xrightarrow{d} & \mathcal{C}_0^\Gamma(\mathbf{S}^n) & & \\ L_1\downarrow & & \downarrow L_0 & & \\ \mathcal{C}_1^\Gamma(D_\Gamma) & \xrightarrow{d} & \mathcal{C}_0^\Gamma(D_\Gamma) & \xrightarrow{\widehat{\theta}} & \mathbf{C} \\ \pi^!\uparrow & & \uparrow\pi^! & & \| \\ \mathcal{C}_1(\Gamma\setminus D_\Gamma) & \xrightarrow{d} & \mathcal{C}_0(\Gamma\setminus D_\Gamma) & \xrightarrow{\theta} & \mathbf{C} \end{array} \qquad (**)$$

Here θ is the *augmentation* defined by

$$\theta(T) = \langle T, \mathbf{1}\rangle$$

where $\mathbf{1}$ is the function identically equal to 1. The bottom row is exact since $\Gamma \setminus D_\Gamma$ is connected; $\widehat{\theta}$ is defined by

$$\widehat{\theta}(T) = \langle T, f\rangle$$

where f is the function given by Lemma 1.4. Let us show the commutativity of the diagram $(**)$. All that need proof is $\pi^! \circ \theta = \widehat{\theta}$.

Recall the arguments in §1 showing the surjectivity of the map $\pi_!$. It says that for $T \in \mathcal{C}_0^\Gamma(D_\Gamma)$

$$\begin{aligned}\langle(\pi^!)^{-1}(T), \mathbf{1}\rangle &= \langle T, f \cdot \pi^*(\mathbf{1})\rangle \\ &= \langle T, f\rangle.\end{aligned}$$

In other words

$$\theta((\pi^!)^{-1}(T)) = \widehat{\theta}(T).$$

Theorem 3.1 *We have*

$$\text{Image}(L_0) \supset \text{Ker}(\widehat{\theta}).$$

Proof. This follows from the surjectivity of L_1 and the exactness of the second row. □

This theorem shows that Γ-invariant currents abound.

Now let us consider the case that Γ is elementary. The simplicity of the situation enables us to determine the image of L_0 completely.

Theorem 3.2 *Suppose that Γ is elementary generated by a single loxodromic element γ with repeller a_+ and attractor a_-. Then*

$$\text{Image}(L_0) = \text{Ker}(\widehat{\theta}).$$

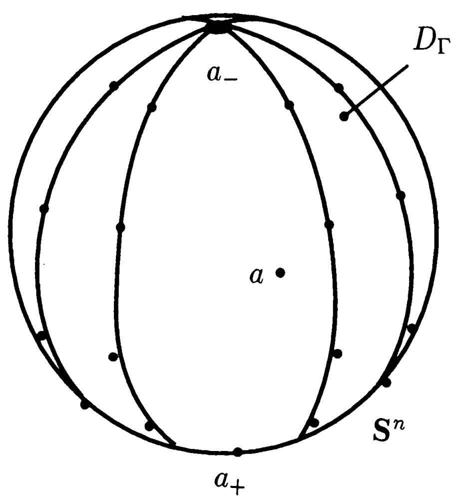

Fig. 5.

Now choose $a \in D_\Gamma$ and set

$$T_a = \sum_{n \in \mathbf{Z}} \delta_{\gamma^n a}$$

where δ_x denotes the Dirac distribution at a point x.

Clearly $T_a \in \mathcal{C}_0^\Gamma(D_\Gamma)$ and $\widehat{\theta}(T_a) = 1$. We are going to construct an element $S_a \in \mathcal{C}_0(\mathbf{S}^n)$ such that $L_0(S_a) = T_a$. But S_a will fail to be Γ-invariant. Thanks to the simplicity of the situation this failure will show Theorem 3.2.

Consider the following sum

$$S_a = \delta_a + \sum_{n>0} (\delta_{\gamma^n a} - \delta_{a_+}) + \sum_{n<0} (\delta_{\gamma^n a} - \delta_{a_-}) .$$

To show that S_a is a well-defined distribution, we only need to show that for any $g \in C^\infty(\mathbf{S}^n)$, $\langle S_a, g\rangle$ converges. But

$$\langle S_a, g\rangle = g(a) + \sum_{n>0} (g(\gamma^n a) - g(a_+)) + \sum_{n<0} (g(\gamma^n a) - g(a_-))$$

and

$$\begin{aligned} \sum_{n>0} |g(\gamma^n a) - g(a_+)| &\leq \text{constant} \sum_{n>0} d(\gamma^n a, a_+) \\ &\leq \text{constant} \sum_{n>0} \lambda^n \quad \text{for some } 0 < \lambda < 1 \\ &< +\infty. \end{aligned}$$

The same estimate holds for the sum $\sum_{n<0} |g(\gamma^n a) - g(a_+)|$, which proves that S_a is a distribution. Clearly $L_0(S_a) = T_a$.

Now let us compute $\gamma_*(S_a)$. We have

$$\begin{aligned} \langle \gamma_*(S_a), g\rangle &= \langle S_a, g \circ \gamma\rangle \\ &= g(\gamma a) + \sum_{n>0} (g(\gamma^{n+1} a) - g(a_+)) \\ &\quad + \sum_{n<0} (g(\gamma^{n+1} a) - g(a_-)). \end{aligned}$$

So

$$\begin{aligned} &\langle \gamma_*(S_a) - S_a, g\rangle \\ &\quad = g(\gamma a) - g(a) \end{aligned}$$

$$+\left\{\sum_{n>0}(g(\gamma^{n+1}a)-g(a_+))-\sum_{n>0}(g(\gamma^n a)-g(a_+))\right\}$$
$$+\left\{\sum_{n<0}(g(\gamma^{n+1}a)-g(a_-))-\sum_{n<0}(g(\gamma^n a)-g(a_-))\right\}$$
$$=\sum_{n\in\mathbf{Z}}(g(\gamma^{n+1}a)-g(\gamma^n a))$$
$$=g(a_+)-g(a_-).$$

For the proof of the last equality, consider the partial sum

$$\sum_{n=-N}^{N-1}(g(\gamma^{n+1}a)-g(\gamma^n a))$$
$$=g(\gamma^N a)-g(\gamma^{-N}a)\overset{N+\infty}{\longrightarrow}g(a_+)-g(a_-).$$

Thus we have

$$\gamma_*(S_a)-S_a=\delta_{a_+}-\delta_{a_-}.$$

Now let us embark upon the proof of Theorem 3.2. By Theorem 3.1 we have already $\mathrm{Ker}(\widehat{\theta})\subset\mathrm{Image}(L_0)$. For absurdity assume $L_0(S)=T_a$ for some $S\in\mathcal{C}_0^\Gamma(\mathbf{S}^n)$. Consider $U=S_a-S$. Then $\mathrm{supp}(U)\subset\{a_+,a_-\}$ and $\gamma_*(U)-U=\delta_{a_+}-\delta_{a_-}$.

Let $\mathbf{1}_+$ be a bump function, equal to 1 near a_+ and 0 near a_-. Then $\langle U,\mathbf{1}_+\circ\gamma\rangle=\langle U,\mathbf{1}_+\rangle$. Thus

$$\langle\gamma_*(U)-U,\mathbf{1}_+\rangle=0.$$

But we also have

$$\langle\delta_{a_+}-\delta_{a_-},\mathbf{1}_+\rangle=1.$$

This is a contradiction.

Now let $T\in\mathcal{C}_0^\Gamma(D_\Gamma)-\mathrm{Ker}(\widehat{\theta})$. Then $T-\widehat{\theta}(T)T_a\in\mathrm{Ker}(\widehat{\theta})$; so there exists an element $S\in\mathcal{C}_0^\Gamma(\mathbf{S}^n)$ such that

$$T-\widehat{\theta}(T)T_a=L_0(S).$$

This implies that T is not an element of $\mathrm{Image}(L_0)$. So we have necessarily

$$\mathrm{Ker}(\widehat{\theta})=\mathrm{Image}(L_0)$$

which proves the theorem.

4. Cross section of the localization map

As before $X = \Gamma \setminus D_\Gamma$. In the previous section, we have shown that the localization map $L_0 : \mathcal{C}_0^\Gamma(\mathbf{S}^n) \longrightarrow \mathcal{C}_0(X)$ is surjective onto $\mathrm{Ker}(\theta)$ for an elementary Kleinian group generated by a single loxodromic transformation γ. That is, given a distribution $T \in \mathcal{C}_0(X)$, such that $\langle T, \mathbf{1}\rangle = 0$, one can choose $S \in \mathcal{C}_0^\Gamma(\mathbf{S}^n)$ such that $L_0(S) = T$. However since the argument there is indirect, one cannot construct S explicitely even when T is given concretely. In this section we shall solve this problem by constructing a cross-section of L_0. The construction has two steps. Denote by $\overline{C}^\infty(\mathbf{S}^n)$ the space of C^∞-functions which vanish on the fixed points a_+ and a_- of γ and by $\overline{\mathcal{C}}(\mathbf{S}^n)$ its topological dual. Denote by $\overline{\mathcal{C}}^\Gamma(\mathbf{S}^n)$ the subspace of $\overline{\mathcal{C}}(\mathbf{S}^n)$ consisting of the elements U such that $\langle U, \gamma^*\varphi - \varphi\rangle = 0$ for any $\varphi \in \overline{C}^\infty(\mathbf{S}^n)$.

The inclusion $\overline{C}^\infty(\mathbf{S}^n) \hookrightarrow C^\infty(\mathbf{S}^n)$ defines the projection

$$p : \mathcal{C}_0^\Gamma(\mathbf{S}^n) \longrightarrow \overline{\mathcal{C}}^\Gamma(\mathbf{S}^n).$$

Also we have the localization map

$$\overline{L}_0 : \overline{\mathcal{C}}^\Gamma(\mathbf{S}^n) \longrightarrow \mathcal{C}_0(X).$$

Clearly we have $L_0 = \overline{L}_0 \circ p$.

The first step is to construct a cross section

$$s : \mathcal{C}_0(X) \longrightarrow \overline{\mathcal{C}}^\Gamma(\mathbf{S}^n).$$

This will be carried out on the whole $\mathcal{C}_0(X)$, not only on $\mathrm{Ker}(\theta)$.

Define $\overline{\theta} : \overline{\mathcal{C}}^\Gamma(\mathbf{S}^n) \longrightarrow \mathbf{C}$ also by $\overline{\theta}(U) = \langle U, f\rangle$. The second step is the construction of a cross section

$$t : \mathrm{Ker}(\overline{\theta}) \longrightarrow \mathcal{C}_0^\Gamma(\mathbf{S}^n).$$

Then $t \circ s$ is the desired cross section of L_0.

1°-*First step*

For any $\psi \in \overline{C}^\infty(\mathbf{S}^n)$, consider the series

$$\Psi = \sum_{n\in\mathbf{Z}} \psi \circ \gamma^n.$$

Lemma 4.1 *The series Ψ converges in the C^∞-topology on compact subset in D_Γ and defines a function $\Psi \in C^\infty(X)$.*

Define a map $\sigma : \overline{C}^{\infty}(\mathbf{S}^n) \longrightarrow C^{\infty}(X)$ by $\sigma(\psi) = \Psi$.

Lemma 4.2 *The map σ is linear, continuous and surjective.*

For the surjectivity, given $\Psi \in C^{\infty}(X)$ we have $\Psi = \sigma(f\Psi)$. The proof of the other parts consists of estimations of derivatives. They are more or less the same as those in §2 and of course based upon the fact that ψ vanishes on the fixed points of γ. The details are left to the reader.

Now the cross section

$$s : \mathcal{C}_0(X) \longrightarrow \overline{\mathcal{C}}^{\Gamma}(\mathbf{S}^n)$$

is defined as the dual of σ.

2°-*Second step*

Choose $U \in \overline{\mathcal{C}}^{\Gamma}(\mathbf{S}^n)$ such that $\langle U, f\rangle = 0$. Let

$$g_- = \sum_{n \geq 0} f \circ \gamma^n.$$

This function can be extended differentiably to $\mathbf{S}^n$, to yield a bump function, constant by 1 around a_- and 0 around a_+. Let us define

$$t : \overline{\mathcal{C}}^{\Gamma}(\mathbf{S}^n) \longrightarrow \mathcal{C}_0^{\Gamma}(\mathbf{S}^n)$$

by the following formula. For $\varphi \in C^{\infty}(\mathbf{S}^n)$, let

$$\langle t(U), \varphi\rangle = \langle U, \varphi_0\rangle$$

where $\varphi_0 = \varphi - \varphi(a_-)g_- - \varphi(a_+)(1 - g_-)$. Clearly $t(U) \in \mathcal{C}^{\Gamma}(\mathbf{S}^n)$. Let us show that $t(U)$ is Γ-invariant. Let

$$\begin{aligned}
&\langle t(U), \varphi \circ \gamma - \varphi\rangle \\
&\quad = \langle U, \varphi \circ \gamma - \varphi(a_-)g_- - \varphi(a_+)(1 - g_-) \\
&\qquad - \varphi_0 \circ \gamma + \varphi_0 \circ \gamma - \varphi_0\rangle \\
&\quad = \langle U, \varphi \circ \gamma - \varphi(a_-)g_- - \varphi(a_+)(1 - g_-) \\
&\qquad - \{\varphi \circ \gamma - \varphi(a_-)g_- \circ \gamma - \varphi(a_+)(1 - g_- \circ \gamma)\} \\
&\qquad + (\varphi_0 \circ \gamma - \varphi_0)\rangle \\
&\quad = (\varphi(a_-) - \varphi(a_+))\,\langle U, g_- \circ \gamma - g_-\rangle + \langle U, \varphi_0 \circ \gamma - \varphi_0\rangle.
\end{aligned}$$

The first term vanishes since $g_- \circ \gamma - g_- = -f$ and the second term vanishes since $\varphi_0 \in \overline{C}^{\infty}(\mathbf{S}^n)$. This completes the construction of the cross section t of the projection p.

All that we proved in this paragraph are in fact applicable to a more general situation.

Let M^n be a manifold and let $\gamma : M \longrightarrow M$ be a diffeomorphism with a finte set $\Sigma = A \cup R$ of fixed points. Assume that

(1) *all the points of A are attractors, that is, the spectral radius of the derivatives at these points is smaller than* 1;

(2) *all the points of R are repellers*;

(3) *γ acts freely and properly discontinuously on $M - \Sigma$.*

The method of constructing s and t works if γ satisfies (1), (2) and (3).

There are examples on $\mathbf{S}^1$ in which there exist the same number of attractors and repellers, placed alternatively.

Also on $\mathbf{S}^n$, there are examples with one attractor and one repeller. Let us show that they are exhausting. Let $n \geq 2$. Consider a small sphere S centered at an attractor. Denote by Q the closed region bounded by S and γS. Then $\langle\gamma\rangle \setminus Q$ is a closed manifold, homeomorphic to $\mathbf{S}^1 \times \mathbf{S}^{n-1}$. Now $\langle\gamma\rangle \setminus (M - \Sigma)$ is also a manifold by (3). Since $n \geq 2$, it is connected. Therefore we have

$$\langle\gamma\rangle \setminus Q = \langle\gamma\rangle \setminus (M - \Sigma).$$

Now it is easy to show that $M = \mathbf{S}^n$ and that there are only one attractor and only one repeller. The case $n = 1$ is left to the reader. But let us give an example:

Let $\tilde{\gamma} : \mathbf{R} \longrightarrow \mathbf{R}$ be the diffeomorphism given by $\tilde{\gamma}(x) = x + \alpha \sin(2\pi n x)$ where $n \in \mathbf{N}^*$ and $\alpha \in]0, \frac{1}{2\pi n}[$. Then $\tilde{\gamma}$ satisfies the relation $\tilde{\gamma}(x+1) = \tilde{\gamma}(x) + 1$ and hence induces a diffeomorphism γ of the circle $\mathbf{S}^1 = \mathbf{R}/\mathbf{Z}$. It has $2n$ fixed point

$$\Sigma = \left\{0, \frac{1}{2n}, \frac{2}{2n}, \frac{3}{2n}, \ldots, \frac{2n-1}{2n}\right\}.$$

The manifold $\mathbf{S}^1 - \Sigma$ is a disjoint union of $2n$ intervalles I_k, $k = 1, \ldots, 2n$.

Let $A = \{\frac{2k-1}{2n} \mid k = 1, \ldots, n\}$ and $R = \{\frac{k}{n} \mid k = 0, \ldots, n-1\}$. The spectral radius $\rho_x(\gamma)$, for $x \in A$ and $x \in R$ are respectively equal to $1 - 2\pi n \alpha$ and $1 + 2\pi n \alpha$.

Furthermore the action generated by γ on $M - \Sigma$ is free and properly discontinuous. The quotient manifold $X = \langle\Gamma\rangle \setminus (M - \Sigma)$ is a disjoint union of $2n$ copies $(X_l)_{l=1,\dots,2n}$ of the circle. □

5. Weakly invariant distributions

Here we shall treat a nonelementary group by the same method as in the previous section. However what we get is a weaker result. For this we need the concept of weakly Γ-invariant distribution.

Definition 5.1 *A group Γ is called a Schottky group if it is generated by s elements $\gamma_1, \dots, \gamma_s$ such that for mutually disjoint closed balls $A_1, \dots, A_s$, $B_1, \dots, B_s$, we have $\gamma_i(A_i) = \overline{\mathbf{S}^n - B_i}$.*

The following facts are well known.

(1) $\Gamma \simeq \langle \gamma_1 \rangle * \cdots * \langle \gamma_s \rangle$.
(2) Γ acts on D_Γ freely.
(3) $\Gamma \setminus D_\Gamma$ is homeomorphic to $\#_s(\mathbf{S}^1 \times \mathbf{S}^{n-1})$.
(4) Γ is convex-cocompact and thus by [Su]: $\delta(\Gamma) = d_H(\Lambda_\Gamma)$.
(5) Λ_Γ is a tame Cantor set.
(6) Any element of Γ is loxodromic.

Definition 5.2 *A distribution $T \in \mathcal{C}_0(\mathbf{S}^n)$ is said to be weakly Γ-invariant if for any $\gamma \in \Gamma$,* $\operatorname{supp}(\gamma_*(T) - T)$ *is contained in Λ_Γ.*

Let us denote weakly Γ-invariant distributions by $\mathcal{C}_0^{(\Gamma)}(\mathbf{S}^n)$. Clearly the localization map L_0 carries $\mathcal{C}_0^{(\Gamma)}(\mathbf{S}^n)$ into $\mathcal{C}_0^{\Gamma}(D_\Gamma)$.

Theorem 5.3 *If Γ is a Schottky group such that $d_H(\Lambda_\Gamma) < \frac{1}{2}$, then*

$$L_0 : \mathcal{C}_0^{(\Gamma)}(\mathbf{S}^n) \longrightarrow \mathcal{C}_0^{\Gamma}(D_\Gamma)$$

is a surjection.

Proof. By Theorem 3.1, we have

$$L_0(\mathcal{C}_0^{(\Gamma)}(\mathbf{S}^n)) \supset L_0(\mathcal{C}_0^{\Gamma}(\mathbf{S}^n)) \supset \operatorname{Ker}(\widehat{\theta}).$$

So we need only to show hat $T_a \in L_0(\mathcal{C}_0^{(\Gamma)}(\mathbf{S}^n))$, where

$$T_a = \sum_{\gamma \in \Gamma} \delta_{\gamma a} \quad a \in D_\Gamma.$$

In fact, for any $T \in \mathcal{C}_0^{\Gamma}(\mathbf{S}^n)$ we have a decomposition

$$T = (T - \widehat{\theta}(T) \cdot T_a) + \widehat{\theta}(T) \cdot T_a.$$

The first summand lies in $\mathrm{Ker}(\widehat{\theta})$ since $\widehat{\theta}(T_a) = 1$. Thus we will have $T \in L_0(\mathcal{C}_0^{(\Gamma)}(\mathbf{S}^n))$.

Now any element $\gamma \in \Gamma' = \Gamma - \{e\}$ is loxodromic. Let $a(\gamma)$ be the attractor of γ. For T_a define S_a as follows.

$$S_a = \delta_a + \sum_{\gamma \in \Gamma'} (\delta_{\gamma a} - \delta_{a(\gamma)}).$$

Notice that except for a finite number of γ, γa and $a(\gamma)$ lie in the isometric sphere $I(\gamma^{-1})$. For a test function $g \in C^\infty(\mathbf{S}^n)$,

$$\langle S_a, g \rangle = g(a) + \sum_{\gamma \in \Gamma'} \{g(\gamma a) - g(a(\gamma))\}$$

and

$$\begin{aligned} \sum_{\gamma \in \Gamma'} |g(\gamma a) - g(a(\gamma))| &\leq \text{constant} \sum_{\gamma \in \Gamma'} |\text{ radius } I(\gamma^{-1})| \\ &\leq \text{constant} \sum_{\gamma \in \Gamma'} |\gamma'(0)|^{\frac{1}{2}} \\ &< +\infty \end{aligned}$$

since $d_H(\Lambda_\Gamma) < \frac{1}{2}$. Thus S_a is a distribution. Clearly $L_0(S_a) = T_a$ and the Γ-invariance of T_a shows that $S_a \in L_0(\mathcal{C}_0^{(\Gamma)}(\mathbf{S}^n))$. □

6. Application to a bigraded cohomology with compact support

We will apply the preceding results to compute a *bigraded cohomology with compact support* of a foliation obtained by suspending one of all the groups Γ considered in the above sections. First let us recall some definitions and useful properties.

6.1. Cohomology of groups

Let Γ be a discrete group acting on a module E and denote by $C^k(\Gamma, E)$ the set of all the maps $\Gamma^k \longrightarrow E$. We define $d : C^k(\Gamma, E) \longrightarrow C^{k+1}(\Gamma, E)$ by

$$\begin{aligned} (dc)(\gamma_1, \dots, \gamma_{k+1}) &= \gamma_1 \cdot c(\gamma_2, \dots, \gamma_{k+1}) \\ &\quad + \sum_{i=1}^{k} (-1)^i c(\gamma_1, \dots, \gamma_i \gamma_{i+1}, \dots, \gamma_{k+1}) \\ &\quad + (-1)^{k+1} c(\gamma_1, \dots, \gamma_k). \end{aligned}$$

The operator d is linear and satisfies $d^2 = 0$; so the image $B^k(\Gamma, E)$ of this operator $d : C^{k-1}(\Gamma, E) \longrightarrow C^k(\Gamma, E)$ is an ideal of the kernel $Z^k(\Gamma, E)$ of $d : C^k(\Gamma, E) \longrightarrow C^{k+1}(\Gamma, E)$. The quotients

$$H^k(\Gamma, E) = Z^k(\Gamma, E)/B^k(\Gamma, E) \text{ for } k \in \mathbf{N}$$

are called the *cohomology groups* of Γ with values in the Γ-module E.

6.2. Bigraded cohomology

Let $\mathcal{F}$ a codimension n foliation on a manifold N of dimension $m + n$. Denote by $T\mathcal{F}$ the tangent bundle of $\mathcal{F}$ and $\nu\mathcal{F} = TN/T\mathcal{F}$ its normal bundle. Let $\Lambda^q T^*\mathcal{F}$ and $\Lambda^p \nu^*\mathcal{F}$ be the vector bundles of exterior q-forms and exterior p-forms associated respectively to $T^*\mathcal{F}$ and $\nu^*\mathcal{F}$. Let $A^{pq}_{\mathcal{F}}$ be the space of global sections of the bundle $\Lambda^q T^*\mathcal{F} \otimes \Lambda^p \nu^*\mathcal{F}$. An element of $A^{pq}_{\mathcal{F}}$ is considered to be a $\Lambda^p \nu^*\mathcal{F}$-valued q-form along the leaves. Because $\Lambda^p \nu^*\mathcal{F}$ is a foliated vector bundle we can define the *exterior derivative* along the leaves $d_{\mathcal{F}} : A^{pq}_{\mathcal{F}} \longrightarrow A^{p,q+1}_{\mathcal{F}}$ by

$$\begin{aligned} & d_{\mathcal{F}}\eta(X_1, \ldots, X_{q+1}) \\ & \quad = \sum_i (-1)^i X_i \cdot \eta(X_1, \ldots, \widehat{X_i}, \ldots, X_{q+1}) \\ & \qquad + \sum_{i<j} (-1)^{i+j} \eta([X_i, X_j], X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_{q+1}). \end{aligned}$$

An easy computation shows that $d^2_{\mathcal{F}} = 0$ and thus we obtain a differential complex

$$0 \longrightarrow A^{p0}_{\mathcal{F}} \xrightarrow{d_{\mathcal{F}}} A^{p1}_{\mathcal{F}} \xrightarrow{d_{\mathcal{F}}} \cdots \xrightarrow{d_{\mathcal{F}}} A^{pm}_{\mathcal{F}} \longrightarrow 0.$$

Its homology $H^{p,*}(N, \mathcal{F})$ is called the *bigraded cohomology* (*foliated cohomology* when $p = 0$) of the foliated manifold $(N, \mathcal{F})$.

We can also define the *bigraded cohomology with compact support* as the homology $H^{p,*}_c(N, \mathcal{F})$ of the differential complex

$$0 \longrightarrow \Omega^{p0}_{\mathcal{F}}(M) \xrightarrow{d_{\mathcal{F}}} \Omega^{p1}_{\mathcal{F}}(M) \xrightarrow{d_{\mathcal{F}}} \cdots \xrightarrow{d_{\mathcal{F}}} \Omega^{pm}_{\mathcal{F}}(M) \longrightarrow 0$$

where $\Omega^{p,*}_{\mathcal{F}}(M)$ is the space of sections of compact support of the vector bundle $\Lambda^* T^*\mathcal{F} \otimes \Lambda^p \nu^*\mathcal{F}$.

6.3. The case of a suspension

Let W be a compact manifold and suppose that there exists an faithful representation $\rho : \Gamma = \pi_1(W) \longrightarrow \mathrm{Diff}(M)$ where $\mathrm{Diff}(M)$ is the diffeomorphism group of a manifold M. Let $\widetilde{W}$ be the universal covering of W. The foliation $\tilde{\mathcal{F}}$ on $\widetilde{W} \times M$ defined by the second projection is invariant by the diagonal action of Γ, thus it induces a foliation $\mathcal{F}$ on the manifold $N = \Gamma \backslash (\widetilde{W} \times M)$ transverse to the locally trivial fibration $M \hookrightarrow N \longrightarrow W$. By using the same method as in [ET] we can prove that we have an isomorphism

$$H_c^{p,*}(N, \mathcal{F}) \cong H^*(W, \Omega^p(M))$$

where $\Omega^p(M)$ has a structure of a Γ-module defined by the induced action of Γ on M. We have also

$$H_c^{p,*}(N, \mathcal{F}) \cong H^*(\Gamma, \Omega^p(M)) \quad \text{for } * = 0 \text{ and } * = 1. \qquad (\mathcal{R})$$

Let us show that for a free group Γ, acting on M in a certain way, the dimension of $H^1(\Gamma, \Omega^p(M))$ is infinite.

Now $Z^1(\Gamma, \Omega^p(M))$ consists of twisted homomorphisms, that is, all the maps $c : \Gamma \longrightarrow \Omega^p(M)$ such that for $\gamma_1, \gamma_2 \in \Gamma$

$$c(\gamma_1\gamma_2) = \gamma_1 c(\gamma_2) + c(\gamma_1).$$

The space $B^1(\Gamma, \Omega^p(M))$ consists of those twisted homomorphisms c such that for some $\omega \in \Omega^p(M)$

$$c(\gamma) = \gamma\omega - \omega, \quad \text{for all } \gamma \in \Gamma.$$

Therefore there exists a natural map

$$r : H^1(\Gamma, \Omega^p(M)) \longrightarrow \mathrm{Hom}(\Gamma, \Omega^p(M)/K^p),$$

where K^p is the submodule of $\Omega^p(M)$ consisting of $\sum_{i=1}^{s}(\gamma_i\omega_i - \omega_i)$ where $\gamma_i \in \Gamma$ and $\omega_i \in \Omega^p(M)$.

Let us show that for a free group $\Gamma = \mathbf{Z} * \cdots * \mathbf{Z}$, r is a surjection.

Let $a_1, \ldots, a_n$ be free generators. For any $\omega_1, \ldots, \omega_n \in \Omega^p(M)$, we claim that there exists uniquely a twisted homomorphism c such that

$$c(e) = 0 \text{ and } c(a_i) = \omega_i \text{ for } i = 1, \ldots, n.$$

Clearly the surjectivity of r follows from this.

This homomorphism is explicitly defined as follows. First let

$$c(a_i^{-1}) = -a_i\omega_i.$$

For a reduced word $\gamma = \gamma_1\gamma_2\cdots\gamma_n$, where γ_i is either a_i or a_i^{-1}, let

$$\begin{aligned} c(\gamma_1\cdots\gamma_n) &= \gamma_1\gamma_2\cdots\gamma_{n-1}c(\gamma_n) + \\ &\qquad \cdots + \gamma_1\gamma_2 c(\gamma_3) + \gamma_1 c(\gamma_2) + c(\gamma_1). \end{aligned}$$

The verification that c is actually a twisted homomorphism is left to the reader. □

Now from the surjectivity of r we get the following

Proposition 6.4 *Let Γ be a free group acting on a manifold M. Assume either of the followings*

(1) *Γ acts on M freely and properly.*

(2) *$M = \mathbf{S}^n$, Γ is a Kleinian group and $n \geq p > \delta(\Gamma) - 1$.*

Then we have $\dim\{H^1(\Gamma, \Omega^p(M))\} = +\infty$.

Proof. Since the dual of $\Omega^p(M)/K^p$ is $\mathcal{C}_p^\Gamma(M)$, it suffices to show that the dimension of the space $\mathcal{C}_p^\Gamma(M)$ is $+\infty$.

The case (1) follows from Proposition 1.6. Let us show the case (2). Suppose $n = p$. It is well known that $\delta(\Gamma) \leq n$. Therefore the proposition follows from Theorem 2.2. So suppose $n - 1 \geq p \geq \delta(\Gamma) - 1$. Consider the following diagram

$$\begin{array}{ccc} \mathcal{C}_{p+1}^\Gamma(\mathbf{S}^n) & \xrightarrow{d} & \mathcal{C}_p^\Gamma(\mathbf{S}^n) \\ L_{p+1}\downarrow & & \downarrow L_p \\ \mathcal{C}_{p+1}(\Gamma\setminus D_\Gamma) & \xrightarrow{d} & \mathcal{C}_p^\Gamma(\Gamma\setminus D_\Gamma) \end{array}$$

Surjectivity of L_{p+1} (Theorem 2.2) implies that

$$d\{\mathcal{C}_{p+1}(\Gamma\setminus D_\Gamma)\} \subset \mathrm{Im}(L_p).$$

But it is well known, easy to show, that $\dim\{d(\mathcal{C}_{p+1}(\Gamma\setminus D_\Gamma))\} = +\infty$. Therefore we have $\dim\{\mathcal{C}_p^\Gamma(\mathbf{S}^n)\} = +\infty$. □

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