Hokkaido Mathematical Journal Vol. 26 (1997) p. 163–175

Weighted inequalities for multilinear oscillatory singular integrals

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(Received July 12, 1995; Revised March 29, 1996)

Abstract. For a class of multilinear oscillatory singular integral operators T^A, we show the following weighted norm inequalities:

$$\int_{R^n} |T^A f(x)|^p w(x)dx \leq C \int_{R^n} |f(x)|^p w(x)dx, \quad 1 < p < \infty,$$

if $w \in A_p$ (Muckenhoupt weight class).

Key words: multilinear operator, oscillatory integral, A_p weight.

1. Introduction

A classical result due to Coifman and Fefferman [3] states that the Calderón-Zygmund singular integral operator T satisfies the following inequality for $w \in A_p$ with $1 < p < \infty$,

$$\int_{R^n} |Tf(x)|^p w(x) \leq C \int_{R^n} |f(x)|^p w(x), \tag{1.1}$$

where C is independent of f. A weight w in R^n will always be a non-negative locally integrable function.

In this paper, we consider a kind of multilinear oscillatory singular integral operators as follows.

$$T^A f(x) = p.v. \int_{R^n} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+m-1}} R_m(A; x, y) f(y)dy, \qquad n \geq 2, \tag{1.2}$$

where $m \geq 2$ is an integer, $P(x, y)$ is any real-valued polynomial defined on $R^n \times R^n$, Ω is homogeneous of degree zero and satisfies the moment condition

$$\int_{S^{n-1}} \theta^\nu \Omega(\theta)d\theta = 0, \quad |\nu| = m - 1,$$

1991 Mathematics Subject Classification : 42B20, 42B25.

[1]Research was supported by NNSF of China.

$R_m(A;x,y)$ denotes the m-th remainder of Taylor series of A at x expanded about y, more precisely,

$$R_m(A;x,y)=A(x)-\sum_{|\alpha|<m}\frac{1}{\alpha!}D^\alpha A(y)(x-y)^\alpha,$$

and $D^\alpha A\in \mathrm{BMO}(R^n)$ for all multi-indices of magnitude $|\alpha|=m-1$.

T^A is closely related to the multilinear operator which was first studied by Cohen [1], and then by Cohen and Gosselin [2]. The operator they studied is defined by

$$\tilde{T}^A f(x)=p.v.\int_{R^n}\frac{\Omega(x-y)}{|x-y|^{n+m-1}}R_m(A;x,y)f(y)dy. \tag{1.3}$$

Using the method of "good-λ" inequality controlled by the maximal function, Cohen and Gosselin [2] showed that if $\Omega\in \mathrm{Lip}_1(S^{n-1})$ and $w\in A_p$, then

$$\|\tilde{T}^A f\|_{p,w}\le C\sum_{|\alpha|=m-1}\|D^\alpha A\|_{\mathrm{BMO}}\|f\|_{p,w},\quad 1<p<\infty.$$

When Ω satisfies only a size condition, Hofmann [4] formulated a version of T1 theorem and proved that if $\Omega\in L^\infty(S^{n-1})$ and $w\in A_p$, then

$$\|\tilde{T}^A f\|_{p,w}\le C\sum_{|\alpha|=m-1}\|D^\alpha A\|_{\mathrm{BMO}}\|f\|_{p,w},\quad 1<p<\infty.$$

Unfortunately, the oscillatory factor $e^{iP(x,y)}$ prevents us from making use of Hofmann's technique to obtain the weighted norm inequality (1.1) for oscillatory integral operators T^A.

The purpose of this paper is to establish the weighted norm inequality of the form (1.1) for T^A by means of the interpolation theorem with change of measure [7].

Now, we state the result of this paper:

Theorem *Let T^A be defined as (1.2). If $\Omega\in L^\infty(S^{n-1})$, then for $w\in A_p$, $1<p<\infty$, we have*

$$\|T^A f\|_{p,w}\le C(n,p,A_p,\deg P)\sum_{|\alpha|=m-1}\|D^\alpha A\|_{\mathrm{BMO}}\|f\|_{p,w},$$

where the constant $C(n,p,A_p,\deg P)$ depends only on the dimension n, the *exponent p, the A_p constant of w and the total degree* $\deg P$ *of polynomial* $P(x,y)$.

2. Proof of Theorem

In order to prove the theorem, we will use some lemmas.

Lemma 1 (See [2]) *Let $b(x)$ be a function on R^n with m-th order derivatives in $L^q(R^n)$, $q>n$. Then*

$$|R_m(b;x,y)| \le C_{m,n}|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|I_x^y|}\int_{I_x^y} |D^\alpha b(z)|^q dz\right)^{1/q},$$

where I_x^y is the cube centered at x, with sides parallel to the axes and whose diameter is $2\sqrt{n}|x-y|$.

Lemma 2 *Let Ω be homogeneous of degree zero and belongs to $L^\infty(S^{n-1})$. Supppose that A has derivatives of order $m-1$ in* $\mathrm{BMO}(R^n)$, *and*

$$M_\Omega^A f(x) = \sup_{r>0} r^{-(n+m-1)} \int_{|x-y|<r} |\Omega(x-y)R_m(A;x,y)f(y)|dy.$$

If $w\in A_p$, $1<p<\infty$, then

$$\|M_\Omega^A f\|_{p,w} \le C \sum_{|\alpha|=m-1} \|D^\alpha A\|_{\mathrm{BMO}} \|f\|_{p,w}.$$

Proof. It suffices to prove the lemma for $\tilde{M}_\Omega^A$, a variant of M_Ω^A,

$$\tilde{M}_\Omega^A f(x) = \sup_{r>0} r^{-(n+m-1)} \int_{r/2<|x-y|<r} |\Omega(x-y)R_m(A;x,y)f(y)|dy.$$

For fixed $x\in R^n$, $r>0$, let $\tilde{Q}(x,r)$ be the cube centered at x and have side length $10\sqrt{n}r$. Set

$$\tilde{A}(y) = A(y) - \sum_{|\alpha|=m-1} \frac{1}{\alpha!} m_{\tilde{Q}(x,r)}(D^\alpha A) y^\alpha,$$

where $m_{\tilde{Q}(x,r)}(D^\alpha A)$ denotes the mean value of $D^\alpha A$ on $\tilde{Q}(x,r)$. By an observation in [2], we have

$$R_m(A;x,y) = R_m(\tilde{A};x,y).$$

So,

$$\begin{aligned}
&\tilde{M}_\Omega^A f(x)\\
&= \sup_{r>0} r^{-(n+m-1)} \int_{r/2<|x-y|<r} |\Omega(x-y) R_m(\tilde{A}; x, y) f(y)| dy\\
&\le \sup_{r>0} r^{-(n+m-1)} \int_{r/2<|x-y|<r} |\Omega(x-y) R_{m-1}(\tilde{A}; x, y) f(y)| dy\\
&\quad + \sup_{r>0} r^{-(n+m-1)} \int_{r/2<|x-y|<r} |\Omega(x-y)\\
&\qquad \sum_{|\alpha|=m-1} \frac{1}{\alpha!} D^\alpha \tilde{A}(y)(x-y)^\alpha || f(y)| dy\\
&= I + II.
\end{aligned}$$

Lemma 1 tells us that

$$\begin{aligned}
I &\le \sup_{r>0} r^{-n} \int_{r/2<|x-y|<r} |\Omega(x-y) f(y)| \frac{|R_{m-1}(\tilde{A}; x, y)|}{|x-y|^{m-1}} dy\\
&\le C \sup_{r>0} r^{-n} \int_{r/2<|x-y|<r} \Bigg[\sum_{|\alpha|=m-1} \frac{1}{|I_x^y|} \int_{I_x^y} |D^\alpha A(z)\\
&\qquad\qquad - m_{\tilde{Q}(x,r)}(D^\alpha A)|^q dz)^{1/q} \Bigg] |f(y)| dy\\
&\le C \sum_{|\alpha|=m-1} \|D^\alpha A\|_{\mathrm{BMO}} M f(x),
\end{aligned}$$

where $q > n$, Mf denotes the Hardy-Littlewood maximal function of f. For any t, $1 < t < \infty$, let $M_t f(x) = [M(|f|)^t(x)]^{1/t}$. By Hölder's inequality, we have

$$\begin{aligned}
II &\le C \sup_{r>0} \Bigg(r^{-n} \int_{r/2<|x-y|<r} \sum_{|\alpha|=m-1} |D^\alpha A(y)\\
&\qquad\qquad - m_{\tilde{Q}(x,r)}(D^\alpha A)|^{t'} dy \Bigg)^{1/t'}\\
&\quad \times \sup_{r>0} \Bigg(r^{-n} \int_{r/2<|x-y|<r} |f(y)|^t dy \Bigg)^{1/t}\\
&\le C \sum_{|\alpha|=m-1} \|D^\alpha A\|_{\mathrm{BMO}} M_t f(x),
\end{aligned}$$

where $1 < t' < \infty$ such that $1/t' + 1/t = 1$. Thus for any $1 < t < \infty$, we obtain

$$\tilde{M}_{\Omega}^{A} f(x) \leq C \sum_{|\alpha|=m-1} \|D^{\alpha} A\|_{\mathrm{BMO}} M_t f(x).$$

By the reverse Hölder's inequality, we know that $w \in A_p$ with $1 < p < \infty$ implies $w \in A_{p/t}$ for some t, $1 < t < p$. The well-known weighted norm inequality for Mf tells us that if $w \in A_p, 1 < p < \infty$, then

$$\begin{aligned}
&\left(\int_{R^n} |\tilde{M}_{\Omega}^{A} f(x)|^p w(x) dx\right)^{1/p} \\
&\quad \leq C \sum_{|\alpha|=m-1} \|D^{\alpha} A\|_{\mathrm{BMO}} \left(\int_{R^n} [M(|f|)^t(x)]^{p/t} w(x) dx\right)^{1/p} \\
&\quad \leq C \sum_{|\alpha|=m-1} \|D^{\alpha} A\|_{\mathrm{BMO}} \left(\int_{R^n} |f(x)|^p w(x) dx\right)^{1/p}.
\end{aligned}$$

This is the desired estimate. □

Lemma 3 *Let $K(x,y)$ be a distribution which agrees with a function away from the diagonal $\{x = y\}$ satisfying*

$$|K(x,y)| \leq \frac{|\Omega(x-y)|}{|x-y|^{n+m-1}} |R_m(A;x,y)|$$

and let Ω, A be the same as the assumption in Theorem. Suppose that the operator

$$Tf(x) = p.v. \int_{R^n} K(x,y) f(y) dy$$

is bounded on $L^p(w), 1 < p < \infty$, when $w \in A_p$. Then the truncated operator

$$T_0 f(x) = \int_{|x-y|\leq 1} K(x,y) f(y) dy$$

is also bounded on $L^p(w)$ with bound $C(\|T\| + \sum_{|\alpha|=m-1} \|D^{\alpha} A\|_{\mathrm{BMO}})$, where C is independent of T, and $\|T\|$ denotes the $L^p(w) \to L^p(w)$ operator norm of T.

Proof. If we can prove

$$\int_{|x-h|<1/4} |T_0 f(x)|^p w(x) dx \leq C \int_{|y-h|\leq 5/4} |f(y)|^p w(y) dy \qquad (2.1)$$

holds for all $h \in R^n$ with bound independent of h, then integrating the above inequality with respect to h yields that

$$\int_{R^n} |T_0 f(x)|^p w(x)dx \le C \int_{R^n} |f(y)|^p w(y)dy.$$

So it suffices to prove (2.1). For any fixed $h \in R^n$, we split f into three parts $f = f_1 + f_2 + f_3$, where

$$\begin{aligned} f_1(y) &= f(y)\chi_{\{|y-h|<1/2\}}(y), \\ f_2(y) &= f(y)\chi_{\{1/2\le|y-h|<5/4\}}(y), \end{aligned}$$

and

$$f_3(y) = f(y)\chi_{\{5/4\le|y-h|\}}(y).$$

Because $|x-h| < 1/4$ and $|y-h| < 1/2$ imply $|x-y| < 1$, it is obvious that $T_0 f_1(x) = Tf_1(x)$ when $|x-h| < 1/4$. In light of the boundedness of T on $L^p(w)$, we have

$$\begin{aligned} \int_{|x-h|<1/4} |T_0 f_1(x)|^p w(x)dx &= \int_{|x-h|<1/4} |Tf_1(x)|^p w(x)dx \\ &\le \|T\|^p \int_{R^n} |f_1(y)|^p w(y)dy = \|T\|^p \int_{|y-h|<1/2} |f(y)|^p dy \\ &\le \|T\|^p \int_{|y-h|<5/4} |f(y)|^p w(y)dy. \end{aligned}$$

If $1/2 \le |y-h| < 5/4$ and $|x-h| < 1/4$, then $|x-y| > 1/4$. Thus

$$\begin{aligned} |T_0 f_2(x)| &\le \int_{|x-y|\le 1} \frac{|\Omega(x-y)|}{|x-y|^{n+m-1}} |R_m(A;x,y) f_2(y)| dy \\ &\le C M_\Omega^A f_2(x). \end{aligned}$$

Lemma 2 tells us that

$$\begin{aligned} &\int_{|x-h|<1/4} |T_0 f_2(x)|^p w(x)dx \\ &\quad \le C \int_{R^n} |M_\Omega^A f_2(x)|^p w(x)dx \\ &\quad \le C \sum_{|\alpha|=m-1} \|D^\alpha A\|_{\mathrm{BMO}}^p \int_{|y-h|<5/4} |f(y)|^p w(y)dy. \end{aligned}$$

Note that $|x-h|<1/4$ and $|y-h|>5/4$ imply $|x-y|>1$. Clearly $T_0 f_3(x)=0$ when $|x-h|<1/4$. Thus we establish (2.1), and then complete the proof of Lemma 3. □

Now we turn our attention to proving Theorem.

We shall carry out the argument by a double induction on the degree in x and y of the polynomial . By the result of Hofmann [4], it is obvious that Theorem holds if the polynomial $P(x,y)$ depends only on x or only on y. Let k and l are two positive integer. Suppose that the polynomial $P(x,y)$ has degree k in x and l in y. We assume that Thoerem is true for all polynomials which are sums of monomials of degree less than k in x times monomials of any degree in y, together with monomials which are of degree k in x times monomials which are of degree less than l in y .

We proceed to the proof of the inductive step. Write

$$P(x,y)=\sum_{|\beta|=k,|\gamma|=l} a_{\beta\gamma}x^\beta y^\gamma + R(x,y),$$

where $R(x,y)$ satisfies the inductive hypothesis. By dilation- invariance, we may assume that $\sum_{|\beta|=k,|\gamma|=l}|a_{\beta\gamma}|=1$. Decompose T^A as

$$\begin{aligned}
&T^A f(x)\\
&\le \int_{|x-y|\le 1} e^{iP(x,y)}\frac{\Omega(x-y)}{|x-y|^{n+m-1}}R_m(A;x,y)f(y)dy\\
&\quad+\sum_{j=1}^{\infty}\int_{2^{j-1}<|x-y|\le 2^j} e^{iP(x,y)}\frac{\Omega(x-y)}{|x-y|^{n+m-1}}R_m(A;x,y)f(y)dy\\
&=T_0^A f(x)+\sum_{j=1}^{\infty}T_j^A f(x).
\end{aligned}$$

For $h\in R^n$, rewriting $P(x,y)$ as

$$P(x,y)=\sum_{|\beta|=k,|\gamma|=l} a_{\beta\gamma}(x-h)^\beta (y-h)^\gamma + R(x,y,h),$$

where the inductive hypothesis applies to $R(x,y,h)$. We split $T_0^A f$ into

$$\begin{aligned}
T_0^A f(x)\le \int_{|x-y|\le 1} & e^{i[R(x,y,h)+\sum a_{\beta\gamma}(y-h)^{\beta+\gamma}]}\\
&\frac{\Omega(x-y)}{|x-y|^{n+m-1}}R_m(A;x,y)f(y)dy
\end{aligned}$$

$$
\begin{aligned}
&+\int_{|x-y|\le 1}\{e^{iP(x,y)}-e^{i[R(x,y,h)+\sum a_{\beta\gamma}(y-h)^{\beta+\gamma}]}\}\\
&\qquad\frac{\Omega(x-y)}{|x-y|^{n+m-1}}R_m(A;x,y)f(y)dy\\
&= T_{01}^A f(x)+T_{02}^A f(x).
\end{aligned}
$$

By the inductive hypothesis and Lemma 3, we get

$$
\begin{aligned}
&\int_{R^n}|T_{01}^A f(x)|^p w(x)dx\\
&\qquad\le C\sum_{|\alpha|=m-1}\|D^\alpha A\|_{\mathrm{BMO}}^p\int_{R^n}|f(x)|^p w(x)dx. \qquad (2.2)
\end{aligned}
$$

When $|x-h|<1/4$ and $|x-y|\le 1$, it is easy to see that

$$
\begin{aligned}
&|e^{iP(x,y)}-e^{i[R(x,y,h)+\sum a_{\beta\gamma}(y-h)^{\beta+\gamma}]}|\\
&\qquad\le C\sum_{|\beta|=k,|\gamma|=l}|a_{\beta\gamma}||x-y|=C|x-y|.
\end{aligned}
$$

If we denote $f_h(y)=f(y)\chi_{\{|y-h|\le 5/4\}}(y)$, then $T_{02}^A f(x)=T_{02}^A f_h(x)$ when $|x-h|<1/4$. Thus

$$
\begin{aligned}
|T_{02}^A f(x)| &\le C\int_{|x-y|\le 1}\frac{|\Omega(x-y)|}{|x-y|^{n+m-2}}|R_m(A;x,y)f_h(y)|dy\\
&\le CM_\Omega^A f_h(x).
\end{aligned}
$$

It follows from Lemma 2 that

$$
\begin{aligned}
&\int_{|x-h|<1/4}|T_{02}^A f(x)|^p w(x)dx\\
&\qquad\le C\sum_{|\alpha|=m-1}\|D^\alpha A\|_{\mathrm{BMO}}^p\int_{|y-h|\le 5/4}|f(y)|^p w(y)dy.
\end{aligned}
$$

Integrating the above inequality with respect to h yields that

$$
\begin{aligned}
&\int_{R^n}|T_{02}^A f(x)|^p w(x)dx\\
&\qquad\le C\sum_{|\alpha|=m-1}\|D^\alpha A\|_{\mathrm{BMO}}^p\int_{R^n}|f(x)|^p w(x)dx. \qquad (2.3)
\end{aligned}
$$

Combining (2.2) with (2.3), we get the desired estimate for $T_0^A f$.

Now we consider $T_j^A f$, $j \geq 1$. Obviously

$$|T_j^A f(x)| \leq \int_{2^{j-1}<|x-y|\leq 2^j} \frac{|\Omega(x-y)|}{|x-y|^{n+m-1}} |R_m(A;x,y)f(y)|dy$$
$$\leq CM_\Omega^A f(x),$$

where C is independent of j. When $w \in A_p$, $1 < p < \infty$, by the reverse Hölder's inequality again, there exists an $\epsilon > 0$ such that $w^{1+\epsilon} \in A_p$. It follows from Lemma 2 that

$$\|T_j^A f\|_{p,w^{1+\epsilon}} \leq C \sum_{|\alpha|=m-1} \|D^\alpha A\|_{\mathrm{BMO}} \|f\|_{p,w^{1+\epsilon}}, \tag{2.4}$$

where C is independent of j. If we can obtain a refined L^p estimate for $T_j^A f$ as follows.

$$\|T_j^A f\|_p \leq C 2^{-j\delta} \sum_{|\alpha|=m-1} \|D^\alpha A\|_{\mathrm{BMO}} \|f\|_p, \tag{2.5}$$

where $\delta > 0$, and C depends only on the total degree of $P(x,y)$, then when $w \in A_p$, $1 < p < \infty$, by the interpolation theorem with change of measure [7] between (2.4) and (2.5), we get

$$\|T_j^A f\|_{p,w} \leq C 2^{-j\theta\delta} \sum_{|\alpha|=m-1} \|D^\alpha A\|_{\mathrm{BMO}} \|f\|_{p,w}, \tag{2.6}$$

with $0 < \theta < 1$. Summing the above inequality over all $j \geq 1$, together with the estimate for $T_0^A f$ gives that

$$\|T^A f\|_{p,w} \leq C \sum_{|\alpha|=m-1} \|D^\alpha A\|_{\mathrm{BMO}} \|f\|_{p,w},$$

where C depends only on n, p, A_p constant of w and the total degree $\deg P$.

Now the proof of Theorem reduces to prove the estimate (2.5). By Lemma 2, it is easy to see that

$$\|T_j^A f\|_p \leq C \sum_{|\alpha|=m-1} \|D^\alpha A\|_{\mathrm{BMO}} \|f\|_p, \quad 1 < p < \infty.$$

So, in order to prove (2.5), we only to prove

$$\|T_j^A f\|_2 \leq C 2^{-j\delta} \sum_{|\alpha|=m-1} \|D^\alpha A\|_{\mathrm{BMO}} \|f\|_2. \tag{2.7}$$

To prove (2.7), we may assume $\sum_{|\alpha|=m-1} \|D^\alpha A\|_{\mathrm{BMO}} = 1$. Define

$$\tilde{T}_j^A f(x) = \int_{1<|x-y|\le 2} e^{iP(2^{j-1}x, 2^{j-1}y)} \frac{\Omega(x-y)}{|x-y|^{n+m-1}} R_m(A;x,y) f(y)dy.$$

By dilation invariance, it is enough to prove that

$$\|\tilde{T}_j^A f\|_2 \le C 2^{-\varepsilon j} \|f\|_2. \tag{2.8}$$

Decompose $\mathbf{R}^n$ into $\mathbf{R}^n = \bigcup I_i$, where I_i is a cube with side length 1, and the cubes have disjoint interiors. Set $f_i = f\chi_{I_i}$. Since the support of $\tilde{T}_j^A f_i$ is contained in a fixed multiple of I_i, so that the supports of the various terms $\tilde{T}_j^A f_i$ have bounded overlaps. Thus we have the "almost orthogonality" property

$$\|\tilde{T}_j^A f\|_2^2 \le C \sum_i \|\tilde{T}_j^A f_i\|_2^2,$$

and therefore it suffices to show

$$\|\tilde{T}_j^A f_i\|_2^2 \le C 2^{-\varepsilon j} \|f_i\|_2^2. \tag{2.9}$$

For fixed i, denote $\tilde{I}_i = 100nI_i$. Let $\phi_i(x) \in C_0^\infty(\mathbf{R}^n)$ such that $0 \le \phi_i \le 1$, ϕ_i is identically one on $10\sqrt{n}I_i$ and vanishes outside of $50\sqrt{n}I_i$, $\|D^\gamma \phi_i\|_\infty \le C_\gamma$ for all multi-index γ. Let x_0 be a point on the boundary of $80\sqrt{n}I_i$. Denote

$$A^{\phi_i}(y) = R_{m-1}(A(\cdot) - \sum_{|\alpha|=m-1} \frac{1}{\alpha!} m_{\tilde{I}_i}(D^\alpha A)(\cdot)^\alpha; y, x_0)\phi_i(y)$$

and for multi-index α, define

$$\tilde{T}_j^{A,\alpha} f(x) = \int_{1<|x-y|\le 2} e^{iP(2^{j-1}x, 2^{j-1}y)} \frac{\Omega(x-y)}{|x-y|^{n+m-1}} (x-y)^\alpha f(y)dy.$$

It is easy to see that

$$\tilde{T}_j^A f_i(x) = \int_{1<|x-y|\le 2} e^{iP(2^{j-1}x, 2^{j-1}y)} \frac{\Omega(x-y)}{|x-y|^{n+m-1}} R_m(A^{\phi_i};x,y) f_i(y)dy$$

$$
\begin{aligned}
&= A^{\phi_i}(x)\tilde{T}_j^{A,0} f_i(x) - \sum_{|\alpha|<m-1} \frac{1}{\alpha!}\tilde{T}_j^{A,\alpha}(D^\alpha A^{\phi_i} f_i)(x) \\
&\quad - \sum_{|\alpha|=m-1} \frac{1}{\alpha!}\tilde{T}_j^{A,\alpha}(D^\alpha A^{\phi_i} f_i)(x) \\
&= \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}
$$

Before we estimate these terms, let us state a lemma.

Lemma 4 *There exists a positive constant $\delta = \delta(n, \deg P)$ such that for any $j \geq 1$ and multi-index α,*

$$
\begin{aligned}
&\left\| \int_{2^{j-1}\leq|x-y|<2^j} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+m-1}}(x-y)^\alpha f(y)dy \right\|_p \\
&\qquad \leq C2^{-(\delta+m-|\alpha|)j}\|f\|_p, \quad 1<p<\infty,
\end{aligned}
$$

where constant C is independent of j, f and coefficients of $P(x,y)$.

Recall that $P(x,y) = \sum_{|\beta|\leq k, |\gamma|\leq l} a_{\beta\gamma}x^\beta y^\gamma$ and $\sum_{|\beta|=k,|\gamma|=l}|a_{\beta\gamma}| = 1$. Lemma 4 can be proved by an argument used in [5]. We omit the details here.

We return to the estimates of I, II and III. Note that for multi-index β, $|\beta| < m-1$,

$$
\begin{aligned}
D^\beta A^{\phi_i}(y) = \sum_{\beta=\mu+\nu} & C_{\mu,\nu} R_{m-|\mu|-1}(D^\mu(A(\cdot) \\
& - \sum_{|\alpha|=m-1} \frac{1}{\alpha!} m_{\tilde{I}_i}(D^\alpha A)(\cdot)^\alpha); y, x_0) \\
& \times D^\nu \phi_i(y).
\end{aligned}
$$

Since that supp $\phi_i \subset 50\sqrt{n}I_i$, by Lemma 1, we have

$$
\begin{aligned}
|D^\beta A^{\phi_i}(y)| &\leq C \sum_{|\alpha|=m-1} \left(\frac{1}{|I_{x_0}^y|} \int_{I_{x_0}^y} |D^\alpha A(z) - m_{\tilde{I}_i}(D^\alpha A)|^t dz \right)^{1/t} \\
&\leq C,
\end{aligned}
$$

where $t > n$. Thus, it follows from Lemma 4 that

$$
\|\mathrm{I}\|_2 \leq \|A^{\phi_i}\|_\infty \|\tilde{T}_j^{A,0} f_i\|_2 \leq C2^{-\delta j}\|f_i\|_2.
$$

Similarly, we have

$$\|\mathrm{II}\|_2 \le C2^{-\delta j}\|f_i\|_2.$$

It remains to estimate the third term III. Note that for any $0<\gamma<n$,

$$\begin{aligned} |\tilde{T}_j^{A,\alpha} f(x)| &\le C\int_{1<|x-y|\le 2}|\Omega(x-y)f(y)|dy \\ &\le C_\gamma\|\Omega\|_{L^q(S^{n-1})}\left(\int_{1<|x-y|\le 2}\frac{|f(y)|^{q'}}{|x-y|^{n-\gamma}}dy\right)^{1/q'} \\ &\le C_\gamma\|\Omega\|_{L^q(S^{n-1})}[I_\gamma(|f|^{q'})(x)]^{1/q'}, \end{aligned}$$

where I_γ denotes the usual fractional integral of order γ. If $p>q'$ and $\sigma>0$, we take a γ such that $0<\gamma<n/p$, and $1/(p+\sigma)=1/p-\gamma/n$. By the Hardy-Littlewood-Sobolev theorem [6], we get

$$\|\tilde{T}_j^{A,\alpha} f\|_{p+\sigma}\le C\|\Omega\|_{L^q(S^{n-1})}\|f\|_p,\quad p>q',\quad \sigma>0.$$

By the last inequality and Lemma 4, an interpolation will give

$$\|\tilde{T}_j^{A,\alpha} f\|_{p+\sigma}\le C2^{-\tilde{\sigma}j}\|f\|_p,\quad 1<p<\infty,\quad \sigma>0, \tag{2.10}$$

where $\tilde{\sigma}$ is a positive constant. On the other hand, if $|\beta|=m-1$, then,

$$\begin{aligned} D^\beta A^{\phi_i}(y) = &\sum_{\beta=\mu+\nu,|\mu|<m-1} C_{\mu,\nu}R_{m-1-|\mu|}(D^\mu(A(\cdot) \\ &-\sum_{|\alpha|=m-1}\frac{1}{\alpha!}m_{\tilde{I}_i}(D^\alpha A)(\cdot)^\alpha);y,x_0) \\ &\times D^\nu\phi_i(y) \\ &+\sum_{|\alpha|=m-1}(D^\alpha A(y)-m_{\tilde{I}_i}(D^\alpha A))\phi_i(y). \end{aligned}$$

Thus, it follows that

$$|D^\beta A^{\phi_i}(y)|\le C\Big(1+\sum_{|\alpha|=m-1}|D^\alpha A(y)-m_{\tilde{I}_i}(D^\alpha A)|\Big),$$

and this shows that for any $t>1$,

$$\|D^\beta A^{\phi_i}\|_t\le C_t.$$

Combining the above inequality and (2.10), we obtain

$$\begin{aligned}\|\mathrm{III}\|_2 &\le C2^{-\tilde{\delta}j} \sum_{|\alpha|=m-1} \|D^\alpha A^{\phi_i} f_i\|_{2-\sigma} \\ &\le C2^{-\tilde{\delta}j} \sum_{|\alpha|=m-1} \|D^\alpha A^{\phi_i}\|_t \|f_i\|_2 \\ &\le C2^{-\tilde{\delta}j} \|f_i\|_2,\end{aligned}$$

where we choose $\sigma > 0$ and $1 < t < \infty$ such that $1/2 + 1/t = 1/(2-\sigma)$.

All above estimates imply that (2.7) is true.

Acknowledgment Authors are very grateful to Dr. Hu for providing us the proof of estimate (2.7).

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