Hokkaido Mathematical Journal Vol. 26 (1997) p. 141–155

Automorphisms and conjugate connections

Mitsuhiro ITOH

(Received October 30, 1995)

Abstract. We study how the automorphism group of a Lie group G acts on the space of gauge-equivalence classes of connections on a principal G-bundle P provided P is reducible to H-subbundles. The action is investigated in terms of conjugate connections and holonomy groups.

Key words: conjugate connection, automorphism group, holonomy group.

1. Introduction

Let P be a principal bundle over a manifold M with a structure group G.

We consider the situation that the structure group G of the bundle P happens to be reduced to a closed subgroup H.

Let σ be an automorphism of G leaving fixed all elements of H.

It turns out then that σ induces via the subgroup reduction a transformation of the space of connections on P, called a conjugation, a generalization of the notion of conjugate affine connection in affine differential geometry ([7]). The notion of conjugate connection in a principal bundle is due to S. Kobayashi and E. Shinozaki ([5]).

It is a principal question how this conjugation relates to the gauge theory of connections on a principal bundle.

S. Kobayashi and Shinozaki gave in [5] a substantial answer to this question as follows.

Theorem ([5]) *Let* $\mathrm{Aut}(G,H)$ *be the group of automorphisms of* G *leaving fixed each element of* H. *Let* $\mathcal{G}(P)$ *be the group of gauge transformations of* P, *and let* $\mathcal{C}(P)$ *be the space of connections on* P. *Then*

(i) *the group* $\mathrm{Aut}(G,H)$ *acts on* P, $\mathcal{G}(P)$ *and* $\mathcal{C}(P)$ *in a natural way so that it induces an action on* $\mathcal{C}(P)/\mathcal{G}(P)$,

and

(ii) *for a fixed Riemannian metric on an oriented* M *and for an au-*

1991 Mathematics Subject Classification : 53C05, 58E15.

tomorphism invariant inner product of $\mathbf{g}$, *the Lie algebra of* G, $\mathrm{Aut}(G, H)$ *acts on the space of Yang-Mills connections and on the moduli space* $\mathcal{M}(P)$ *of Yang-Mills connections.*

The subject of this article is to understand in a more definite way the action of $\mathrm{Aut}(G, H)$ on $\mathcal{C}(P)/\mathcal{G}(P)$ and on $\mathcal{M}(P)$.

Let $\mathrm{Int}(G, H)$ be the subgroup of $\mathrm{Aut}(G, H)$ consisting of inner automorphisms. Then, as will be shown, an inner automorphism acts on each connection as a constant gauge transformation so that the action of $\mathrm{Aut}(G, H)$ restricted to $\mathrm{Int}(G, H)$ on the space $\mathcal{C}(P)/\mathcal{G}(P)$ turns out to be trivial. Namely,

Theorem 1 *The action of* $\mathrm{Aut}(G, H)$ *on* $\mathcal{C}(P)/\mathcal{G}(P)$ *and on* $\mathcal{M}(P)$ *induces an action of the group of outer automorphisms* $\mathrm{Aut}(G, H)/\mathrm{Int}(G, H)$ *on* $\mathcal{C}(P)/\mathcal{G}(P)$ *and on* $\mathcal{M}(P)$.

Furthermore we have

Theorem 2 *The action of* $\mathrm{Aut}(G, H)/\mathrm{Int}(G, H)$ *restricted to* $\mathcal{C}^*(P)/\mathcal{G}(P)$ *and on* $\mathcal{M}^*(P)$ *is free. Here* $\mathcal{C}^*(P)$ *denotes the space of connections on* P *whose holonomy group is* G, *and* $\mathcal{M}^*(P) = \mathcal{M}(P) \cap (\mathcal{C}^*(P)/\mathcal{G}(P))$.

In several cases, a bundle P has non-equivalent reductions to H-subbundles. If P admits, for instance, two non-equivalent H-subbundles Q_1, Q_2, then from Theorem 1 the free group generated by two copies of the group $\mathrm{Aut}(G, H)/\mathrm{Int}(G, H)$ inherits an action on the space $\mathcal{C}(P)/\mathcal{G}(P)$. It is shown in Proposition 6 in section 5 that the action is not effective and the quotient of this free group by a certain normal subgroup acts effectively on $\mathcal{C}(P)/\mathcal{G}(P)$ and freely on $\mathcal{C}^*(P)/\mathcal{G}(P)$.

2. Conjugate connections

If the structure group G of P is reducible to a closed subgroup H, then there exist an open covering $\{U_i\}$ of M giving local trivializations of P and sections $\{s_i : U_i \longrightarrow P\}$ such that the transition functions a_{ij} given by

$$s_j(x) = s_i(x)\, a_{ij}(x), \quad x \in U_i \cap U_j \tag{1}$$

take values in $H \subset G$ (refer to Proposition 5.3, p. 52 in [4]).

Note In the principal bundle P the subset $\bigcup_{x \in M}\{s_i(x)a;\ a \in H\}$ defines a subbundle Q whose structure group is H. The fibre of Q over x is $\{s_i(x)a;\ a \in H\}$, while the fibre of P is $\{s_i(x)a; a \in G\}$.

The actions of $\mathrm{Aut}(G, H)$ on P, $\mathcal{G}(P)$ and on $\mathcal{C}(P)$ are defined by making use of the sections $\{s_i : U_i \longrightarrow P\}$ and the transition functions $\{a_{ij} : U_i \cap U_j \longrightarrow H\}$.

The formulations we adopt here were first given in [5]. See Remark in §2 in [5].

Let $\sigma \in \mathrm{Aut}(G, H)$. Then σ induces a transformation of P, $\sigma_Q : P \longrightarrow P$, satisfying

$$\sigma_Q \circ R_a = R_{\sigma(a)} \circ \sigma_Q, \quad a \in G \tag{2}$$

We write an arbitrary u in P as $u = s_i(x)\ a$ for some s_i and $a \in G$ and define

$$\sigma_Q(u) = s_i(x)\ \sigma(a) \tag{3}$$

Suppose that u has another representation $u = s_j(x)\ a'$. Then from (1) we have $a_{ij}(x)\ a' = a$ and

$$s_j(x)\ \sigma(a') = s_i(x)\ a_{ij}(x)\ \sigma(a') = s_i(x)\ \sigma(a) \tag{4}$$

so that the definition of σ_Q does not depend on any choice of local trivializing sections s_i.

Note that σ_Q leaves fixed all points of Q.

A gauge transformation of P is a bundle automorphism of P, namely a diffeomorphism of P commuting with the right action R_a and inducing the identity transformation id_M of the base manifold M.

The group of automorphisms $\mathrm{Aut}(G, H)$ induces the action on $\mathcal{G}(P)$ in the following way

$$g \in \mathcal{G}(P) \longmapsto g^\sigma \in \mathcal{G}(P) \tag{5}$$

where

$$g^\sigma(u) = (\sigma_Q \circ g \circ (\sigma_Q)^{-1})(u), \quad u \in P \tag{6}$$

Obviously, g^σ is a diffeomorphism of P commuting with the right action. It induces id_M on M.

The action of $\mathrm{Aut}(G, H)$ on the space of connections is defined by

$$\omega^\sigma = ((\sigma_Q)^{-1})^*(\sigma \circ \omega) \tag{7}$$

Note $(\sigma_Q)^{-1} = (\sigma^{-1})_Q$

The right hand side needs to be explained. $\sigma \circ \omega$ is a $\mathbf{g}$-valued 1-form on P, composed of ω and $\sigma \in \mathrm{Aut}(\mathbf{g})$.

$((\sigma_Q)^{-1})^*$ is the pull-back of 1-forms on P induced by the transformation $(\sigma_Q)^{-1}$ of P.

To see that ω^σ is a connection we verify

$$\omega^\sigma(X^\sharp) = X, \quad X \in \mathbf{g} \tag{8}$$

where $X^\sharp$ is the fundamental vector field on P induced by the right action $R_{a(t)}$, $a(t) = \exp tX$,
and

$$R_a^*\, \omega^\sigma = Ad(a^{-1})\omega^\sigma, \quad a \in G \tag{9}$$

To simplify the argument we set $\tau = \sigma^{-1}$.

Since at $u \in P$ $X_u^\sharp = \frac{d}{dt}(u \exp tX)|_{t=0}$ it follows from $\tau_Q(ua) = \tau_Q(u)\, \tau(a)$ that $(\tau_Q)_*(X_u^\sharp) = (\tau(X))^\sharp_{\tau_Q(u)}$. So at $\tau_Q(u)$ $(\tau_Q^*\omega)(X^\sharp) = \tau(X)$ and hence $\omega^\sigma(X^\sharp) = \sigma(\tau X) = X$.

To show

$$(R_a^*\, \omega^\sigma)(Y) = Ad(a^{-1})\omega^\sigma(Y), \quad Y \in T_uP \tag{10}$$

we have $\tau_Q \circ R_a = R_{\tau(a)} \circ \tau_Q$ and hence

$$(R_{\tau(a)})_*((\tau_Q)_*Y) = (\tau_Q)_*((R_a)_*(Y)) \tag{11}$$

so that

$$\begin{aligned}(R_a^*\tau_Q^*\omega)(Y) &= \omega((\tau_Q)_*(R_a)_*(Y)) = \omega((R_{\tau(a)})_*(\tau_Q)_*Y) \\ &= Ad((\tau(a))^{-1})\omega((\tau_Q)_*Y)\end{aligned} \tag{12}$$

Since $\sigma((\tau(a))^{-1}) = (\sigma\tau(a))^{-1} = a^{-1}$

$$\begin{aligned}R_a^*\omega^\sigma(Y) &= \sigma(Ad((\tau(a))^{-1})\omega((\tau_Q)_*(Y)) \\ &= Ad(a^{-1})\sigma(\omega((\tau_Q)_*Y) = Ad(a^{-1})\omega^\sigma(Y).\end{aligned} \tag{13}$$

Definition We call ω^σ the σ-conjugate of ω for $\sigma \in \mathrm{Aut}(G, H)$.

On the H-subbundle Q a connection and a gauge transformation are naturally regarded as ones on P. To see this we represent connections on

P and gauge transformations of P in terms of local data on M by applying the local sections $s_i : U_i \longrightarrow P$.

In fact, let ω be a connection on P. Then ω has a local representation

$$\omega = \{\omega_i\} \tag{14}$$

where each $\omega_i = s_i^*\omega$ is a $\mathbf{g}$-valued 1-form defined over $U_i \subset M$, the pull-back of ω by the section s_i.

Over $U_i \cap U_j$ they satisfy the compatibility condition

$$\omega_j = a_{ij}^{-1}\omega_i a_{ij} + a_{ij}^{-1} da_{ij}. \tag{15}$$

As a routine business shows, a family of $\mathbf{g}$-valued 1-forms defined over each local trivializing neighborhood U_i satisfying the compatibility condition yields conversely a connection on P whose pull-back by each s_i is again itself.

The σ-conjugate ω^σ is represented

$$\omega^\sigma = \{\sigma(\omega_i)\}, \tag{16}$$

since $\sigma_Q(s_i(x)) = s_i(x),\ x \in U_i$.

Let ω be a connection on Q. Since each s_i may be a section of Q, each local representative of $\omega = \{\omega_i\}$, $\omega_i = s_i^*(\omega)$, is now a $\mathbf{h}$-valued 1-form. $\{\omega_i\}$ satisfies the compatibility condition (15) so that we can regard it as a connection on P, whose σ-conjugate ω^σ is ω for any $\sigma \in \mathrm{Aut}(G, H)$.

This means that the set $\mathcal{C}(Q)$ of all connections on the H-subbundle Q is a fixed-point set under the $\mathrm{Aut}(G, H)$-action when regarded as connections on P;

$$\mathcal{C}(Q) \subset \mathcal{C}^{\mathrm{Aut}(G,H)}(P). \tag{17}$$

Remark. If $H = \{a \in G; \sigma(a) = a, \sigma \in \mathrm{Aut}(g, H)\}$, then

$$\mathcal{C}(Q) = \mathcal{C}^{\mathrm{Aut}(G,H)}(P). \tag{18}$$

A gauge transformation of P can be also represented in a local form;

$$g = \{g_i; U_i \longrightarrow G\} \tag{19}$$

satisfying the compatibility condition

$$g_j(x) = a_{ij}^{-1}(x)\ g_i(x)\ a_{ij}^{-1}, \quad x \in U_i \cap U_j. \tag{20}$$

Here g_i is given by

$$g(s_i(x)) = s_i(x)g_i(x), \quad x \in U_i \tag{21}$$

From the definition (6) the σ-conjugate g^σ of a $g \in \mathcal{G}(P)$ is then represented as

$$g^\sigma = \{\sigma(g_i); U_i \longrightarrow G\} \tag{22}$$

A gauge transformation of Q is regarded as a $g \in \mathcal{G}(P)$ by requiring $g(ua) = g(u)a$, $u \in Q, a \in G$.

So, a gauge transformation g of Q satisfies from (22) $g^\sigma = g$ for all $\sigma \in \mathrm{Aut}(G, H)$, since such a g has a local form $\{g_i : U_i \longrightarrow G\}$ taking values in H. On the other hand suppose that $H = \{a \in G; \sigma(a) = a, \sigma \in \mathrm{Aut}(G, H)\}$. Then, $g \in \mathcal{G}(P)$ is fixed under the $\mathrm{Aut}(G, H)$-action if and only if g is a gauge transformation of Q.

We will briefly understand in Theorem given by Kobayashi and Shinozaki how the group $\mathrm{Aut}(G, H)$ acts on $\mathcal{C}(P)/\mathcal{G}(P)$ and on the moduli space $\mathcal{M}(P)$.

We say that connections $\omega_1, \omega_2 \in \mathcal{C}(P)$ are gauge equivalent if $\omega_2 = g^*\omega_1$ for some $g \in \mathcal{G}(P)$.

We apply $((\sigma_Q)^{-1})^*$ and $\sigma \in \mathrm{Aut}(\mathbf{g})$ to $\omega_2 = g^*\omega_1$ to have

$$\omega_2^\sigma = ((\sigma_Q)^{-1})^* g^*(\sigma(\omega_1)) = ((\sigma_Q)^{-1})^* g^* \sigma_Q^* ((\sigma_Q^{-1})^*(\sigma(\omega_1)) \tag{23}$$

which is just $(g^\sigma)^*(\omega_1^\sigma)$. So,

Proposition 1 *Assume that ω_1, ω_2 are gauge equivalent connections on P, namely, $\omega_2 = g^*(\omega_1)$, $g \in \mathcal{G}(P)$. Then, their conjugate ω_1^σ and ω_2^σ, $\sigma \in \mathrm{Aut}(G, H)$, are gauge equivalent under $g^\sigma \in \mathcal{G}(P)$.*

Therefore, the automorphism group $\mathrm{Aut}(G, H)$ acts naturally on the space of gauge equivalence classes of connections on P;

$$\begin{aligned} \mathrm{Aut}(G, H) \times \mathcal{C}(P)/\mathcal{G}(P) &\longrightarrow \mathcal{C}(P)/\mathcal{G}(P) \\ (\sigma, [\omega]) &\longmapsto [\omega^\sigma] \end{aligned} \tag{24}$$

The first part of Theorem ([5]) follows from this proposition.

Let $\Omega = d\omega + \frac{1}{2}[\omega \wedge \omega]$ be the curvature form of a connection ω on P.

If we consider the connection as a family $\omega = \{\omega_i\}$ of local $\mathbf{g}$-valued 1-forms, then the curvature form $\Omega = \{\Omega_i\}$, $\Omega_i = d\omega_i + \frac{1}{2}[\omega_i \wedge \omega_i]$, is a

$\mathbf{g}_P$-valued, globally defined 2-form on M. Here $\mathbf{g}_P = P \times_{Ad} \mathbf{g}$ is the adjoint bundle of P.

We provide a Riemannian metric on an oriented manifold M and an automorphism invariant positive definite inner product $\langle\ .\ ,\ .\ \rangle$ on $\mathbf{g}$. The existence of such an inner product is guaranteed if G is a compact Lie group.

The Yang-Mills functional $\mathcal{YM}(\omega)$ is defined by the integral over M of an n-form $\langle \Omega \wedge *\Omega \rangle$ ($n = \dim M$), which is over U_i given by $\langle \Omega_i \wedge *\Omega_i \rangle$.

Since over $U_i \cap U_j$

$$\Omega_j = a_{ij}^{-1} \Omega_i\ a_{ij}^{-1} \tag{25}$$

and the inner product is automorphism invariant, the two n-forms $\langle \Omega_i \wedge *\Omega_i \rangle$ and $\langle \Omega_j \wedge *\Omega_j \rangle$ coincide over $U_i \cap U_j$ so that the functional $\mathcal{YM}$ is well defined.

Let ω^σ be the conjugate of ω. Then it has the curvature form Ω^σ;

$$\Omega^\sigma = ((\sigma_Q^{-1})^*(\sigma(\Omega)) \tag{26}$$

whose local representation is $(\Omega^\sigma)_i = \sigma(\Omega_i)$. It follows that

$$\langle (\Omega^\sigma)_i \wedge *(\Omega^\sigma)_i \rangle = \langle \Omega_i \wedge *\Omega_i \rangle \tag{27}$$

which implies that $\mathcal{YM}(\omega^\sigma) = \mathcal{YM}(\omega)$, $\sigma \in \mathrm{Aut}(G, H)$, that is, the Yang-Mills functional $\mathcal{YM}$ is $\mathrm{Aut}(G, H)$-invariant.

Therefore, if ω is a Yang-Mills connection, so is ω^σ.

Remarks. (i) A Yang-Mills connection ω is a solution of the Yang-Mills equation

$$d_\omega(*\ \Omega) = 0 \tag{28}$$

where $\Omega \in \Gamma(\mathbf{g}_P \otimes \Lambda_M^2)$ and d_ω is the covariant exterior derivative associated to the connection ω. It is obviously seen that the Yang-Mills equation is $\mathrm{Aut}(G, H)$-invariant, that is,

$$d_{\omega^\sigma}(*\ \Omega^\sigma) = (d_\omega(*\ \Omega))^\sigma. \tag{29}$$

So we assert again that the moduli space $\mathcal{M}(P)$ is invariant under the action.

(ii) Suppose a base manifold M is an oriented Riemannian 4-manifold. Since $\Omega_i^\sigma = \sigma(\Omega_i)$, the curvature form $\{\Omega_i\}$ is self-dual (or anti-self-dual) if and only if so is the curvature form of the σ-conjugate connection. Thus, when the structure group G of P is reducible to a closed subgroup $H \subset G$, $\mathrm{Aut}(G, H)$ acts naturally on the moduli space of self-dual connections on P (or of anti-self-dual connections).

(iii) Generalizations of the notion of self-duality are given for a Kähler manifold and for a quaternionic Kähler manifold (for instances see [8] and [6]). We observe that over these manifolds the group $\mathrm{Aut}(G, H)$ acts on the moduli space of respective self-dual connections on P provided the structure group G of P is reducible to H

(iv) Similarly as the case of connections the automorphism group $\mathrm{Aut}(G, H)$ enjoys an action on the space of Yang-Mills-Higgs fields (ω, Φ) (or monopoles) modulo the gauge transformations of P. Here a Higgs field Φ is a section of $\mathbf{g}_P$ (for Yang-Mills-Higgs fields refer for examples to [2] and [3]). The covariant derivatives of Φ satisfy in terms of the action of $\mathrm{Aut}(G, H)$

$$\nabla^\sigma \Phi^\sigma = (\nabla \Phi)^\sigma \tag{30}$$

Here ∇ and ∇^σ mean the covariant derivatives with respect to ω and ω^σ, respectively.

3. Inner automorphisms

Before stating the $\mathrm{Aut}(G, H)$-action on $\mathcal{C}(P)/\mathcal{G}(P)$ in terms of holonomy groups, we make a brief observation that any inner automorphism $\sigma \in \mathrm{Aut}(G, H)$ induces naturally a constant gauge transformation of P.

In fact, if there is an element $b \in G$ such that

$$\sigma(a) = b\ a\ b^{-1}, \quad a \in G, \tag{31}$$

then $R_b \circ \sigma_Q$ gives a gauge transformation, because

$$(R_b \circ \sigma_Q)(ua) = \sigma_Q(u)\sigma(a)b = \sigma_Q(u)ba \tag{32}$$

which is $R_a(R_b \circ \sigma_Q(u))$.

It is easily checked that the gauge transformation $R_b \circ \sigma_Q$ has a local representation consisting of constant mappings $\{g_i = b; U_i \longrightarrow G\}$. Thus, $R_b \circ \sigma_Q$ may be a "constant" gauge transformation.

Furthermore for any $\omega \in \mathcal{C}(P)$ the σ-conjugate ω^σ is gauge equivalent to ω with respect to the gauge transformation $g_\sigma = R_b \circ \sigma_Q$. This is because

ω^σ is written as

$$R_b^*(g_\sigma^{-1})^*\sigma(\omega) = \sigma(Ad(b^{-1})(g_\sigma^{-1})^*\omega) = (g_\sigma^{-1})^*\omega. \tag{33}$$

Here we used $\sigma(Ad(b^{-1})(X) = X, \quad X \in \mathbf{g}$.

Thus we get the following

Proposition 2 *Let $\sigma \in \mathrm{Int}(G,H)$, the group of inner automorphisms of G leaving fixed all elements of H. Then for each $\omega \in \mathcal{C}(P)$ the σ-conjugate ω^σ is gauge equivalent to ω.*

Therefore the action of $\mathrm{Aut}(G,H)$ is trivial on $\mathcal{C}(P)/\mathcal{G}(P)$ when restricted to the subgroup $\mathrm{Int}(G,H)$.

So, Theorem 1 in Introduction follows from this proposition, namely the action of $\mathrm{Aut}(G,H)$ induces in a natural way an action of the outer automorphism group $\mathrm{Aut}(G,H)/\mathrm{Int}(G,H)$ on $\mathcal{C}(P)/\mathcal{G}(P)$ and also on the moduli space of Yang-Mills (respective self-dual) connections on P.

4. Conjugate connections and holonomy groups

We investigate how the conjugation on connections affect the holonomy groups.

The holonomy group is defined in terms of horizontal curves [4].

Let x_t, $0 \le t \le 1$ be a closed curve in M starting at a point $x \in M$.

Let u be a point of P over x. We say that a lift u_t of x_t, a curve in P whose projection is x_t, starting at u ω-horizontal when it satisfies

$$\omega\left(\frac{d}{dt}u_t\right) = 0 \tag{34}$$

The existence of a horizontal lift u_t of x_t starting at u is unique.

When $t = 1$, u_1 is over x so that there is an $a \in G$ such that $u_1 = ua$.

The set of $a \in G$ given in this way where x_t moves over all closed curves starting at x is a subgroup of G which we call the holonomy group $\Phi(\omega) = \Phi_u(\omega)$ of ω with a reference point u.

If we change a reference point as $v = ua$, then $\Phi_v(\omega) = a^{-1}\ \Phi_u(\omega)\ a$ (see Proposition 4.1, p.72, [4]).

Proposition 3 *Let $\sigma \in \mathrm{Aut}(G,H)$. Then, u_t is a ω-horizontal curve if and only if $\sigma_Q(u_t)$ is a ω^σ-horizontal curve.*

Therefore, for a fixed reference point $u = s_i(x), x \in U_i$ σ induces an isomorphism of $\Phi_u(\omega)$ to $\Phi_u(\omega^\sigma)$;

$$\sigma : \Phi_u(\omega) \longrightarrow \Phi_u(\omega^\sigma); \quad a \longmapsto \sigma(a) \tag{35}$$

Proof. Since $\frac{d}{dt}(\sigma_Q(u_t)) = (\sigma_Q)_*(\frac{d}{dt}u_t)$, it follows from the definition (7) of the σ-conjugation

$$\omega^\sigma\left(\frac{d}{dt}(\sigma_Q(u_t))\right) = \sigma\left(\omega\left(\frac{d}{dt}u_t\right)\right). \tag{36}$$

So the first statement is shown.

Take a reference point $u = s_i(x)$. Then u_t and $\sigma_Q(u_t)$ are lifts of the same curve x_t and have the same starting point u. So, for $a \in \Phi_u(\omega)$ $u_1 = s_i(x)a$ and thus

$$\sigma_Q(u_1) = \sigma_Q(s_i(x)a) = u\sigma(a) \tag{37}$$

This implies that σ induces an isomorphism between the holonomy groups. □

The following lemma is plainly clear. But we have no references to refer to. So we are going to state

Lemma *Let $\omega \in \mathcal{C}(P)$ and $g \in \mathcal{G}(P)$. Let u_t be a lift of a closed curve x_t in M.*

Then u_t is ω-horizontal if and only if $g^{-1}(u_t)$ is a $g^*(\omega)$-horizontal curve.

Proposition 4 *Let $\sigma \in \mathrm{Aut}(G, H)$ and let $\omega \in \mathcal{C}(P)$ be a connection whose holonomy group $\Phi_u(\omega)$ is G.*

If ω^σ is gauge equivalent to ω, then σ must be an inner automorphism.

Proof. Suppose $\omega^\sigma = g^*(\omega)$ for a $g \in \mathcal{G}(P)$.

For $a \in \Phi_u(\omega)$ let u_t be a ω-horizontal curve for which $u_1 = ua$.

From Proposition 3 $\sigma_Q(u_t)$ is ω^*-horizontal, while from the above lemma $g^{-1}(u_t)$ is a $g^*(\omega)$-horizontal curve starting at $g^{-1}(u)$. Since u and $g^{-1}(u)$ are on the same fibre, we may set $g^{-1}(u) = ub$, $b \in G$.

From the right translation invariance $R_{b^{-1}}g^{-1}(u_t)$ is $g^*(\omega)$-horizontal and starts at $ubb^{-1} = u$.

Since $\omega^\sigma = g^*(\omega)$, from the uniqueness of solution of the horizontal curve equation (34) with respect to an initial point it holds

$$\sigma_Q(u_t) = R_{b^{-1}} g^{-1}(u_t), \quad 0 \le t \le 1 \tag{38}$$

for which, if $t = 1$,

$$\sigma_Q(u_1) = R_{b^{-1}} g^{-1}(u_1). \tag{39}$$

We take a reference point $u = s_i(x), x \in U_i$. Then $\sigma_Q(u_1) = u\sigma(a)$ and $R_{b^{-1}} g^{-1}(u_1) = g^{-1}(u)ab^{-1} = ubab^{-1}$. Thus

$$\sigma(a) = bab^{-1}, \quad a \in \Phi_u(\omega) \tag{40}$$

Since $\Phi_u(\omega) = G$, σ is an inner automorphism. □

We write $\overline{\sigma} \in \mathrm{Aut}(G, H)/\mathrm{Int}(G, H)$ for the equivalence class represented by $\sigma \in \mathrm{Aut}(G, H)$.

It suffices for the proof of Theorem 2 in Introduction to show that if

$$[\omega]^{\overline{\sigma}} = [\omega] \tag{41}$$

for some $\omega \in \mathcal{C}^*(P)$, then $\overline{\sigma}$ is the identity, that is, $\sigma \in \mathrm{Int}(G, H)$.

Here we define the action of $\overline{\sigma}$ as $[\omega]^{\overline{\sigma}} = [\omega^\sigma]$.

The condition (41) for $\overline{\sigma}$ is equivalent to that

$$\omega^\sigma = g^*(\omega) \tag{42}$$

for a $g \in \mathcal{G}(P)$. But this is just the assumption of Proposition 4. So, Theorem 2 is shown.

Remark. Let ω be a connection having holonomy group not necessarily to be the whole group G. If it has some σ-conjugate ω^σ gauge equivalent to ω, i.e., $\omega^\sigma = g^*(\omega)$, then the $\sigma \in \mathrm{Aut}(G, H)$ induces from (40) a group conjugation between two subgroups $\Phi_u(\omega)$ and $\Phi_u(\omega^\sigma)$, which is a condition on an outer automorphism $\overline{\sigma}$ having a fixed point on $\mathcal{C}(P)/\mathcal{G}(P)$.

Of course $\mathrm{Aut}(G, H)/\mathrm{Int}(G, H)$ leaves fixed the subspace $\mathcal{C}(Q)/\mathcal{G}(P)$ pointwise.

5. Further remarks

A principal bundle P of structure group G may have many non-equivalent reductions to H-subbundles even if a subgroup H is fixed. Here H-subbundles Q_1, Q_2 are equivalent in P if there is a $g \in \mathcal{G}(P)$ sending Q_1

onto Q_2.

For instance, a principal $SU(2)$ bundle P has the number m of non-equivalent $SO(2)$-reductions when the base manifold M is a 4-dimension oriented closed manifold;

$$m = \frac{1}{2}\sharp\{\alpha \in H^2(M; \mathbf{Z}); \alpha \cdot \alpha = k\}. \tag{43}$$

Here $k = c_2(P)[M]$ is the second Chern number of the associated vector bundle $E = P \times_\rho \mathbf{C}^2$ and $\alpha \cdot \alpha$ is the quadratic form of α.

Consider the case that P has, for example, two non-equivalent H-reductions, Q_1 and Q_2.

Then the group $\mathrm{Aut}(G, H)$ has the two actions on P derived from the reductions Q_1 and Q_2.

Proposition 5 (i) *Let $\sigma, \tau \in \mathrm{Aut}(G, H)$. If $\sigma\tau = e$ in $\mathrm{Aut}(G, H)$, then $\sigma_{Q_1} \circ \tau_{Q_2}$ and $\sigma_{Q_2} \circ \tau_{Q_1}$ act on P as gauge transformations so that $\sigma_{Q_1} \circ \tau_{Q_2}$ and $\sigma_{Q_2} \circ \tau_{Q_1}$ induce the trivial action on $\mathcal{C}(P)/\mathcal{G}(P)$.*

(ii) *More generally, if $\sigma\tau \in \mathrm{Int}(G, H)$, then $\sigma_{Q_1} \circ \tau_{Q_2}$ and $\sigma_{Q_2} \circ \tau_{Q_1}$ act trivially on $\mathcal{C}(P)/\mathcal{G}(P)$.*

Proof. For $a \in G$

$$(\sigma_{Q_1}\tau_{Q_2})(ua) = \sigma_{Q_1}(\tau_{Q_2}(u)\tau(a)) = \sigma_{Q_1}\tau_{Q_2}(u)a, \tag{44}$$

since $\sigma\tau = e$ in $\mathrm{Aut}(G, H)$. So, (i) follows.

To show (ii) we let $\mu^{-1} = \sigma\tau$ and have from (i) and $\sigma\tau\mu = e$ that $\sigma_{Q_1}(\tau\mu)_{Q_2} = \sigma_{Q_1}\tau_{Q_2}\mu_{Q_2}$ is a gauge transformation of P so that $\sigma_{Q_1}\tau_{Q_2}\mu_{Q_2}$ acts trivially on $\mathcal{C}(P)/\mathcal{G}(P)$. Since μ is an inner automorphism, μ_{Q_2} and hence $\sigma_{Q_1} \circ \tau_{Q_2}$ act on $\mathcal{C}(P)/\mathcal{G}(P)$ also trivially. □

We denote by $\overline{\sigma}_{Q_i}$ the transformation of $\mathcal{C}(P)/\mathcal{G}(P)$ given by $\overline{\sigma} \in \mathrm{Aut}(G, H)/\mathrm{Int}(G, H)$ via the H-reduction Q_i, $i = 1, 2$ and let

$$\overline{\mathrm{Aut}(G, H)}_{Q_i} = \{\overline{\sigma}_{Q_i}; \overline{\sigma} \in \mathrm{Aut}(G, H)/\mathrm{Int}(G, H)\}. \tag{45}$$

Then we get a free group generated by $\overline{\mathrm{Aut}(G, H)}_{Q_i}$, $i = 1, 2$ and denote it by $\overline{\mathrm{Aut}(G, H)}_{Q_1} * \overline{\mathrm{Aut}(G, H)}_{Q_2}$.

From the above proposition the action of this free group on $\mathcal{C}(P)/\mathcal{G}(P)$ induced by each factor $\overline{\mathrm{Aut}(G, H)}_{Q_i}$ is not effective.

In fact, this free group has a canonical homomorphism

$$\varphi ; \overline{\mathrm{Aut}(G,H)}_{Q_1} * \overline{\mathrm{Aut}(G,H)}_{Q_2} \longrightarrow \mathrm{Aut}(G,H)/\mathrm{Int}(G,H), \qquad (46)$$

$$\varphi((\overline{\sigma_1})_{Q_1}(\overline{\sigma_2})_{Q_2}(\overline{\sigma_3})_{Q_1}\cdots(\overline{\sigma_k})_{Q_2}) = \overline{\sigma_1}\ \overline{\sigma_2}\cdots\overline{\sigma_k} \qquad (47)$$

whose kernel we denote by S acts on the space $\mathcal{C}(P)/\mathcal{G}(P)$ trivially. So,

Proposition 6 *The actions of* $\mathrm{Aut}(G,H)$ *via the* H*-reductions* Q_1 *and* Q_2 *induce in a natural manner the action on* $\mathcal{C}(P)/\mathcal{G}(P)$ *of the quotient group* $\overline{\mathrm{Aut}(G,H)}_{Q_1} * \overline{\mathrm{Aut}(G,H)}_{Q_2}/S$ *of the free group by the normal subgroup* S.
This action is free on the subspace $\mathcal{C}^*(P)/\mathcal{G}(P)$.

Proof. To verify that this action is free on $\mathcal{C}^*/\mathcal{G}(P)$ let

$$\overline{\rho} = (\overline{\sigma_1})_{Q_1}(\overline{\sigma_2})_{Q_2}\cdots(\overline{\sigma_k})_{Q_2} \qquad (48)$$

be an element of $\overline{\mathrm{Aut}(G,H)}_{Q_1} * \overline{\mathrm{Aut}(G,H)}_{Q_2}$ and assume

$$[\omega]^{\overline{\rho}} = [\omega] \qquad (49)$$

for an $\omega \in \mathcal{C}^*(P)$.

To show $\overline{\rho} \in S$ we choose in $\overline{\rho}$ a representative $(\sigma_1)_{Q_1}(\sigma_2)_{Q_2}\cdots(\sigma_k)_{Q_2}$ acting on P as a diffeomorphism. Then (49) is equivalent to

$$(((\omega^{(\sigma_k)_{Q_2}})^{(\sigma_{k-1})_{Q_1}}\ldots)^{(\sigma_2)_{Q_2}})^{(\sigma_1)_{Q_1}} = g^*(\omega) \qquad (50)$$

for a $g \in \mathcal{G}(P)$.

Since the holonomy group $\Phi_u(\omega)$ is the whole group G, we iterate the argument in the proof of Proposition 4 and have

$$\sigma_1\sigma_2\cdots\sigma_k \ : G \longrightarrow G \qquad (51)$$

must be inner. So, $\overline{\rho}$ is in $S = \ker\varphi$. □

We finish this article with several remarks on outer automorphism groups.

An automorphism of G induces an automorphism of its Lie algebra $\mathbf{g}$. When G is simply connected, an automorphism of $\mathbf{g}$ induces conversely an automorphism of the group G (see for example Theorem 3.27 in [9]). So the outer automorphism group $\mathrm{Aut}(G)/\mathrm{Int}(G)$ is isomorphic to that of its Lie algebra $\mathbf{g}$, namely to $\mathrm{Aut}(\mathbf{g})/\mathrm{Int}(\mathbf{g})$.

The following two facts on outer automorphism groups are well known.

The outer automorphism group $\mathrm{Aut}(G)/\mathrm{Int}(G)$ of a compact Lie group G is compact. If G is compact and semi-simple, then $\mathrm{Aut}(G)/\mathrm{Int}(G)$ is a finite group.

The second fact stems from that every derivation of a semi-simple Lie algebra is an inner derivation.

Any automorphism of a Lie algebra $\mathbf{g}$ canonically extends as an automorphism of the complexification $\mathbf{g}^{\mathbf{C}}$ commuting with the $\mathbf{C}$-conjugation. Thus the automorphism group $\mathrm{Aut}(\mathbf{g})$ of $\mathbf{g}$ is the subgroup in $\mathrm{Aut}(\mathbf{g}^{\mathbf{C}})$ consisting of automorphisms commuting with the $\mathbf{C}$-conjugation.

For complex simple Lie algebras the outer automorphism groups $\mathrm{Aut}(\mathbf{g}^C)/\mathrm{Int}(\mathbf{g}^C)$ are completely determined in terms of the Dynkin diagrams.

If we fix a Cartan subalgebra in $\mathbf{g}^{\mathbf{C}}$, then any coset in $\mathrm{Aut}(\mathbf{g}^{\mathbf{C}})$ modulo $\mathrm{Int}(\mathbf{g}^{\mathbf{C}})$ contains an automorphism leaving the Cartan subalgebra invariant. By using the structure theorem of simple Lie algebras the outer automorphism group of $\mathbf{g}^{\mathbf{C}}$ is then isomorphic to the finite group consisting of linear transformations of the Cartan subalgebra leaving a simple root system, and furthermore isomorphic to automorphisms of the Dynkin diagram of $\mathbf{g}^{\mathbf{C}}$.

In fact, the outer automorphism group is trivial for $\mathbf{g}^C$ of type A_1, B_ℓ, C_ℓ, E_7, E_8, F_4 or G_2, isomorphic to $\mathbf{Z}_2 \cong \{1, -1\}$ for $\mathbf{g}^C$ of type $A_\ell (\ell \geq 2)$, D_ℓ $(\ell \geq 5)$ or E_6 and isomorphic to S_3, the symmetric group of degree 3 for $\mathbf{g}^C$ of type D_4 (see Theorem 3.29, Ch. X in [1]).

We exhibit one example of outer automorphisms of $SU(n)$ fixing a subgroup.

Let

$$\sigma : \ \mathbf{su}(n) \longrightarrow \mathbf{su}(n); \quad \sigma(X) = \overline{X} \tag{52}$$

is an involution yielding a Riemannian symmetric pair $(SU(n), SO(n))$.

When $n \geq 3$ the automorphism σ is outer, since $\mathrm{rank}\,\mathbf{su}(n) > \mathrm{rank}\,\mathbf{so}(n)$ and we have the following theorem. Let $\mathbf{g}$ be a compact semi-simple Lie algebra and let $\theta \in \mathrm{Aut}(\mathbf{g})$ be an involution. Let $\mathbf{k}$ denote the set of fixed points of θ. Then $\theta \in \mathrm{Int}(\mathbf{g})$ if and only if $\mathrm{rank}\,\mathbf{g} = \mathrm{rank}\,\mathbf{k}$ (see Theorem 5.6, Ch IX, [1]).

So, when $n \geq 3$, $\mathbf{su}(n) \cong A_{n-1}$ and therefore the outer automorphism group

$$\mathrm{Aut}(SU(n), SO(n))/\mathrm{Int}(SU(n), SO(n)) \cong \mathbf{Z}_2, \tag{53}$$

(σ is a generator) acts effectively on the space of gauge equivalence classes of connections $\mathcal{C}(P)/\mathcal{G}(P)$, provided an $SU(n)$ principal bundle P is reducible to an $SO(n)$-subbundle.

For the $n = 2$ case, contrarily to the $n \geq 3$ case, from the above arguments σ is an inner automorphism and $\mathrm{Aut}(SU(2), SO(2))/\mathrm{Int}(SU(2), SO(2))$ must be trivial so that the moduli space of self-dual $SU(2)$-connections does not admit any outer automorphism action.

After preparing this manuscript, the author received a paper written by S. Kobayashi and E. Shinozaki in which they obtained a theorem quite similar to our theorems.

References

[1] Helgason S., *Differential Geometry, Lie Groups, and Symmetric Spaces.* Academic Press, New York, 1978.

[2] Itoh M., *Generalized magnetic monopoles over contact manifolds.* J. Math, Phys. **36**(2) (1995), 742–749.

[3] Jaffe A. and Taubes C., *Vortices and Monopoles.* Birkhäuser, Boston, 1980.

[4] Kobayashi S. and Nomizu K., *Foundations of Differential Geometry, Vol. I.* Interscience Publishers, New York, 1963.

[5] Kobayashi S. and Shinozaki E., *Conjugate connections in principal bundles.* preprint, 1994.

[6] Nagatomo Y., *Rigidity of c_1-self-dual connections on quaternionic Kähler manifolds.* J. Math, Phys. **33**(12) (1992), 4020–4025.

[7] Nomizu K. and Sasaki T., *Affine Differential Geometry.* Shokabo, Tokyo, in Japanese, 1994.

[8] Suh Y., *On the anti-self-duality of the Yang-Mills connection over higher dimensional Kaehler manifold.* Tsukuba J. Math. **14** (1990), 505–512.

[9] Warner F., *Foundations of Differential Manifolds and Lie Groups.* Scott, Foresman and Company, Glenview, Illinois, 1981.

Institute of Mathematics
University of Tsukuba, 305 Japan
E-mail: itohm@sakura.cc.tsukuba.ac.jp