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Symmetry of isometric embeddings of Riemannian manifolds and local scalar invariants*

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Abstract. We study the infinitesimal symmetry of the isometric embeddings of a Riemannian manifold M^n into $\mathbb{R}^{n+d}$, $n \geq 2$, $d \geq 1$. Then we define a notion of scalar invariant for submanifolds in $\mathbb{R}^{n+d}$ in terms of this symmetry. As an example, we show by calculation that the Gaussian curvature of a surface is an invariant.

Key words: isometric embedding, infinitesimal symmetry, scalar invariant, Killing field.

Introduction

Let M be a smooth (C^∞) manifold of dimension n, $n \geq 2$, with Riemannian metric g. A mapping $u = (u^1, \ldots, u^{n+d}) : M \to \mathbb{R}^{n+d}$, $d \geq 1$, is a local isometric embedding if u satisfies

$$\langle du, du \rangle = g.$$

In terms of local coordinates $x = (x^1, \ldots, x^n)$ of M, the above equation is written as

$$\sum_{\alpha=1}^{n+d} \frac{\partial u^\alpha}{\partial x^i} \frac{\partial u^\alpha}{\partial x^j} = g_{ij}, \quad \text{for each} \quad i, j = 1, \ldots, n, \tag{2.5}$$

where $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$.

A (local) scalar invariant of M is a real valued function defined on an open subset of M which is invariant under local isometries. The scalar curvature is the simplest scalar invariant. If $\mathrm{vol}_M(p, r)$ is the volume of the geodesic ball of radius r centered at a point $p \in M$, then for sufficiently small $r \geq 0$

$$\frac{\mathrm{vol}_M(p, r)}{\mathrm{vol}_{\mathbb{R}^n}(0, r)} = 1 - c\kappa_2(p) r^2 + \sum_{n \geq 4} \kappa_n(p) r^n,$$

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where c is a constant depending only on n and κ_2 is the scalar curvature. The functions 1, κ_2 and $\kappa_n, n \geq 4$, are scalar invariants. We refer the readers to Chapter 2 of [Gil] for the theory of local invariants.

For an isometric embedding $u : M \to \mathbb{R}^{n+d}$, a local scalar invariant κ of M can be expressed as a function of finite jet of u:

$$\kappa(x) = A(u^{(m)}), \tag{3.5}$$

where $u^{(m)}$ denotes all the partial derivatives of $u = (u^1, \dots, u^{n+d})$ of order $\leq m$. (3.5) is a partial differential equation of order m that an isometric embedding u satisfies. The scalar curvature $\kappa_2(x)$ is a function of the second jet of u. In the case $d = 1$, it is shown in [H1] that under a certain condition on the scalar curvature the first derivatives of (2.1) and (3.5) with $\kappa = \kappa_2$, $m = 2$ form a non-linear system of elliptic partial differential equations of second order for $u = (u^1, \dots, u^{n+1})$.

In the present paper we study the symmetry of (2.1) and define notions of extrinsic and intrinsic invariants of a submanifold M^n of $\mathbb{R}^{n+d}$ in terms of symmetry of (2.1).

Let M be a submanifold in $\mathbb{R}^{n+d}$ given by

$$u^{n+\sigma} = h^\sigma(u_1, \dots, u_n), \quad \sigma = 1, \dots, d.$$

Let $h = (h^1, \dots, h^d)$. A real valued function $a(h^{(m)})$ of finite jet of h is an extrinsic invariant if it is invariant under rigid motions of $\mathbb{R}^{n+d}$ (Definition 3.2). Principle curvatures of hypersurfaces in a euclidean space are extrinsic invariants. $a(h^{(m)})$ is an intrinsic invariant if it is invariant under the symmetry of (2.1) (Definition 3.5). The scalar curvature and all the scalar invariants of the classical theory are intrinsic invariants in our sense. By expressing $a(h^{(m)})$ in terms of $u^{(m)}$ we obtain (3.5), which is a compatibility equation of (2.1).

It seems to the author that this method of obtaining compatibility equations works equally well for the embeddings of conformal and CR structures.

Our problems are purely local so that we assume that M is an open subset of $\mathbb{R}^n$ with the standard coordinates $(x^1, \dots, x^n)$ and that $(g_{ij})_{i,j=1,\dots,n}$ is a C^∞, symmetric, positive definite matrix valued function defined on M. All the manifolds and mappings are assumed to be C^∞ unless stated otherwise.

In §1, we review some rudiments of the jet theory and then in §2, we study the infinitesimal symmetry (or the Lie-Bäcklund transformation) of

(2.1). A solution of the evolution equation (3.4) , where $Q^\alpha(x, u^{(m)})$ are the components of an infinitesimal symmetry is a 1-parameter family of embeddings,that is , a "bending". Bending of submanifolds has been studied in [Ja1], [Ja2] and [Ten]. We show that the evolutionary expression of Killing fields of M and the infinitesimal rigid motions of $\mathbb{R}^{n+d}$ are infinitesimal symmetries (Theorem 2.1 and Theorem 2.2).

In §3, we define the extrinsic and intrinsic invariants for n-dimensional submanifolds in $\mathbb{R}^{n+d}$. Then as an example , we show by calculation that the Gaussian curvature is an intrinsic invariant.

1. Preliminaries

In this section, we recall some rudiments of the theory of jets and symmetry of differential equations. We adopt from [Olv] the basic definitions and notations. However, the difference is that [Olv] treats the locally solvable equations while (2.1) is overdetermined if $d < n(n-1)/2$, and therefore, not locally solvable. Nevertheless, there exists symmetry in (2.1). Thus we define the infinitesimal symmetry as in Definition 1.4.

Let $X = \{(x^1, \ldots, x^n)\}$ be an open subset of $\mathbb{R}^n$ and $U = \{(u^1, \ldots, u^q)\}$ be an open subset of $\mathbb{R}^q$. Let $U^{(m)}$ be an open subset of a Euclidean space whose coordinates represent all the partial derivatives of smooth maps $u(x) = (u^1(x), \ldots, u^q(x))$ from X to U of all orders 0 through m. A multi-index of order r is an unordered r-tuple of integers $J = (j_1, \ldots, j_r)$ with $1 \leq j_s \leq n$. The order of multi-index J is denoted by $|J|$. A typical point in $U^{(m)}$ is denoted by $u^{(m)}$, so that

$$u^{(m)} = (u_J^\alpha), \quad 1 \leq \alpha \leq q, \quad 0 \leq |J| \leq m.$$

Then $U^{(m)}$ is an open subset of the Euclidean space of dimension $q \cdot \binom{n+m}{m}$. The product space $X \times U^{(m)}$ is called the m-th order jet space and is denoted by $J^m(X, U)$.

Let $F = (f^1, \ldots, f^q)$ be a smooth map from X into U. For each $x \in X$ let

$$j_x^m F = (\partial_J f^\alpha(x)), \quad 1 \leq \alpha \leq q, \quad 0 \leq |J| \leq m.$$

Then the map $j^m F : X \to J^m(X, U)$ defined by $x \mapsto (x, j_x^m F)$, is a section of $J^m(X, U)$, and is called the m-th graph of F. By $j_x F$ we denote the jet of F at x of unspecified order. Let A be the set of real valued smooth functions

$\mathfrak{a}(x, u^{(m)})$ of some finite, but unspecified order m. An element of A is called a differential function and denote by $\mathfrak{a}[u]$. The order of a differential function is the order of the highest derivatives that occurs. Then A is a commutative algebra and the subset $A^{(m)}$ of A consisting of the differential functions of order less than or equal to m is a subalgebra.

Now consider a system of m-th order differential equations

$$\Delta^\nu(x, u^{(m)}) = 0, \quad 1 \le \nu \le l, \tag{1.1}$$

for unknown functions $u = (u^1, \dots, u^q)$ of n variables $x = (x^1, \dots, x^n)$.

Let I be the set of all differential functions of the form

$$\sum_{|J|\ge 0} \sum_{\nu=1}^{l} P_\nu^J[u](D_J\Delta^\nu), \quad P_\nu^J[u] \in A,$$

where $D_J = D_{(j_1,\dots,j_r)} = D_{j_1} \circ \cdots \circ D_{j_r}$ is a composition of total differential operators. Then we see that I is an ideal of A and that I is closed under total differentiation, namely

$$D_J I \subset I, \quad \text{for any multi-index} \quad J. \tag{1.2}$$

For each $m = 1, 2, \dots$, $I^{(m)} := I \cap A^{(m)}$ is an ideal of $A^{(m)}$. If F is a solution of (1.1) of differentiability class C^m, $\mathfrak{a}(j^m F) = 0$, for all $\mathfrak{a} \in I^{(m)}$.

Let

$$V = \sum_{i=1}^{n} \xi^i(x, u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi^\alpha(x, u)\frac{\partial}{\partial u^\alpha} \tag{1.3}$$

be a vector field on $X \times U$. The m-th prolongation of V is a vector field on $J^m(X, U)$ defined by

$$pr^{(m)}V = V + \sum_{1\le|J|\le m} \phi_J^\alpha \frac{\partial}{\partial u_J^\alpha},$$

where the coefficients ϕ_J^α are given by the prolongation formula

$$\phi_J^\alpha = D_J\Big(\phi^\alpha - \sum_{i=1}^{n} \xi^i u_i^\alpha\Big) + \sum_{i=1}^{n} \xi^i u_{J,i}^\alpha. \tag{1.4}$$

By a straight-forward calculation one can show

Proposition 1.1 *If V and W are vector fields on $X \times U$, then*

$$pr^{(m)}[V, W] = [pr^{(m)}V, pr^{(m)}W],$$

where $[\ ,\ ]$ *is the Lie bracket.*

We also have

Proposition 1.2 *Suppose that* $\mathcal{L}_0$ *is a finite dimensional Lie algebra of vector fields on* $X \times U$, *with generators* V_j, $j = 1, \ldots, N$. *Suppose that* $pr^{(m)}V_j$, $j = 1, \ldots, N$, *are linearly independent everywhere for some non-negative integer* m. *Then there exist* $(n + q\binom{n+m}{m} - N)$ *functionally independent differential functions of order* m *annihilated by* $pr^{(m)}V_j$, $j = 1, \ldots, N$.

Proof. The number of independent variables $(x, u^{(m)})$ is $n + q\binom{n+m}{m}$, so the assertion follows from the Frobenius theorem. □

Now let $Q = (Q^1[u], \ldots, Q^q[u])$ be a q-tuple of differential functions. An evolutionary vector field with the characteristic Q is an expression of the form

$$V_Q = \sum_{\alpha=1}^{q} Q^\alpha[u] \frac{\partial}{\partial u^\alpha}, \tag{1.5}$$

which can be regarded as a vector field in the jet space $J^m(X, U)$, for a sufficiently large m. Let V be a vector field on $X \times U$ as in (1.3). Let

$$Q^\alpha[u] = \phi^\alpha - \sum_{i=1}^{n} \xi^i u_i^\alpha, \quad \alpha = 1, \ldots, q. \tag{1.6}$$

Then the evolutionary vector field $V_Q = \sum_{\alpha=1}^{q} Q^\alpha[u] \frac{\partial}{\partial u^\alpha}$ with the characteristic $Q[u]$ given by (1.6) is called the evolutionary representative of V. The prolongation formula (1.4) applied to the evolutionary vector field (1.5) yields

$$prV_Q = V_Q + \sum_J (D_J Q^\alpha) \frac{\partial}{\partial u_J^\alpha}. \tag{1.7}$$

prV_Q is a linear differential operator that acts on differential functions. When it acts on a differential function of order m only finitely many terms with $|J| \leq m$ in the summation of (1.7) virtually act as partial differential operators. Finally, we define the infinitesimal symmetry of the systems that are not nocally solvable:

Definition 1.4 An evolutionary vector field V_Q is an infinitesimal symmetry (or Lie-Bäcklund transformation) of a system (1.1) if

$$(prV_Q)\Delta^\nu \in I, \quad \text{for each} \quad \nu = 1, \dots, l.$$

2. Infinitesimal symmetries for isometric embeddings

Let (M, g) be an n-dimensional manifold with Riemannian metric g. A C^1 mapping $u = (u^1, \dots, u^{n+d})$ of M into a Euclidean space $\mathbb{R}^{n+d}$ is a local isometric embedding if and only if u satisfies

$$\sum_{\alpha=1}^{n+d} \frac{\partial u^\alpha}{\partial x^i} \frac{\partial u^\alpha}{\partial x^j} = g_{ij}(x), \quad 1 \leq i, \quad j \leq n, \tag{2.1}$$

where $(x^1, \dots, x^n)$ is a local coordinate system of M and $g_{ij}(x) = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$. Since $g_{ij} = g_{ji}$, the number of equations in (2.1) is $n(n+1)/2$ and then the system (2.1) is underdetermined if $d > n(n-1)/2$ and overdetermined if $d < n(n-1)/2$. For each $i, j = 1, \dots, n$, let

$$\Delta^{ij} = \sum_{\alpha=1}^{n+d} \frac{\partial u^\alpha}{\partial x^i} \frac{\partial u^\alpha}{\partial x^j} - g_{ij}(x). \tag{2.2}$$

In this section we denote by script letters the jet theoretic notions associated with (2.1) : $\mathcal{A}$ is the algebra of differential functions in the arguments

$$(x^1, \dots, x^n, u^1, \dots, u^{n+d}, u_i^\alpha, u_{ij}^\alpha, \cdots),$$

where $u_i^\alpha = \frac{\partial u^\alpha}{\partial x^i}$, $u_{ij}^\alpha = \frac{\partial^2 u^\alpha}{\partial x^i \partial x^j}$, and so forth. $\mathcal{A}^{(m)}$ is the subalgebra of $\mathcal{A}$ consisting of the differential functions of order less than or equal to m. $\mathcal{I}$ is an ideal of $\mathcal{A}$ consisting of all the differential functions of the form

$$\sum_J \sum_{i,j=1}^{n} P_{ij}^J[u](D_J \Delta^{ij}), \quad P_{ij}^J[u] \in \mathcal{A}. \tag{2.3}$$

For each non-negative integer m,

$$\mathcal{I}^{(m)} = \mathcal{I} \cap \mathcal{A}^{(m)},$$

and so forth. An evolutionary vector field $V_Q = \sum_{\alpha=1}^{n+d} Q^\alpha[u] \frac{\partial}{\partial u^\alpha}$ is an infinitesimal symmetry of (2.1) if $Q = (Q^1, \dots, Q^{n+d})$ satisfies

$$\sum_{\alpha=1}^{n+d} \{(D_j Q^\alpha) u_i^\alpha + (D_i Q^\alpha) u_j^\alpha\} = 0, \quad \text{mod } \mathcal{I},$$

$$\text{for each } i, j = 1, \ldots, n. \tag{2.4}$$

Now we mention two special kinds of infinitesimal symmetries : the evolutionary representative of a Killing field and the infinitesimal rigid motions of $\mathbb{R}^{n+d}$. A vector field V on M is a Killing vector field if

$$L_V g = 0, \tag{2.5}$$

where L is the Lie derivative and g is the Riemannian metric. Write $V = \sum_{k=1}^{n} \xi^k(x) \frac{\partial}{\partial x^k}$ in terms of the coordinates. Then (2.5) is a system of first order linear partial differential equations

$$\sum_{k=1}^{n} \left(g_{jk} \frac{\partial \xi^k}{\partial x^i} + g_{ik} \frac{\partial \xi^k}{\partial x^j} + \xi^k \frac{\partial g_{ij}}{\partial x^k} \right) = 0, \quad i, j = 1, \ldots, n, \tag{2.6}$$

for the unknowns $(\xi^1(x), \ldots, \xi^n(x))$.
The evolutionary representative of V has characteristic

$$Q^\alpha[u] = -\sum_{k=1}^{n} \xi^k(x) u_k^\alpha, \quad \alpha = 1, \ldots, n+d. \tag{2.7}$$

Theorem 2.1 *If $V = \sum_{k=1}^{n} \xi^k(x) \frac{\partial}{\partial x^k}$ is a Killing field on M its evolutionary representative V_Q is an infinitesimal symmetry of (2.1). Conversely, if an evolutionary vector field $\sum_{\alpha=1}^{n+d} Q^\alpha[u] \frac{\partial}{\partial u^\alpha}$, with Q^α as in (2.7) is an infinitesimal symmetry of (2.1) and if there exists a solution F of (2.1) then $V = \sum_{k=1}^{n} \xi^k \frac{\partial}{\partial x^k}$ is a Killing field of M.*

Proof. Suppose that V is a Killing field. Let Q^α be as in (2.7). Then

$$\begin{aligned} &\sum_{\alpha=1}^{n+d} ((D_j Q^\alpha) u_i^\alpha + (D_i Q^\alpha) u_j^\alpha) \\ &\quad = -\sum_{\alpha=1}^{n+d} \sum_{k=1}^{n} \{ (\xi_j^k u_k^\alpha + \xi^k u_{kj}^\alpha) u_i^\alpha + (\xi_i^k u_k^\alpha + \xi^k u_{ki}^\alpha) u_j^\alpha \}. \end{aligned}$$

Since

$$\sum_{\alpha=1}^{n+d} u_k^\alpha u_i^\alpha = g_{ki}, \quad \bmod \mathcal{I},$$

and

$$\sum_{\alpha=1}^{n+d}(u_{kj}^{\alpha}u_i^{\alpha}+u_{ki}^{\alpha}u_j^{\alpha})=\frac{\partial g_{ij}}{\partial x^k},\quad \bmod \mathcal{I},$$

the above expression equals to

$$\begin{aligned}&-\sum_k\left(\xi_j^k g_{ki}+\xi_i^k g_{kj}+\xi^k\frac{\partial g_{ij}}{\partial x^k}\right),\quad \bmod \mathcal{I},\\&=0,\quad \bmod \mathcal{I},\quad \text{by (2.6)}.\end{aligned}$$

Thus V_Q is an infinitesimal symmetry.

Conversely, if

$$V_Q=\sum_{\alpha=1}^{n+d}\left(-\sum_{k=1}^{n}\xi^k(x)u_k^{\alpha}\right)\frac{\partial}{\partial u^{\alpha}}$$

is an infinitesimal symmetry of (2.1), then (2.4) implies that

$$-\sum_{\alpha=1}^{n+d}\sum_{k=1}^{n}\{\xi_j^k u_k^{\alpha}u_i^{\alpha}+\xi_i^k u_k^{\alpha}u_j^{\alpha}+\xi^k D_k(u_i^{\alpha}u_j^{\alpha})\}=0,\ \bmod \mathcal{I}.\tag{2.8}$$

If $F=(f^1,\dots,f^{n+d})$ is a solution, evaluation of (2.8) on j^1F yields

$$\sum_{\alpha=1}^{n+d}\sum_{k=1}^{n}\{\xi_j^k f_k^{\alpha}f_i^{\alpha}+\xi_i^k f_k^{\alpha}f_j^{\alpha}+\xi^k D_k(f_i^{\alpha}f_j^{\alpha})\}=0.$$

Substitute

$$\sum_{\alpha=1}^{n+d}f_i^{\alpha}f_j^{\alpha}=g_{ij},\quad i,j=1,\dots,n,$$

to get (2.6), which implies that $\sum_{k=1}^{n}\xi^k(x)\frac{\partial}{\partial x^k}$ is a Killing vector field of M. □

We denote by $\mathcal{L}$ the Lie algebra of infinitesimal symmetries of (2.1) and by $\mathcal{L}_0$ the set of infinitesimal rigid motions of $\mathbb{R}^{n+d}$. Then $\mathcal{L}_0$ is a Lie algebra of dimension $(n+d)(n+d+1)/2$ with the standard basis consisting of $n+d$ translations

$$T_k=\frac{\partial}{\partial u^k},\quad k=1,\dots,n+d\tag{2.9}$$

and $(n+d)(n+d-1)/2$ rotations

$$R_{jk} = u^j \frac{\partial}{\partial u^k} - u^k \frac{\partial}{\partial u^j}, \quad j,k = 1,\dots,n+d, \quad j \neq k. \tag{2.10}$$

$\mathcal{L}_0$ is a Lie subalgebra of $\mathcal{L}$ by the following.

Theorem 2.2 *An infinitesimal rigid motion in $\mathbb{R}^{n+d}$ is an infinitesimal symmetry of* (2.1).

Proof. Since (2.4) is linear in $(Q^1,\dots,Q^{n+d})$, it is enough to show that the translations T_i and the rotations R_{ij} satisfy (2.4), and it is easy to see that the left hand side of (2.4) is identically equal to zero for T_i and R_{ij}. $\square$

Now let $\tilde{M}$ be an n-dimensional submanifold of $\mathbb{R}^{n+d}$. A vector field Z on $\tilde{M}$, not necessary tangent to $\tilde{M}$, is called an infinitesimal bending of $\tilde{M}$ if

$$\langle \nabla'_X Z, Y\rangle + \langle X, \nabla'_Y Z\rangle = 0, \tag{2.11}$$

for all vectors X and Y tangent to $\tilde{M}$, where ∇' is the covariant differentiation of $\mathbb{R}^{n+d}$ (see [Sp]). Now we have

Theorem 2.3 *Suppose that $V_Q = \sum_{\alpha=1}^{n+d} Q^\alpha[u]\frac{\partial}{\partial u^\alpha}$ is an infinitesimal symmetry of* (2.1) *and that F is a solution of* (2.1). *Then V_Q evaluated on the jet of F*

$$V_Q(j_xF) = \sum_{\alpha=1}^{n+d} Q^\alpha(j_xF)\frac{\partial}{\partial u^\alpha}$$

is an infinitesimal bending of $F(M)$ at $x \in M$.

Proof. Suppose that $Q^\alpha[u]$, $\alpha = 1,\dots,n+d$, satisfies (2.4) and $F = (f^1,\dots,f^{n+d})$ is a solution of (2.1). Then

$$\sum_{\alpha=1}^{n+d}\left\{\frac{\partial}{\partial x^j}Q^\alpha(j_xF)\frac{\partial f^\alpha}{\partial x^i} + \frac{\partial}{\partial x^i}Q^\alpha(j_xF)\frac{\partial f^\alpha}{\partial x^j}\right\} = 0,$$
$$i,j = 1,\dots,n. \tag{2.12}$$

Let $X_i = F_*(\frac{\partial}{\partial x^i})$, $i = 1,\dots,n$, and let $Z = \sum_{\alpha=1}^{n+d} Q^\alpha(j_xF)\frac{\partial}{\partial u^\alpha}$. Then (2.12) is equivalent to

$$\langle \nabla'_{X_j} Z, X_i\rangle + \langle X_j, \nabla'_{X_i} Z\rangle = 0, \quad i,j = 1,\dots,n,$$

which implies that Z satisfies (2.11). $\square$

3. Local invariants for isometric embeddings

Let $\tilde{M}$ be an n-dimensional C^∞ submanifold of $\mathbb{R}^{n+d} = \{(u^1, \ldots, u^{n+d})\}$. After Euclidean motions, $\tilde{M}$ is locally given by

$$u^{n+\sigma} = h^{n+\sigma}(u^1, \ldots, u^n), \quad \sigma = 1, \ldots, d. \tag{3.1}$$

A local invariant of order m is a differential function of m-th jet of

$$h(u^1, \ldots, u^n) = (h^1(u^1, \ldots, u^n), \ldots, h^d(u^1, \ldots, u^n)),$$

which is invariant under isometries of $\tilde{M}$.

First, we consider differential functions of m-th jet of h which are invariant under the action of $\mathcal{L}_0$, the infinitesimal isometries of $\mathbb{R}^{n+d}$. Since $\mathcal{L}_0$ is of dimension $(n+d)(n+d+1)/2$ with generators (2.9) and (2.10) and the dimension of the m-th jet space of h is $n + d\binom{n+m}{m}$, by Proposition 1.2 we have

Theorem 3.1 *In the algebra of differential functions of m-th jet, $m \geq 2$, of a system of d functions $(h^1, \ldots, h^d)$ of n independent variables $(u^1, \ldots, u^n)$, there are $n + d\binom{n+m}{m} - (n+d)(n+d+1)/2$ functionally independent differential functions which are annihilated by $\mathcal{L}_0$.*

Now let M be as in §2. Since the problem is local, we may regard M as an open subset of $\mathbb{R}^n$ with the standard coordinates $(x^1, \ldots, x^n)$ and let $U = \mathbb{R}^{n+d} = \{(u^1, \ldots, u^{n+d})\}$. For each positive integer m we define a map π from an open subset Ω^m of $J^m(M, \mathbb{R}^{n+d})$ to the m-th jet space $J^m(\mathbb{R}^n, \mathbb{R}^d)$ of (3.1) as follows. For $m = 1$, consider the Chain rule

$$\frac{\partial u^{n+\sigma}}{\partial x^i} = \sum_{k=1}^{n} h_k^\sigma \frac{\partial u^k}{\partial x^i}, \quad i = 1, \ldots, n, \tag{3.2}$$

for the function $u^{n+\sigma} = h^{n+\sigma}(u^1, \ldots, u^n)$, $\sigma = 1, \ldots, d$. Let Ω^1 be the subset of $J^1(M, \mathbb{R}^{n+d})$ on which $[\frac{\partial u^k}{\partial x^i}]_{i,k=1,\ldots,n}$ is non-singular. Then on Ω^1, we solve (3.2) for h_k^σ in terms of u_i^α, so define $\pi : \Omega^1 \to J^1(\mathbb{R}^n, \mathbb{R}^d)$ by

$$\begin{aligned} \pi : (x, u, u_i^\alpha : \alpha = 1, \ldots, n+d,\ i = 1, \ldots, n) \\ \mapsto (u, h_k^\sigma : \sigma = 1, \ldots, d,\ k = 1, \ldots, n). \end{aligned}$$

For $m = 2$, to define $\pi : \Omega^2 \subset J^2(M, \mathbb{R}^{n+d}) \to J^2(\mathbb{R}^n, \mathbb{R}^d)$, differentiate

(3.2) to get

$$\frac{\partial^2 u^{n+\sigma}}{\partial x^i \partial x^j} = \sum_{k,s=1}^{n} h^\sigma_{ks} \frac{\partial u^k}{\partial x^i} \frac{\partial u^s}{\partial x^j} + \sum_{k=1}^{n} h^\sigma_k \frac{\partial^2 u^k}{\partial x^i \partial x^j}. \tag{3.3}$$

Let Ω^2 be the subset of $J^2(M, \mathbb{R}^{n+d})$ on which $[\frac{\partial u^k}{\partial x^i}]_{i,k=1,\ldots,n}$ is non-singular. Then on Ω^2, we solve (3.2) and (3.3) for h^σ_k, h^σ_{ks}, in terms of u^α_i, u^α_{ij}, so define $\pi : \Omega^2 \to J^2(\mathbb{R}^n, \mathbb{R}^d)$ by

$$\begin{aligned}\pi : (x, u, u^\alpha_i, u^\alpha_{ij} : \alpha = 1, \ldots, n+d,\ i,j = 1, \ldots, n) \\ \mapsto (u, h^\sigma_k, h^\sigma_{ks} : \sigma = 1, \ldots, d,\ k, s = 1, \ldots, n).\end{aligned}$$

We define $\pi : \Omega^m \to J^m(\mathbb{R}^n, \mathbb{R}^d)$ inductively for each positive integer m.

Now we define the notions of invariants for isometric embedding. We shall call a differential function as in Theorem 3.1 an extrinsic invariant:

Definition 3.2 An extrinsic invariant for (2.1) of order $m \geq 1$, is a differential function $\mathbf{a}$ defined on $J^m(\mathbb{R}^n,\ \mathbb{R}^d)$ such that

$$(pr^{(m)} V)(\mathbf{a} \circ \pi) = 0, \quad \mod \mathcal{I},$$

for all $V \in \mathcal{L}_0$.

To define another notion of invariant, consider the cases in which there exists a 1-parameter family of solutions of (2.1). A basic fact on the infinitesimal symmetries is the following.

Proposition 3.3 *Suppose that $V_Q = \sum_{\alpha=1}^{n+d} Q^\alpha[u] \frac{\partial}{\partial u^\alpha}$ is an infinitesimal symmetry and that $u = F(x)$ is a solution of (2.1). Suppose that a mapping*

$$v = (v^1, \ldots, v^{n+d}) : M \times (-\epsilon, \epsilon) \to \mathbb{R}^{n+d}$$

satisfies

$$\begin{cases} \frac{\partial v^\alpha(x,t)}{\partial t} = Q^\alpha(x, v^{(m)}),\ \alpha = 1, \ldots, n+d, \\ v(x, 0) = F(x), \end{cases} \tag{3.4}$$

where $v^{(m)} = \{(\partial/\partial x^1)^{\alpha_1} \cdots (\partial/\partial x^n)^{\alpha_n} v : \alpha_1 + \cdots + \alpha_n \leq m\}$. Then for each $t \in (-\epsilon, \epsilon)$, $v(\cdot, t)$ is a solution of (2.1).

Proof. See [Olv]. □

A solution $v(x,t)$ to the evolution equation (3.4) is a 'bending' of the embedding F. An infinitesimal invariant is a differential function which remains unchanged under any bending. However, a solution of (3.4) does not always exist, and thus we are led to define the following

Definition 3.4 A differential function $\mathbf{a}$ defined on $J^m(\mathbb{R}^n, \mathbb{R}^d)$ is an infinitesimal invariant of order m of (2.1) if

$$(prV_Q)(\mathbf{a} \circ \pi) = 0, \quad \bmod \mathcal{I},$$

for any infinitesimal symmetry V_Q of (2.1).

Remark. In Definition 3.4 if the characteristic $Q[u] = (Q^1[u], \ldots, Q^{n+d}[u])$ of an infinitesimal symmetry V_Q are differential functions of order r then

$$(prV_Q)(\mathbf{a} \circ \pi) = (pr^{(m)}V_Q)(\mathbf{a} \circ \pi)$$

is a differential function of order $m+r$, defined on an open subset Ω^{m+r} of $J^{m+r}(M, \mathbb{R}^{n+d})$. (2.1) is a condition for an embedding $u = (u^1, \ldots, u^{n+d})$ and thus

$$\left[\frac{\partial u^\alpha}{\partial x^i}\right]_{i=1,\ldots,n}^{\alpha=1,\ldots,n+d}$$

is of maximal rank n on the solution subvariety S^{m+r} of $J^{m+r}(M, \mathbb{R}^{n+d})$ given by the ideal $\mathcal{I}^{m+r}$, therefore $S^{m+r} \cap \Omega^{m+r}$ is non-empty.

Definition 3.5 A differential function $\mathbf{a}$ defined on $J^m(\mathbb{R}^n, \mathbb{R}^d)$ is an intrinsic invariant of order m of (2.1) if $\mathbf{a} \circ \pi = k(x) \bmod \mathcal{I}$, for some function $k(x)$. We call $k(x)$ the intrinsic expression of $\mathbf{a}$.

If $u = (u^1, \ldots, u^{n+d}) : M \to \mathbb{R}^{n+d}$ is a solution of (2.1) such that $[\frac{\partial u^\alpha}{\partial x^i}]_{\alpha,i=1,\ldots,n}$ is non-singular we put $(\mathbf{a} \circ \pi)(u^{(m)}) = A(u^{(m)})$. Then we have

$$\kappa(x) = A(u^{(m)}), \tag{3.5}$$

which is a partial differential equation that a C^m solution of (2.1) satisfies. For surfaces in $\mathbb{R}^3$, principal curvatures are extrinsic invariants and the Gaussian curvature is an intrinsic invariant. The latter is the Theorema Egregium of Gauss. Relations among the three notions of invariants are the following.

Theorem 3.6 *Let* $\mathbf{a}$ *be a differential function defined on* $J^m(\mathbb{R}^n, \mathbb{R}^d)$ *of order* m. *If* $\mathbf{a}$ *is an infinitesimal invariant of* (2.1), *then* $\mathbf{a}$ *is an extrinsic* *invariant of* (2.1). *If* $\mathbf{a}$ *is an intrinsic invariant of* (2.1), *then* $\mathbf{a}$ *is an infinitesimal invariants of* (2.1).

Proof. The first assertion is obvious, for an infinitesimal rigid motion of $\mathbb{R}^{n+d}$ is an infinitesimal symmetry of (2.1) (Theorem 2.2).

Now suppose that $\mathbf{a}$ is an intrinsic invariant and suppose that

$$V_Q = \sum_{\alpha=1}^{n+d} Q^\alpha[u] \frac{\partial}{\partial u^\alpha}$$

is an infinitesimal symmetry of (2.1). Since $\mathbf{a} \circ \pi = b + k(x)$, for some $b \in \mathcal{I}$,

$$(prV_Q)(\mathbf{a} \circ \pi) = (prV_Q)b + (prV_Q)k(x),$$

where the second term of the right hand side is zero and the first term of right hand side is contained in $\mathcal{I}$, and therefore,

$$(prV_Q)(\mathbf{a} \circ \pi) = 0, \ \ \mathrm{mod}\ \mathcal{I}.$$

□

Finally, we calculate intrinsic invariants of a surface in $\mathbb{R}^3$ given by $u^3 = h(u^1, u^2)$ by means of Definition 3.5.

First, we observe that the Lie algebra $\mathcal{L}_0$ of infinitesimal isometries of $\mathbb{R}^3$ is of dimension 6 and $J^1(\mathbb{R}^2, \mathbb{R})$ is of dimension 5, therefore, there is no extrinsic invariants of order 1. By Theorem 3.1 with $n = 2$, $d = 1$, $m = 2$, there are two functionally independent extrinsic invariants. Principal curvatures $\lambda_i, i = 1, 2$ are those extrinsic invariants. They are given by

$$\frac{(1 + (h_2)^2)h_{11} - 2h_1h_2h_{12} + (1 + (h_1)^2)h_{22} \pm \sqrt{A}}{2(1 + (h_1)^2 + (h_2)^2)^{3/2}}, \tag{3.6}$$

where

$$\begin{aligned} A = {} & [(1 + (h_2)^2)h_{11} - 2h_1h_2h_{12} + (1 + (h_1)^2)h_{22}]^2 \\ & - 4(1 + (h_1)^2 + (h_2)^2)(h_{11}h_{22} - (h_{12})^2). \end{aligned}$$

Any function of λ_i, $i = 1, 2$, is also an extrinsic invariant. Now we show that the Gaussian curvature

$$\lambda_1\lambda_2 = \frac{h_{11}h_{22} - (h_{12})^2}{(1 + (h_1)^2 + (h_2)^2)^2} \tag{3.7}$$

is an intrinsic invariant in the sense of Definition 3.5.

(3.2) and (3.3) with $n = 2$, $d = 1$, are

$$\begin{cases} u^3_x = h_1 u^1_x + h_2 u^2_x, \\ u^3_y = h_1 u^1_y + h_2 u^2_y, \end{cases} \tag{3.8}$$

and

$$\begin{cases} u^3_{xx} = h_{11} u^1_x u^1_x + h_{12}(u^1_x u^2_x + u^2_x u^1_x) + h_{22} u^2_x u^2_x + h_1 u^1_{xx} + h_2 u^2_{xx}, \\ u^3_{xy} = h_{11} u^1_x u^1_y + h_{12}(u^1_x u^2_y + u^2_x u^1_y) + h_{22} u^2_x u^2_y + h_1 u^1_{xy} + h_2 u^2_{xy}, \\ u^3_{yy} = h_{11} u^1_y u^1_y + h_{12}(u^1_y u^2_y + u^2_y u^1_y) + h_{22} u^2_y u^2_y + h_1 u^1_{yy} + h_2 u^2_{yy}. \end{cases} \tag{3.9}$$

By solving (3.8) and (3.9) for h_i and h_{ij}, $i, j = 1, 2$, and substituting in (3.7), we get $(\lambda_1 \lambda_2) \circ \pi$, which is a differential function on $\Omega^2 \subset J^2(\mathbb{R}^2, \mathbb{R}^3)$ as defined in §3. Then eliminate the elements of $\mathcal{I}$ from $(\lambda_1 \lambda_2) \circ \pi$, where $\mathcal{I}$ is the ideal as defined in §2 with $n = 2$, $d = 1$. We used Mathematica® for the symbolic calculations, to get the following.

Theorem 3.7 *The Gaussian curvature* (3.7) *is an intrinsic invariant of order* 2. *In fact,*

$$\begin{aligned} &4 \frac{h_{11} h_{22} - (h_{12})^2}{(1 + (h_1)^2 + (h_2)^2)^2} \circ \pi \\ &= (g_{11} g_{22} - (g_{12})^2)^{-2} [2((g_{12})^2 - g_{11} g_{22})(g_{11,yy} - 2 g_{12,xy} + g_{22,xx}) \\ &\quad + g_{11}(g_{11,y} g_{22,y} - 2 g_{12,x} g_{22,y} + (g_{22,x})^2) \\ &\quad + g_{12}(g_{11,x} g_{22,y} - g_{11,y} g_{22,x} - 2 g_{11,y} g_{12,y} \\ &\qquad\qquad + 4 g_{12,x} g_{12,y} - 2 g_{12,x} g_{22,x}) \\ &\quad + g_{22}(g_{11,x} g_{22,x} - 2 g_{11,x} g_{12,y} + (g_{11,y})^2)], \quad \mod \mathcal{I}, \end{aligned} \tag{3.10}$$

where $g_{11,x} = \frac{\partial g_{11}}{\partial x}$, *and so forth.*

The right hand side of (3.10) is the intrinsic expression of the Gaussian curvature.

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