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Fatou property of harmonic maps from complete manifolds with simple type ends

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Abstract. In this paper, we consider harmonic maps on a class of complete noncompact manifolds, we prove the existence and Fatou property for harmonic maps into convex balls

Key words: complete manifold, harmonic map, Fatou property, convex ball.

1. Introduction and Results

The boundary behavior, so-called Fatou property, of the solution of an elliptic equation has been attracting many mathematicians. But more attention was concentrated on bounded domains. Until the beginning of 1960's (or, earlier, due to Gilbarg and Serrin [4]), J. Moser [10] did not use his Harnack inequality to consider the behavior at infinity of the solution to a divergence-type uniformly elliptic equation on unbounded domains.

Recently, P. Aviles, H. I. Choi and M. Micallef considered the behavior at infinity of harmonic maps from Cartan-Hadamard manifolds with curvature K satisfying $-a^2 \leq K \leq -b^2 < 0$ to convex balls. They found that in this case the result is completely analogous to that on bounded domains. From the point of view of Martin boundary, this phenomenon is very natural. Because the Martin boundary of the above Cartan-Hadamard manifolds is actually S^n [0].

A natural problem is that when the Martin boundary is a point or finitely many points, how is the situation? Does the above phenomenon also appear? On the other hand, the well-known Liouville-type theorem due to Hildebrandt, Jost and Widman tells us that a harmonic map from a simple Riemannian manifold to a geodesiclly convex ball has to be constant. And from Section 2 we know that the Martin boundary of such manifolds is a point. Thus, one should ask if the above Hildebrandt-Jost-Widman's theorem can be interpreted as Fatou property.

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The present paper is a continuation of the previous paper [17], where we discussed Fatou property for the domain manifolds being complete Riemannian manifolds with nonnegative sectional curvature at infinity. Here, we will discuss the case for the following domain manifolds: M^m $(m \geq 3)$, complete noncompact Riemannian manifolds with finitely many ends such that each end is isometric to the complement of a compact subset in a simple Rimannian manifold (See Section 2). Our results are as follows.

Theorem 1 *Let M^m be a complete Riemannian manifold as above, whose ends are $E_1, \ldots, E_s$, $s \geq 2$, N^n be a complete Riemannian manifold, $B_Q(\tau)$ be a geodesic convex ball (See section 2) in N. Then for any given points $p_1, \ldots, p_s \in B_Q(\tau)$, there exists a unique harmonic map $f : M \to B_Q(\tau)$ with $f(x) \to p_\sigma$, as $x \in E_\sigma, x \to \infty$ and $E(f) < \infty$.*

Theorem 2 *Let M^m, $B_Q(\tau)$ be as in Theorem 1, $f : M \to B_Q(\tau)$ be a harmonic map such that the energy density is bounded and the energy is finite. Then Fatou property holds, i.e., there exist $p_1, \ldots, p_s$ such that $f(x) \to p_\sigma$ as $x(\in E_\sigma) \to \infty$.*

Corollary 1 *Let $s = 1$. Then there is no nonconstant harmonic map $f : M \to B_Q(\tau)$ with bounded energy density and finite energy.*

The theorem due to Hildebrandt, Jost and Widman [7] tells us that when M is a simple Riemannian manifold, any harmonic map $f : M \to B_Q(\tau)$ is constant, this makes us believe that the complexity of topology results in the existence of nonconstant harmonic maps. From this point of view, Fatou property is a natural generalization of Liouville type theorem. We think that there exists close relation between Fatou property and Liouville type theorems. The understanding of the behavior at infinity of harmonic maps on various kinds of manifolds makes Liouville type theorems clearer.

Now, we discuss the conditions in Theorem 2. From Theorem 1, the finite energy condition seems to be reasonable. But, thanks to Liouville-type theorem in [7], we believe that the condition can be omitted. In addition, requiring bounded energy density is unsatisfactory, but if for each end $E_\sigma, \sigma = 1, \ldots, s$, Riemannian metric $(\gamma_{\alpha\beta})$ satisfies $|\gamma_{\alpha\beta}|_{C^1} < +\infty$, then for any harmonic map $f : M^m \to B_Q(\tau)$, its energy density is bounded. This is a direct consequence of [3, Proposition 8, 5.13 or Theorem 4(i)] (also see [11]).

This paper is organized as follows: In the section 2, we give some necessary preliminaries, in particular, positive harmonic functions on M^m, and Harnack inequality due to J. Moser. In the section 3 and section 4, we give the proofs of theorem 1 and theorem 2

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2. Preliminaries

A complete Riemannian manifold V^m is called simple if it is diffeomorphic to R^m, and the corresponding metric g_{ij} satisfies that there exist numbers $\Lambda \geq \lambda > 0$,

$$\lambda|\xi|^2 \leq g_{ij}\xi^i\xi^j \leq \Lambda|\xi|^2, \tag{1}$$

where $x \in R^m$, $\xi \in R^m$. Throughout this paper, assuming that M^m $(m \geq 3)$ is a complete Riemannian manifold satisfying that there exists a compact subset $D \subset\subset M$ with s components of $M \backslash D$, denoted by $E_1, \ldots, E_s$ $(s \geq 2)$, each of which is isometric to the complement of a closed ball in some simple Riemannian manifold (unless otherwise specified).

The following lemma might be known, but we are not able to find it in the literature, so give a simple proof.

Lemma 1 *Let V^m be a simple Riemannian manifold, $p \in V$, and $B_p(r_0)$ be a geodesic ball centered at p with radius r_0. Then there exists a unique harmonic function f on $V \setminus B_p(r_0)$ with $\int_M |\nabla f|^2 < \infty$, $f|_{\partial B_p(r_0)} = 1$ and $\lim_{x\to\infty} f(x) = 0$.*

Proof. In the present setting, Laplace-Beltrami operator on V is obviously uniformly elliptic. From [5, Theorem 1.1], one knows that there exists a positive Green function $G(x, y)$ on V satisfying for all $x, y \in V$

$$G(x,y) \leq K_1(m,\lambda,\Lambda)|x-y|^{2-m}, \tag{2}$$

$$G(x,y) \geq K_2(m,\lambda,\Lambda)|x-y|^{2-m}, \tag{3}$$

where K_1, K_2 depend only on m, λ, Λ, and $|\cdot|$ is Euclidean distance. Obviously, fixing y, $\lim_{x\to\infty} G(x,y) = 0$ and $G(x,p)$ is harmonic on $V \setminus B_p(r_0)$. One can now imitate the method of [13, Section 4] to construct the required harmonic function f as follows: Choose a sequence of numbers

$R_0 < R_1 < R_2 < \ldots$ and consider the following Dirichlet problem

$$\begin{cases} \triangle_V f_i = 0, & \text{on } B_p(R_i) \setminus B_p(r_0), \\ f_i|_{\partial B_p(R_i)} = 0, & f_i|_{\partial B_p(r_0)} = 1, \end{cases}$$

where $\triangle_V$ is Laplace-Beltrami operator with respect to Riemannian metric on V. One is able to obtain a sequence of harmonic functions $\{f_i\}_{i=1}^{\infty}$, satisfying $f_i \leq f_{i+1}$ and $f_i \leq 1$ on $B_p(R_i) \setminus B_p(r_0)$ by Maximum principle. By means of the standard elliptic estimates, one has that $\{f_i\}$ uniformly converges to some harmonic function, denoted by f, on any compact subset of $V \setminus B_p(r_0)$, with $f|_{\partial B_p(r_0)} = 1$. Choose a constant C satisfying $CG(x,p) \mid \partial B_p(r_0) \geq 1$, by the harmonicity of $f(x)$, $G(x,p)$ on $V \setminus B_p(r_0)$ and the behavior at infinity of $G(x,p)$, one has

$$f_i(x) \leq CG(x,p), \quad \forall x \in B_p(R_i) \setminus B_p(r_0).$$

Thus, $f(x) \leq CG(x,p)$, $\forall x \in V \setminus B_p(r_0)$, therefore $\lim\limits_{x \to \infty} f(x) = 0$.

Remained is to prove the energy finiteness, which is analogous to that of [12, Theorem 2.1] (also see [17]). By the harmonicity and the boundary value of f, one has

$$\begin{aligned} \int\limits_{B_p(R_i) \setminus B_p(r_0)} |\nabla f_i|^2 &= \int\limits_{B_p(R_i) \setminus B_p(r_0)} \nabla(f_i \nabla f_i) \\ &= - \int\limits_{\partial B_p(r_0)} \frac{\partial f_i}{\partial \gamma}, \end{aligned}$$

where γ is the unit outer normal vector. Fixing $R > r_0$, when $R_i > R$, one has

$$\int\limits_{B_p(R) \setminus B_p(r_0)} |\nabla f_i|^2 \leq - \int\limits_{\partial B_p(r_0)} \frac{\partial f_i}{\partial \gamma}.$$

Let i go to infinity, one has

$$\int\limits_{B_p(R) \setminus B_p(r_0)} |\nabla f|^2 \leq - \int\limits_{\partial B_p(r_0)} \frac{\partial f}{\partial \gamma}.$$

Thus $\int\limits_{V \setminus B_p(r_0)} |\nabla f|^2 < +\infty$. This completes the proof of Lemma 1. □

Remark 1. By the definition of M, there exists a harmonic function f_σ on

E_σ with $\lim_{x(\in E_\sigma)\to\infty} h_\sigma(x) = 0$ and $h_\sigma(x) = 1$ as $x \in \partial E_\sigma$ $(\sigma = 1, \ldots, s)$.

For the sake of convenience, we state Harnack inequality due to J. Moser and its variant. For details see [15].

Theorem (Harnack inequality). *Let u be a nonnegative solution of the equation $\partial_i(a^{ij}\partial_j u) = 0$ on $B_O(R) \subset R^m$, where a^{ij} satisfy (1). $a^{ij} = a^{ji}$; (2). $\exists \lambda, \Lambda > 0$, $\lambda|\xi|^2 \le a^{ij}\xi^i\xi^j \le \Lambda|\xi|^2$, $\forall \xi \in R^m$. Then $\forall \theta \in (0,1)$, one has*

$$\sup_{B_O(\theta R)} u \le C \inf_{B_O(\theta R)} u,$$

where C is a constant depending only on m, $\dfrac{\Lambda}{\lambda}$, and θ.

Corollary *Let u be a nonnegative solution of the equation in the above theorem on $R^m \setminus B_O(1)$. Then for any $R > 3$, one has*

$$\sup_{\partial B_O(R)} u \le C \inf_{\partial B_O(R)} u,$$

where C is constant depending only on $m, \dfrac{\Lambda}{\lambda}$.

Proof. It is easy to see that for any $R > 3$, there exist points $x_1, \ldots, x_k$ on $\partial B_O(R)$ such that

$$\bigcup_{i=1}^{k} B_{x_i}\left(\frac{R}{3}\right) \supset \partial B_O(R),$$

where $k = k(m)$ depends only on m. Remained is a modification of the proof of [13, Section 3, Theorem 3.2], we omit it. $\square$

In order to motivate Fatou property, we observe the distribution of Martin boundary points for the present manifold M^m. To the aim, we firstly need to discuss the Green function on M. By means of [14, Section 2, Remark 1] and the above remark 1, there exists a minimal positive Green function $G(x,y)$ (see [2]) on M. Choosing $x_0 \in M$, considering

$$\frac{G(x,y)}{G(x_0,y)},$$

and using Harnack inequality, the similar discussion of [11, Section 3] deduces

$$\lim_{y(\in E_\sigma)\to\infty} \frac{G(x,y)}{G(x_0,y)} = \frac{f_\sigma(x)}{f_\sigma(x_0)},$$

where f's are the harmonic functions in Lemma 3 below. Thus, from the definition of Martin boundary [12], each end of M, E_σ, corresponds to a Martin boundary point.

From the point of view of Martin boundary, the following lemma is natural.

Lemma 2 *Let h be a nonnegative harmonic function on E_σ. Then there exists a nonnegative constant a with $0 \le a < \infty$ and $\lim_{x\to\infty} h(x) = a$.*

Proof. Using Harnack inequality, completely similar to the proof of [13, Theorem 3.3], one can show that $\lim_{x\to\infty} h(x)$ exists (also see [10, Section 5]). On the other hand, by Lemma 1, there exists a barrier g_σ with $\lim_{x\to\infty} g_\sigma(x) = 1$ and $g_\sigma|_{\partial E_\sigma} = 0$. If $\lim_{x\to\infty} h(x) = \infty$, one can fix $y \in E_\sigma$ and choose a positive number A with $\frac{h(y)}{A} < g_\sigma(y)$. But Maximum principle implies $\frac{h(x)}{A} \ge g_\sigma(x)$, $\forall x \in E_\sigma$. This is a contradiction. □

From the existence of barrier functions in Lemma 1, we also have the following

Lemma 3 (1). *There exists a unique positive harmonic functions f_σ on M satisfying $\lim_{x(\in E_\sigma)\to\infty} f_\sigma(x) = 1$ and $\lim_{x(\notin E_\sigma)\to\infty} = 0$;* (2). *Any bounded harmonic function is a combination of $\{f_1, \dots, f_s\}$.*

Its proof is similar to [13, Section 6, Theorem 6.1]. We omit it.

Finally, we state two lemmas, which are useful in the sequel development. They can be found in [1] and [8] respectively. Firstly, we give a definition: geodesic convex ball $B_Q(\tau)$ of N^n. Let N^n be a Riemannian manifold, $B_Q(\tau)$ be a geodesic convex ball in N^n, i.e., the geodesic ball centered at Q with radius $\tau, \tau < \frac{\pi}{2\sqrt{\kappa}}$, and $B_Q(\tau)$ lies inside the cut-locus of Q, here κ is an upper bound of the sectional curvature of $N, \kappa \ge 0$. In addition, Ω always denotes a bounded domain with smooth boundary in a complete noncompact manifold.

Lemma 4 *Given $\varphi \in C^0(\partial\Omega, B_Q(\tau))$, let $u \in C^0(\bar{\Omega}, B_Q(\tau)) \cap C^\infty(\Omega, B_Q(\tau))$ be a harmonic map on Ω which equals φ on $\partial\Omega$. With respect to geodesic normal coordinates centered at p, φ may also be viewed as being R^n-valued. Let $h : \bar{\Omega} \to R^n$ be the harmonic extension of φ, i.e., $h = (h^1, \ldots, h^n)$, where h^i is a harmonic function for each i and $h|_{\partial\Omega} = \varphi$. Let $v : \bar{\Omega} \to R$ be the harmonic extension of $\frac{1}{2}|\varphi|^2 = \frac{1}{2}\sum_{i=1}^n(\varphi^i)^2$. Then, there exists a constant $C > 0$, depending only on the geometry of $B_Q(\tau)$ such that*

$$[\rho(u(x), h(x))]^2 \leq C(v(x) - \frac{1}{2}|h(x)|^2) \quad x \in \Omega, \tag{4}$$

where ρ is the distance function on N.

Lemma 5 *Let $h = (h^1, \ldots, h^n)$ be normal coordinates on $B_Q(2\tau)$ such that Q has coordinates $(0, \ldots, 0)$. Denote by $g_{ik}(h), \Gamma^l_{ik}(h)$, and $\Gamma_{ikl}(h)$ the metric and Christoffel symbols, respectively, in this coordinates system. Then for all h satisfying $|h| = (\sum_{i=1}^n h^i h^i)^{\frac{1}{2}} \leq 2\tau < \frac{\pi}{\sqrt{\kappa}}$ and all $\xi \in R^n$ we have the following estimates*

$$\Gamma^l_{ik}(h)h^l\xi^i\xi^k \leq \{\delta_{ik} - a_\kappa(|h|)g_{ik}(h)\}\xi^i\xi^k, \tag{5}$$

where

$$a_\kappa(t) = \begin{cases} t\sqrt{\kappa}ctg(t\sqrt{\kappa}), & \kappa > 0,\ 0 \leq t < \frac{\pi}{\sqrt{\kappa}}, \\ 1, & \kappa = 0,\ 0 \leq t < \infty\ . \end{cases}$$

3. The proof of Theorem 1

Similar to the method of [1] and [17], we will use bounded harmonic functions on M obtained in Lemma 3 to approximate harmonic maps. Its key is Lemma 4. Let ends of M be $E_1, \ldots, E_s$ ($s \geq 2$), correspondingly, one has positive harmonic functions $f_1, \ldots, f_s$ satisfying

$$\begin{cases} \lim\limits_{x(\in E_\sigma)\to\infty} f_\sigma(x) = 1, \\ \lim\limits_{x(\notin E_\sigma)\to\infty} f_\sigma(x) = 0. \end{cases}$$

One can fix a normal coordinate on $B_Q(\tau)$ as in Lemma 5 and set the corresponding coordinate of p_σ being $(h^1_\sigma, \ldots, (h^n_\sigma)$. Construct n functions $\sum_{\sigma=1}^s h^k_\sigma f_\sigma$, $1 \leq k \leq n$, which are harmonic. Setting $h = (\sum_{\sigma=1}^s h^1_\sigma f_\sigma, \ldots, \sum_{\sigma=1}^s h^n_\sigma f_\sigma)$, under the above normal coordinate of $B_Q(\tau)$, h defines a map from M to $B_Q(\tau)$, denoted still by h. This is because $\lim\limits_{x(\in E_\sigma)\to\infty} h(x) = p_\sigma$ $(\sigma = 1, \ldots, s)$ and maximum principle implies

$$\sum_{k=1}^{n}\Big(\sum_{\sigma=1}^{s} h_\sigma^k f_\sigma(x)\Big)^2 \leq \tau^2, \quad \forall x \in M.$$

Choosing $\{R_i\}_i^\infty$ with $R_1 < R_2 <, \ldots, < R_l \to \infty$, as $l \to \infty$, and fixing $x_0 \in M$, $B_{x_0}(R_i)$ is a geodesic ball at x_0 with radius R_i. By means of [8, Theorem 1] and [9, Theorem 1], there exists a uniquely harmonic map $u_i : B_{x_0}(R_i) \to B_Q(\tau)$ with $u_i|_{\partial B_{x_0}(R_i)} = h|_{\partial B_{x_0}(R_i)}$. On the other hand, [3, Theorem 4] implies $\{u_i\}_{i=1}^\infty$ uniformly converges to a harmonic map on arbitrary compact subset of $M, u : M \to B_Q(\tau)$. From Lemma 4, one has

$$[\rho(u_i(x), h(x))]^2 \leq C\Big(v_i(x) - \frac{1}{2}|h(x)|^2\Big), \quad \forall x \in B_{x_0}(R_i),$$

where v_i is the harmonic extension of $\frac{1}{2}|h(x)|^2|_{\partial B_{x_0}(R_i)}$, C depends only on the geometry of $B_Q(\tau)$.

Now,we set $w = \frac{1}{2}\sum_{\sigma=1}^{s}\sum_{k=1}^{n}(h_\sigma^k)^2 f_\sigma$, which is harmonic on M and has the same behavior at infinity as $\frac{1}{2}|h(x)|^2$. It is obvious that $v_i(x) - \frac{1}{2}|h(x)|^2 < w(x) - \frac{1}{2}|h(x)|^2$, on $B_{x_0}(R_i)$. Thus, one has

$$[\rho(u_i(x), h(x))]^2 \leq C\Big(w(x) - \frac{1}{2}|h(x)|^2\Big), \quad \forall i.$$

Hence, $[\rho(u(x), h(x))]^2 \leq C(w(x) - \frac{1}{2}|h(x)|^2)$, i.e., u is harmonic and $u(x) \to p_\sigma$ as $x \in E_\sigma, x \to \infty$ $(\sigma = 1, \ldots, s)$.

From [9], we know that u is unique. we interpret as follows: Let u_1 be another harmonic map $M \to B_Q(\tau)$ with $u_1(x) \to p_\sigma$ as $x \in E_\sigma, x \to \infty$ $(\sigma = 1, \ldots, s)$. From [9, Theorem 1], we have that the following function on $B_{x_0}(R_i)$ satisfies maximum principle:

$$\rho_i(x) = \frac{q_k(\rho(u(x), u_1(x)))}{cos(\sqrt{\kappa}\rho(Q, u(x)))cos(\sqrt{\kappa}\rho(Q, u_1(x)))}$$

where $q_k : R \to R$, defined by

$$q_k(t) = \begin{cases} \dfrac{(1 - cos\sqrt{\kappa}t)}{\kappa}, & \kappa > 0 \\ \dfrac{t^2}{2}, & \kappa = 0. \end{cases}$$

Obviously, $\rho_i(x) \to 0$ as $i \to \infty$, i.e., $u = u_1$.

Remained is to prove the energy finiteness. The uniqueness of u_i implies that its energy is minimal i.e.,

$$\int\limits_{B_{x_0}(R_i)} |\nabla u_i|^2 \leq \int\limits_{B_{x_0}(R_i)} |\nabla h|^2, \forall i,$$

hence, one has

$$\int\limits_{B_{x_0}(R)} |\nabla u_i|^2 \leq \int\limits_{M} |\nabla h|^2,$$

for any $R > 0$ with $R_i > R$. Thus

$$\int\limits_{B_{x_0}(R)} |\nabla u|^2 \leq \int\limits_{M} |\nabla h|^2.$$

So, we only need to prove $\int\limits_{M} |\nabla h|^2 < \infty$. To this aim, it is sufficient to prove $\int\limits_{M} |\nabla f_\sigma|^2 < \infty, \sigma = 1, \ldots, s$. Noting the construction of f_σ and Lemma 1, this is easy to prove. Thus we complete the proof of Theorem 1.

Remark 2. In Theorem 1, we assume $s \geq 2$. In case of $s = 1$, by the discussion of the uniqueness, it is easy to see that no nonconstant harmonic map $u : M \to B_Q(\tau)$ with $u(x) \to p \in B_Q(\tau)$ as $x \to \infty$, which is also true for the harmonic map $u : M \to B_Q(\tau)$ with $\lim\limits_{x(\in M)\to\infty} u(x) = p \in B_Q(\tau)$ for $s \geq 2$.

4. The proofs of Theorem 2 and Corollary 1

In order to make Theorem 2 more general than that stated, we consider the Green function on each end E_σ ($\sigma = 1, \ldots, s$) and its properties. Equivalently, we consider the Green function on $V \setminus B_p(r_0)$ as in Lemma 1. Choosing $\{R_i\}_{i=1}^{\infty}$ with $5r_0 < R_1 < R_2 <, \ldots, < R_l \to \infty$, as $l \to \infty$, by means of [5, Theorem 1.1], there exists a Green function $G_i(x, y)$ on $B_p(R_i) \setminus B_p(r_0)$ with respect to Dirichlet boundary value satisfying that for any $x, y \in B_p(R_i) \setminus B_p(r_0)$

$$G_i(x, y) \leq K_1(m, \lambda, \Lambda)|x - y|^{2-m},$$

holds and for any $x, y \in B_p(R_i) \setminus B_p(r_0)$ with $|x-y| \le \frac{1}{2}\mathrm{dist}(y, \partial B_p(R_i) \cup \partial B_p(r_0))$

$$G_i(x,y) \ge K_2(m,\lambda,\Lambda)|x-y|^{2-m},$$

holds. Thus one can use the multiplities of $|x-y|^{2-m}$ as a barrier function and show $\{G_i(x,y)\}_{i=1}^{\infty}$ uniformly converges to a Green function, denoted by $G(x,y)$, on any compact subset of $V \setminus B_p(r_0)$, which satisfies that for $x \in V \setminus B_p(r_0), G(x,y)|_{\partial B_p(r_0)} = 0$ and $\lim\limits_{y\to\infty} G(x,y) = 0$. In addition, for $x, y \in V \setminus B_p(r_0)$

$$G(x,y) \le K_1(m,\lambda,\Lambda)|x-y|^{2-m} \tag{6}$$

holds and for $x, y \in V \setminus B_p(2r_0)$ with $|x-y| \le \frac{1}{2}\mathrm{dist}(y, \partial B_p(r_0))$

$$G(x,y) \ge K_1(m,\lambda,\Lambda)|x-y|^{2-m} \tag{7}$$

holds. Thus, from (6) and (7), one can easily deduce that for any $x \in V \setminus B_p(r_0)$ with $d(x,p) \ge 2r_0$ and some real number $q > 1$

$$\sup_{d(x,p)\ge 2r_0} \max_{\partial B_x(r_0)} G(x,y) < \infty, \tag{8}$$

$$\int\limits_{B_x(r_0)} |G(x,y)|^q < \infty. \tag{9}$$

Remark 3. The estimates (8), (9) are the conditions imposed on the Green function in [17]. In the present setting, because of excellent property of Laplace-Beltrami operator on each end, these conditions are naturally satisfied. It should be pointed out that the conditions in [17] are imposed on the Green function of M. In fact, it is unnecessary. Because each end of the manifolds discussed in [17] is large, on which there exists a positive Green function with respect to Dirichlet boundary value, so we only need that the Green function on each end satisfies (8), (9). Finally, the proof of Fatou property in [17] is completely similar to what we will do here. The following example also shows that such conditions are not reasonable: Suppose that each end is isometric to $R^3 \setminus B_O(1)$. On $R^3 \setminus B_O(1)$, the Green function is

$$G(X, X_0) = \frac{1}{4\pi}\Bigg[\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}$$

$$-\frac{1}{\sqrt{R_0^2R^2-2(xx_0+yy_0+zz_0)+1}}\Bigg],$$

where $X=(x,y,z)$, $X_0=(x_0,y_0,z_0)$, $R_0=|X_0|$, $R=|X|$. It is easy to check that the above Green function satisfies (8), (9). (The case of $R^m\setminus B_O(1)$ is similar.)

Proposition 1 *Let (V,γ) be a simply Riemannian manifold, for $p\in V$, $B_p(r_0)$ be a geodesic ball centered at p with radius r_0. Suppose that $u: V\setminus B_p(r_0)\to B_Q(\tau)$ is harmonic with bounded energy density and finite energy. Then $\exists p\in B_Q(\tau)$, $\lim\limits_{x\to\infty}u(x)=p$.*

Proof. Fix a normal coordinate on $B_Q(\tau)$ as before. Setting

$$|u(x)|^2=\sum_{i=1}^n|u^i(x)|^2,$$

a direct computation shows

$$\begin{aligned}\frac{1}{2}\triangle|u|^2 &= |\nabla u|^2+u^l\triangle u^l\\ &= |\nabla u|^2-u^l\Gamma^l_{ij}u^i_\alpha u^j_\beta\gamma^{\alpha\beta},\end{aligned}$$

where $(\gamma^{\alpha\beta})^{-1}$ is the metric on V, Γ^l_{ij} as in Lemma 5. Using Lemma 5, one has

$$\begin{aligned}\frac{1}{2}\triangle|u|^2 &\ge |\nabla u|^2-\{\delta_{ij}-a_\kappa(|u|)g_{ij}(u)\}u^i_\alpha u^j_\beta\gamma^{\alpha\beta}\\ &= a_\kappa(|u|)e(u)\ge 0,\end{aligned}\tag{10}$$

where $(g_{ij}), a_\kappa$ as in Lemma 5, $e(u)$ is the energy density of $u, a_\kappa(|u|)>0$ for $|u|<\dfrac{\pi}{2\sqrt{\kappa}}$.

Let $G_i(x,y)$ be the Green function at the beginning of this section. Considering $\int\limits_{B_p(R_i)\setminus B_p(r_0)}\triangle|u|^2(y)G_i(x,y)dy$, denoted by f_i, it satisfies

$$\begin{cases}\triangle f_i=-\triangle|u|^2, & \text{on } B_p(R_i)\setminus B_p(r_0),\\ f_i|\partial B_p(R_i)\cup\partial B_p(r_0)=0.\end{cases}$$

Maximum principle implies $f_i\le\tau^2$, so $\int\limits_{V\setminus B_p(r_0)}\triangle|u|^2(y)G(x,y)dy$, denoted by $f(x)$, is not greater than τ^2. By (10), one has

$$\int_{V\backslash B_p(r_0)} e(u)G(x,y)dy < \infty.$$

On the other hand, $|\triangle u^l| \le Ce(u), C$ depending only on the geometry of $B_Q(\tau)$. Thus, one can define

$$f^l = \int_{V\backslash B_p(r_0)} \triangle u^l(y)G(x,y)dy + u^l, \quad 1 \le l \le n. \tag{11}$$

Obviously, f^l is harmonic. Lemma 2 implies there exists a real number a^l with

$$f^l(x) \to a^l, \text{ as } x \to \infty.$$

If we can prove that $\lim_{x\to\infty} \int_{V\backslash B_p(r_0)} \triangle u^l(y)G(x,y)dy$ exists, then the proposition 1 is proved. To this aim, we claim that under the conditions of Proposition 1, one has

$$\int_{V\backslash B_p(r_0)} \triangle u^l(y)G(x,y)dy \to 0, \ 1 \le l \le n, \text{ as } x \to \infty. \tag{12}$$

Since $\int_{V\backslash B_p(r_0)} e(u)dy < \infty$, for any sufficiently small $\epsilon > 0$, there exists sufficiently large $R > 0$ with $\int_{V\backslash B_p(R)} |\triangle u^l| < \epsilon$. Fix $R > 5r_0$ and consider the integral $\int_{V\backslash B_p(R)} \triangle u^l(y)G(x,y)dy$. Assuming that $\mathrm{dist}(p,x)$ is sufficiently large and satisfies $B_x(r_0) \subset V \setminus B_p(R)$, one has

$$\begin{aligned}
&\Big|\int_{V\backslash B_p(R)} \triangle u^l(y)G(x,y)dy\Big| \\
&\le \Big|\int_{(V\backslash B_p(R))\backslash B_x(r_0)} \triangle u^l(y)G(x,y)dy\Big| + \Big|\int_{B_x(r_0)} \triangle u^l(y)G(x,y)dy\Big| \\
&\le \max_{\partial B_x(r_0)} G(x,y) \int_{(V\backslash B_p(R))\backslash B_x(r_0)} |\triangle u^l(y)|dy
\end{aligned}$$

$$+\Big(\int\limits_{B_x(r_0)} |\triangle u^l(y)|^r dy\Big)^{\frac{1}{r}}\Big(\int\limits_{B_x(r_0)} |G(x,y)|^q dy\Big)^{\frac{1}{q}},$$

where $\frac{1}{r}+\frac{1}{q}=1$. Since $|\triangle u^l|<\infty$, using (8), (9), the right hand side of the above inequality is sufficiently small.

In the following, we consider the integral $\int_{B_p(R)\setminus B_p(r_0)} \triangle u^l(y)G(x,y)dy$. Firstly, we observe the behavior $G(x,y)$ with respect to y on $B_p(R)\setminus B_p(r_0)$. Assuming that $d(x,p)$ is sufficiently large, the maximum value of $G(x,y)$ on $B_p(R)\setminus B_p(r_0)$ is achieved on $\partial B_p(R)$, say, point $y_R(x)\in\partial B_p(R)$. Hence,

$$\begin{aligned}\Big|\int\limits_{B_p(R)\setminus B_p(r_0)} \triangle u^l(y)G(x,y)dy\Big| &\le \int\limits_{B_p(R)\setminus B_p(r_0)} |\triangle u^l(y)|G(x,y_R(x))dy\\ &\le CG(x,y_R(x)),\end{aligned}$$

where C is a constant. From (6), one has that the left hand side of the above inequality goes to 0 as $x\to\infty$. Thus, we complete the proof of Proposition 1, also Theorem 2. $\square$

Using Proposition 1, (10) and Maximum principle, Corollary 1 can be easily obtained.

Remark 4. From Proposition 1, we can slightly change the image of the harmonic map u in Theorem 2, by only requiring that $u(E_\sigma)$'s be contained in some geodesic convex balls. It is easy to see that in this case Corollary 1 does not hold by using the work due to Schoen and Yau [16]. When M^m is homeomorphic to R^m, Z. R. Jin obtained some results in some special cases (See [10]).

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