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C^{ℓ}-determinacy of weighted homogeneous germs

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Abstract. We provide new estimates on the degree of C^{ℓ}-G-determinacy (G is one of Mather's groups $\mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$) of weighted homogeneous map germs satisfying a convenient Lojasiewicz condition. The results give an explicit order such that the C^{ℓ} geometrical structure of a weighted homogeneous polynomial map-germ is preserved after higher order perturbations. As an application of our results, we use the degree of C^{1}-determinacy and the Newton diagram to obtain equisingular deformations in the Briançon-Speder example.

Key words: C^{ℓ}-determinacy, weighted homogeneous map-germs, controled vector fields, weighted homogeneous control functions.

Introduction

The determinacy of map-germs is a fundamental subject in singularity theory, and many works are devoted to the characterization of finite (infinite) determinacy and to estimating the order of determinacy, with respect to various equivalence relations. In particular, finding the accurate order of determinacy of a map-germ is important for applications or practical problems as well as for pure mathematical theory.

In this paper we provide new estimates on the degree of C^{ℓ}-G-determinacy (G is one of Mather's groups $\mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$) of weighted homogeneous map germs satisfying a convenient Lojasiewicz condition. We generalize previous results on homogeneous map-germs given by the first author in [9]. The results give an explicit order such that the C^{ℓ} geometrical structure of a weighted homogeneous polynomial map-germ is preserved after higher order perturbations. Our method consists of constructing controled vector fields based on weighted homogeneous standard control functions.

The question of determining the degree of C^{0}-G-determinacy of weighted homogeneous map germs has been considered by several authors (e.g. [3], [4], [8]), but these results do not include the C^{ℓ} case, $0 < \ell < \infty$. As

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an application of our results, we use the degree of C^1-determinacy and the Newton diagram to obtain equisingular deformations in the Briançon-Speder example.

After completing this work, the authors came across a paper of Bromberg and Lópes de Medrano [2] that contains similar results. They consider only germs of functions and the group $G = \mathcal{R}$, but their estimates for the degree of C^ℓ-determinacy apply to germs of class $C^{\ell+1}$, $0 \le \ell < \infty$.

1. Basic definitons

The basic notation is the same as in [9] or [12]. Let $C(n,p)$ be the space of smooth map-germs $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$.

The groups C^ℓ-G, $G = \mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$, $0 \le \ell < \infty$ are defined as the groups $\mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$, taking diffeomorphisms of class C^ℓ, $\ell \ge 1$ or homeomorphisms, when $\ell = 0$. These groups act on the space of C^ℓ map-germs. Our interest however, is rather in the induced equivalence relation, C^ℓ-G-equivalence, in the space $C(n,p)$.

A C^∞ map germ $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ is k-C^ℓ-G-determined, $0 \le \ell < \infty$, ($G = \mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$) if for each germ g such that $j^k g(0) = j^k f(0)$, g is C^ℓ-G-equivalent to f. The ring of C^∞ function-germs $f : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ is denoted by C_n and m_n denotes its maximal ideal.

As in [12], given $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ we denote by $I_{\mathcal{R}} f$ the ideal of C_n generated by the $p \times p$ minors of the jacobean matrix of f, by $I_{\mathcal{C}} f$ the ideal generated by the coordinate functions of f, and by $I_{\mathcal{K}} f$ the ideal $I_{\mathcal{R}} f + I_{\mathcal{C}} f$.

Let $N_{\mathcal{C}} f(x) = |f(x)|^2$, $N_{\mathcal{R}} f(x) = |df(x)|^2 = \sum_j M_j^2$, where the M_j are the generators of $I_{\mathcal{R}} f$ and $N_{\mathcal{K}} f = N_{\mathcal{R}} f + N_{\mathcal{C}} f$. We say that $N_G f$ satisfies a Lojasiewicz condition if there exist constants $c > 0$ and $\alpha > 0$ such that $N_G f(x) \ge c|x|^\alpha$.

Proposition 1.1 [12] *$N_G f(x)$ satisfies a Lojasiewicz condition if and only if f is finitely C^ℓ-G-determined for any ℓ, $0 \le \ell < \infty$.*

Definition 1.2 Given $(r_1, \dots, r_n : d_1, \dots, d_p)$, r_i, $d_j \in \mathbb{Q}^+$, a map germ $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ is weighted homogeneous of type $(r_1, \dots, r_n : d_1, \dots, d_p)$ if for all $\lambda \in \mathbb{R} - \{0\}$:

$$f\left(\lambda^{r_1} x_1, \lambda^{r_2} x_2, \dots, \lambda^{r_n} x_n\right) = \left(\lambda^{d_1} f_1(x), \lambda^{d_2} f_2(x), \dots, \lambda^{d_p} f_p(x)\right).$$

Definition 1.3 Given $(r_1, \dots, r_n)$, for any monomial $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$, we define $\mathrm{fil}(x^\alpha) = \sum_{i=1}^n \alpha_i r_i$.

We define a filtration in the ring C_n via the function defined by $\mathrm{fil}(f) = \inf_\alpha \left\{\mathrm{fil}(x^\alpha) \mid \left(\frac{\partial^\alpha f}{\partial x^\alpha}\right)(0) \neq 0\right\}$, for any germ f in C_n. This definition can be extended to C_{n+r}, the ring of r-parameter families of germs in n-variables, by defining $\mathrm{fil}(x^\alpha t^\beta) = \mathrm{fil}(x^\alpha)$.

For any map germ $f = (f_1, \dots, f_p)$ in $C(n,p)$ we call $\mathrm{fil}(f) = (d_1, \dots, d_p)$, where $d_i = \mathrm{fil}(f_i)$ for each $i = 1, \dots, p$.

2. Estimates for the degree of C^ℓ-determinacy

Definition 2.1 Let $(r_1, r_2, \dots, r_n; 2k)$ be fixed. We define the standard control function $\rho_k(x)$ by $\rho_k(x) = x_1^{2\alpha_1} + x_2^{2\alpha_2} + \cdots + x_n^{2\alpha_n}$, where the α_i are chosen in such a way that the function ρ_k is weighted homogeneous of type $(r_1, r_2, \dots, r_n; 2k)$.

We observe that $\rho_k(x)$ satisfies a Lojasiewicz condition $\rho_k(x) \geq c|x|^{2\alpha}$ for some constants c and α.

Lemma 1 *Let $h(x)$ be a weighted homogeneous polynomial of type $(r_1, \dots, r_n; 2k)$ and $h_t(x)$, $t \in [0,1]$ a deformation of h, which is weighted homogeneous of the same type as h. Then:*

a. *There exists a constant c_1 such that $|h_t(x)| \leq c_1 \rho_k(x)$.*
b. *If there exist constants c and α such that $|h_t(x)| \geq c|x|^\alpha$, then $|h_t(x)| \geq c_2 \rho_k(x)$ for some constant c_2.*

Proof. Let $M = \{(y,t) \in \mathbb{R}^n \times [0,1]$ such that $\rho_k(y) = 1\}$.

To prove (a), we first observe that for each pair (x,t), $x \neq 0$, fixed, there is a pair $(y,t) \in M$, and a real number $\lambda \neq 0$, such that $(x,t) = (\lambda^{r_1} y_1, \dots, \lambda^{r_n} y_n, t)$.

Now, let $c_1 = \sup\{h_t(y)$ such that $(y,t) \in M\}$. Then:

$$h_t(x) = h_t(\lambda y) = \lambda^{2k} h_t(y) \leq \lambda^{2k} c_1 \rho_k(y) = c_1 \rho_k(x).$$

To prove (b), let $c_2 = \inf\{h_t(y)$ such that $(y,t) \in M\}$. From the hypothesis, $c_2 > 0$, hence:

$$c_2 \rho_k(x) = c_2 \lambda^{2k} \rho_k(y) \leq \lambda^{2k} h_t(y) = h_t(x).$$

□

Lemma 2 *Let $h(x)$ be a weighted homogeneous polynomial of type $(r_1, \dots,$*

$r_n; 2k)$, with $r_1 \leq r_2 \leq \cdots \leq r_n$, $\rho(x)$ be the standard control of same type as $h(x)$ and $h_t(x)$ be a deformation of h such that:

$$\operatorname{fil}(h_t) \geq 2k + \ell r_n + 1, \quad t \in [0,1], \quad \ell \geq 1.$$

Then the function $\nu(x) = h_t(x)/\rho(x)$ is differentiable of class C^ℓ.

Proof. We will proceed by induction on the class of differentiability.

First we consider $\ell = 1$. The gradient of $\nu(x)$ is

$$\nabla\nu(x) = \frac{\nabla h_t(x)}{\rho(x)} - \frac{\nabla\rho(x)\cdot h_t(x)}{\rho(x)^2}, \quad \text{with} \quad \inf_i\left\{\operatorname{fil}\Big(\frac{\partial\rho}{\partial x_i}(x)\Big) \geq 2k - r_n\right\}$$

and $\operatorname{fil}(h_t(x)) \geq 2k - r_n + 1$, then $\operatorname{fil}|\nabla\rho(x)\cdot h_t(x)| \geq 4k+1$.

Each term of $\nabla\nu(x)$ is of form $g(x)\cdot m(x)/\rho(x)$, where $m(x)$ is weighted homogeneous of type $(r_1,\dots,r_n;2k)$ and $\lim_{x\to 0} g(x) = 0$. It follows from Lemma 1 that $m(x)/\rho(x)$ is bounded, hence $\nabla\nu(x)$ is continuous.

Let us assume by induction that for all function $\nu = h(x)/\rho(x)$ with $\operatorname{fil}(h) \geq 2k + (\ell-1)r_n + 1$, ν is of class $C^{\ell-1}$.

Let $\nu = h(x)/\rho(x)$ with $\operatorname{fil}(h) \geq 2k + \ell r_n + 1$. Then $\nabla\nu(x) = H(x)/\rho(x)$ with $\operatorname{fil}(H) \geq 2k + (\ell-1)r_n + 1$ is of class $C^{\ell-1}$, and ν is of class C^ℓ. □

Case 1: $G = \mathcal{R}$.

Proposition 2.2 *Let $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ be a weighted homogeneous polynomial map germ of type $(r_1,\dots,r_n\,;d_1,\dots,d_p)$ with $r_1 \leq \cdots \leq r_n$, $d_1 \leq \cdots \leq d_p$, satisfying a Lojasiewicz condition $N_{\mathcal{R}}f(x) \geq c|x|^\alpha$, for constants c and α. Then:*

(a) *Deformations of f defined by $f_t(x) = f(x) + t\Theta(x)$, $\Theta = (\Theta_1,\dots,\Theta_p)$ with $\operatorname{fil}(\Theta_i) \geq d_i - r_1 + \ell r_n + 1$, for all i, $\ell \geq 1$ and $t \in [0,1]$ are C^ℓ-$\mathcal{R}$-trivial.*

(b) *If f_t is a deformation of f which is weighted homogeneous of the same type as f, then the family f_t is C^0-$\mathcal{R}$-trivial for small t.*

We observe that for each $p\times p$ minor M_I of df, there is an s_I such that M_I is weighted homogeneous of type $(r_1,\dots,r_n;s_I)$.

Let $N^*_{\mathcal{R}}f$ be defined by $N^*_{\mathcal{R}}f = \sum_I M_I^{2\alpha_I}$, where $\alpha_I = k/s_I$, and $k =$ *l.c.m.*(s_I). Then, $N^*_{\mathcal{R}}f$ is a weighted homogeneous control function of type $(r_1,\dots,r_n;2k)$.

For deformations f_t of f, we define the control $N^*_{\mathcal{R}}f_t$ by $N^*_{\mathcal{R}}f_t = \sum_I M_{t_I}^{2\alpha_I}$, where M_{t_I} are the $p\times p$ minors of J_{f_t}, and the α_I are the same as above. If f_t is weighted homogeneous of same type as f, then $N^*_{\mathcal{R}}f_t$ is weighted homogeneous of type $(r_1, \ldots, r_n; 2k)$ for all t. If $f_t(x) = f(x) + t\Theta(x)$ and $\text{fil}(\Theta_i) \geq d_i$, it follows that $\text{fil}(N^*_{\mathcal{R}}f_t) \geq \text{fil}(N^*_{\mathcal{R}}f)$.

Lemma 3 *There exist constants c_1 and c_2 such that:*

$$c_2\rho_k(x) \leq N^*_{\mathcal{R}}f_t \leq c_1\rho_k(x).$$

Proof. When f_t is weighted homogeneous of the same type as f, the result follows from Lemma 1.

If $\text{fil}(f_t) > \text{fil}(f)$, we write $N^*_{\mathcal{R}}f_t = N^*_{\mathcal{R}}f + tR(x,t)$ where $R(x,t)$ is a polynomial with $\text{fil}(R(x,t)) > \text{fil}(N^*_{\mathcal{R}}f)$.

Then $N^*_{\mathcal{R}}f \leq N^*_{\mathcal{R}}f_t + |R_t(x)|$, for $0 \leq t \leq 1$. By Lemma 1, there exists a constant c_2 such that: $c_2\rho_k(x) \leq N^*_{\mathcal{R}}f \leq N^*_{\mathcal{R}}f_t + |R_t(x)|$.

Since $\text{fil}(R(x,t)) > \text{fil}(N^*_{\mathcal{R}}f)$, it follows that $\lim_{x\to 0}|R_t(x)|/\rho_k(x) = 0$ (Lemma 2). Thus $c_2\rho_k(x) \leq N^*_{\mathcal{R}}f_t$.

It is easy to see that there exists a constant c_1 such that $N^*_{\mathcal{R}}f_t \leq c_1\rho(x)$ for small t. □

Proof of the Proposition 2.2

(a) Let M_{t_I} a $p \times p$ minor of J_{f_t}, $I = (i_1, i_2, \ldots, i_p) \subset (1, 2, \ldots, n)$. Then, there exists a vector field W_I associated to M_{t_I}, such that: $\dfrac{\partial f_t}{\partial t}M_{t_I} = df(W_I)$, where

$$W_I = \sum_1^n w_i\frac{\partial}{\partial x_i}, \quad \text{with:} \quad \begin{cases} w_i = 0; \quad \text{if } \ i \notin I \\ w_{i_m} = \sum_{j=1}^p N_{ji_m}\left(\frac{\partial f_t}{\partial t}\right)_j; \ \text{if } i_m \in I \end{cases}$$

and N_{ji_m} is the $(p-1) \times (p-1)$ minor cofactor of $\dfrac{\partial f_j}{\partial x_{i_m}}$ in df. (See [4] or [9] for more details).

Since $\text{fil}(w_{i_m}) = \min_{j=1,\ldots,p}\left(\text{fil}(N_{ji_m}) + \text{fil}\left(\frac{\partial f_t}{\partial t}\right)_j\right)$, and $\text{fil}(w_{i_m}) = d - r_I + r_{i_m} - r_1 + \ell r_n + 1$, where $d = d_1 + d_2 + \cdots + d_p$ and $r_I = r_{i_1} + r_{i_2} + \cdots + r_{i_p}$, the least possible filtration of w_i for $i = 1, \ldots, n$ is $\text{fil}(w_1) = d - r + \ell r_n + 1$, where $r = r_1 + r_2 + \cdots + r_p$.

Then $\dfrac{\partial f_t}{\partial t}N^*_{\mathcal{R}}f_t = df_t(W_R)$, where $W_R = \sum_I M_I^{2\alpha_I - 1}w_i$, with $\text{fil}(W_R) = 2k + \ell r_n + 1$.

Let $\nu : \mathbb{R}^n \times \mathbb{R}, 0 \to \mathbb{R}^n \times \mathbb{R}, 0$ be the vector field defined by $\nu(x) = W_R/N^*_{\mathcal{R}}f_t$. By Lemma 2, ν is of class C^ℓ.

The equation $\frac{\partial f_t}{\partial t}(x,t) = (df_t)_x(x,t)(\nu(x,t))$ implies the C^ℓ-$\mathcal{R}$-triviality of the family $f_t(x)$ in a neighbourhood of $t=0$. Since the same argument is true in a neighbourhood of $t=\bar{t}$, for all $t \in [0,1]$, the proof is complete.

(b) The vector field is constructed as in case (a). Here, $\text{fil}(W_R) \geq 2k + r_1$, and $\text{fil}(W_{R_i}) \geq 2k + r_i$, where the W_{R_i} are the components of W_R. Then, the vector field $\nu(x) = W_R/N^*_{\mathcal{R}} f_t$ is continuous. Furthermore, $\nu(x,t) \leq c|x|$, and this condition implies the integrability of the vector field $\nu \cdot$ ([7]) □

Case 2: $G = \mathcal{C}$.

Let $N^*_{\mathcal{C}} f = \sum_{i=1}^p f_i^{2\beta_i}$, where $\beta_i = k/d_i$, and $k = l.c.m.(d_i)$.

Given a deformation f_t of f, $f_t = f + t\Theta$, let $N^*_{\mathcal{C}} f_t = \sum_i f_{ti}^{2\beta_i}$, where each β_i is the same as above.

Proposition 2.3 *Let $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ be a weighted homogeneous polynomial map germ of type $(r_1, \dots, r_n; d_1, \dots, d_p)$ with $r_1 \leq \cdots \leq r_n$, $d_1 \leq \cdots \leq d_p$, satisfying a Lojasiewicz condition $N_{\mathcal{C}}(f(x)) \geq c|x|^\alpha$, for constants c and α. Then:*

(a) *Deformations $f_t = f + t\Theta$ of f, with fil$(\Theta_i) \geq d_p + \ell r_n + 1$, for all $i, t \in [0,1]$ and $\ell \geq 1$ are C^ℓ-$\mathcal{C}$-trivial.*

(b) *Small deformations of f, with fil$(\Theta_i) = d_p + 1$, $i = 1, \dots, p$ are C^0-$\mathcal{C}$-trivial.*

Proof. **(a)** C^ℓ-$\mathcal{C}$-triviality of the family f_t is obtained by constructing map germs V_i of class C^ℓ, $V_i : \mathbb{R}^n \times \mathbb{R}, 0 \to \mathbb{R}^p, 0$; $V_i = (V_{i_1}, V_{i_2}, \dots, V_{i_p})$, with $V_{i_j}(x,0) = \delta_{i_j}(x)$ in such a way that: $\frac{\partial f_t}{\partial t} = \sum_{i=1}^p V_i(x,t)(f_{ti})$.

Since $\dfrac{\partial f_t}{\partial t} = \Big(\dfrac{\partial f_t}{\partial t} \cdot \Big(\sum_{i=1}^p (f_t)_i^{2\beta_i - 1}\Big)/N^*_{\mathcal{C}} f_t\Big)(f_{ti})$, we define:

$W_i(x,t) = (f_{ti})^{2\beta_i - 1}\Big(\dfrac{\partial f_t}{\partial t}\Big)$. Then $\dfrac{\partial f_t}{\partial t} = \Big(\sum_{i=1}^p W_i(x,t)/N^*_{\mathcal{C}} f_t\Big)(f_{ti})$ with fil$\,(W_i(x,t)) \geq 2k + \ell r_n + 1$ for all i.

Let $V : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}, 0 \to \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}, 0$ be the vector field defined by: $(0, V_p, 0)$, where $V_p(x,y,t) = (\sum_{i=1}^p W_i(x,t)/N^*_{\mathcal{C}} f_t)\, y_i$.

As V is of class C^ℓ, the result follows by integrating V. □

Case (b) is analogous to (a).

In order to obtain a better estimate, as in [9] we prove the following lemma:

Lemma 4 *Let c be a constant such that $|f_{ti}(x)|^2 \leq c\rho(x)$, and V and U be neighbourhoods of the region $|y| < c\rho(x)^{1/2}$ in $\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R} - \{0,0,t\}$,*

$$V = \Big\{(x,y,t) \text{ such that } |y| \leq c_1\rho(x)^{1/2}, \text{ with } c_1 > c\Big\}$$

and U is chosen in such a way that $U \subset \overline{U} \subset V$.

There exists a conic bump function $p : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}$ such that: $p_{|\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R} - \{0,0,t\}}$ is smooth, and

$$\begin{cases} p(x,y,t) = 1, & \text{for all } (x,y,t) \in \overline{U} \\ p(x,y,t) = 0, & \text{outside of } V \\ 0 \leq p(x,y,t) \leq 1, & \text{in } V - \overline{U} \\ p(0,0,t) = 0, & \text{for all } t. \end{cases}$$

Proof. We define the function $p(x,y_i) = h(\Theta_i)$, with $\Theta_i = y_i \cdot \rho(x)^{-1/2}$, where for each i, the set $|y_i| \leq c\rho(x)^{1/2}$, is in $\mathbb{R}^n \times \mathbb{R}^p$ and $h : \mathbb{R} \to \mathbb{R}$ is the usual bump function,
$$\begin{cases} h(\Theta) = 1 & , \text{ if } 0 \leq \Theta \leq \Theta_1 \ ; \\ h(\Theta) = 0 & , \text{ if } \Theta \geq \Theta_2 \ ; \\ 0 \leq h(\Theta) \leq 1\,, & \text{ if } \Theta_1 < \Theta < \Theta_2 \ . \end{cases}$$

Since $|f_{t_i}| \leq c\rho(x)^{1/2}$, for a constant c, we have:

$$\begin{cases} h(\Theta_i) \leq 1 & \text{if } |y_i| \leq c\rho(x)^{1/2} \\ 0 \leq h(\Theta_i) \leq 1 & \text{if } c\rho(x)^{1/2} \leq |y_i| \leq c_1\rho(x)^{1/2} \\ h(\Theta_i) = 0 & \text{if } c_1\rho(x)^{1/2} \leq |y_i| \ . \end{cases}$$

The desired conic bump function is defined by:

$$p(x,y,t) = p(x,y_1)p(x,y_2)\cdots p(x,y_p).$$

□

Proposition 2.4 *Let $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ be a weighted homogeneous polynomial map germ of type $(r_1, \ldots, r_n; d_1, \ldots, d_p)$ with $r_1 \leq \cdots \leq r_n$, $d_1 \leq \cdots \leq d_p$, satisfying a Lojasiewicz condition: $N_{\mathcal{C}}|f(x)| \geq c|x|^\alpha$, for constants c and α. Then:*

(a) *Deformations $f_t = f + t\Theta$ of f, with fil$(\Theta_i) \geq d_p + \ell r_n$, $i = 1, \ldots, p$, $t \in [0,1]$ and $\ell \geq 1$ are C^ℓ-$\mathcal{C}$-trivial.*

(b) *Small deformations of f, with fil$(\Theta_i) = d_p$, $i = 1, \ldots, p$ are C^0-$\mathcal{C}$-trivial.*

Proof. **(a)** Let V be the vector field defined by $V = W/N_{\mathcal{C}}^{*} f_t$, where W is defined in a neighbourhood of $\{0,0,t\}$ in $\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}$, with $W_j(x,y,t) = p(x,y,t)w_j(x,y,t)$ for $i \leq j \leq p$, and w_j are defined as in Proposition 1.3.

We just have to check the class of differentiability of $W_j = (w_{j1}, \dots, w_{jp})$ where: $w_{ji} = (h_{ji}/\rho_k(x))\, p(x,y,t)y_i$, and $h_{ji} = \Theta_i f_{tj}^{2\beta_j - 1}$. Then, fil$(h_{ji}) \geq 2k + \ell r_n$, and

$$|W_{ji}| = |h_{ji}/\rho(x)|\,|py_j| \leq |h_{ji}/\rho(x)|\,(\rho(x))^{1/2}.$$

Applying Lemmas 4 and 2, we see that each W_{ji} is of class C^ℓ and W is of class C^ℓ. As in proposition 2.3, case (b) is analogous to (a). □

Case 3: $G = \mathcal{K}$.

Proposition 2.5 *Let $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ be a weighted homogeneous polynomial map germ of type $(r_1, \dots, r_n; d_1, \dots, d_p)$ with $r_1 \leq \dots \leq r_n$, $d_1 \leq \dots \leq d_p$, satisfying a Lojasiewicz condition $N_{\mathcal{K}} f(x)| \geq c|x|^\alpha$, for constants c and α. Then:*

(a) *Deformations $f_t = f + t\Theta$ of f, with fil$(\Theta_i) \geq d_p + \ell r_n$, $i = 1, \dots, p$, $t \in [0,1]$ and $\ell \geq 1$ are C^ℓ-$\mathcal{K}$-trivial.*

(b) *Small deformations of f, with fil$(\Theta_i) = d_p$, $i = 1, \dots, p$ are C^0-$\mathcal{K}$-trivial.*

Proof. Since the group C^ℓ-$\mathcal{K}$ is the semi-direct product of the groups C^ℓ-$\mathcal{R}$ and C^ℓ-$\mathcal{C}$, the vector fields are defined as in cases $G = \mathcal{R}$ and $\mathcal{C}$, and the control function $N_{\mathcal{K}}^{*} f$ is defined by: $N_{\mathcal{K}}^{*} f = N_{\mathcal{R}}^{*} f^\alpha + N_{\mathcal{C}}^{*} f^\beta$ where α and β are constants such that $N_{\mathcal{K}}^{*} f$ is weighted homogeneous. □

As a consequence of the above results, we obtain a general estimate for the degree of C^ℓ-G-determinacy ($G = \mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$).

Proposition 2.6 *Let $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ be a weighted homogeneous polynomial map germ of type $(r_1, \dots, r_n; d_1, \dots, d_p)$, $r_1 \leq \dots \leq r_n$, $d_1 \leq \dots \leq d_p$, satisfying a Lojasiewicz condition: $N_G f(x)| \geq c|x|^\alpha$, for constants c and α.*

(a) *f is k-C^ℓ-G-determined, where k is the least integer bigger than or equal to: $((d_p + \ell r_n + 1 - 2r_1)/r_1)$, $0 < \ell < \infty$,*

(b) *f is k-C^0-G-determined, where $k = d_p/r_1$,*

(c) *Small deformations of f, of degree d_p/r_1 are C^0-G-trivial.*

Example.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x,y) = (x^a - y^b; xy)$, with $a \geq b$ even integers. f is a weighted homogeneous map germ of type $(r, s; d, r+s)$, where $d = l.c.m.(a, b)$ and $r = d/a$, $s = d/b$.

Let $f_t = f + t\Theta$,with $\Theta = (\Theta_1, \Theta_2)$ a deformation of f.

If $\mathrm{fil}(\Theta_1) = d$ and $\mathrm{fil}(\Theta_2) = s + r$, the family f_t is C^0-$\mathcal{R}$-trivial for small t.

The family f_t is C^ℓ-$\mathcal{R}$-trivial for all t if $\mathrm{fil}(\Theta_1) \geq d - r + \ell s + 1$ and $\mathrm{fil}(\Theta_2) \geq (\ell + 1)s + 1$.

The function f is k-C^0-G-determined, where $k = sup(d, r+s)/r$, for all $G = \mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$.

If $0 < \ell < \infty$, the function f is k-C^ℓ-G-determined, where k is the least integer bigger than or equal to: $(sup(d, r+s) + \ell s - 2r + 1)/r$.

The Briançon-Speder example

Let$f: k^3, 0 \to k, 0$, ($k = \mathbb{R}$ or $\mathbb{C}$) be defined by $f(x, y, z) = z^5 + y^7 x + x^{15}$, which is a weighted homogeneous map-germ of type (1,2,3;15).

When $k = \mathbb{C}$, the family $F(x, y, z, t) = z^5 + y^7 x + x^{15} + ty^6 z$ is topologically trivial, since the Milnor number $\mu(f_t)$ is constant for all t. Briançon and Speder showed in [1] that the variety $F^{-1}(0)$ in $\mathbb{C}^4$ is not equisingular along the parameter space $0 \times \mathbb{C}$ at 0. A complete description of all equisingular deformations of f is given in [10]: the variety $F^{-1}(0)$, defined by $F(x, y, z, t) = f(x, y, z) + tx^a y^b z^c$ is equisingular along the parameter space at 0, if and only if the monomial $x^a y^b z^c$ is in the Newton polyhedron determined by the points $\{(15, 0, 0), (0, 8, 0), (0, 0, 5), (1, 7, 0)\}$.

We consider here the analogous question for the real family $F: \mathbb{R}^3 \times \mathbb{R}, 0 \to \mathbb{R}, 0$; $F(x, y, z, t) = f(x, y, z) + tx^a y^b z^c$. Applying the Proposition 2.2 we obtain that deformations $f_t(x, y, z) = f(x, y, z) + tx^a y^b z^c$ are topologically trivial for t small, if $\mathrm{fil}(x^a y^b z^c) = a + 2b + 3c \geq 15$ and are C^ℓ-$\mathcal{R}$-trivial, $(\ell \geq 1)$ if $\mathrm{fil}(x^a y^b z^c) = a + 2b + 3c \geq 15 + 3\ell$. Hence the variety $F^{-1}(0)$ is Whitney-equisingular along the parameter space $0 \times \mathbb{R}$ at 0, whenever $a + 2b + 3c \geq 18$.

It is easy to check that, for sufficiently small values of t, the deformations $f_t(x, y, z) = f(x, y, z) + tx^a y^b z^c$, with (a, b, c) equal to $(15, 0, 0)$, $(0, 0, 5)$ and $(1, 7, 0)$ are indeed C^∞-trivial, hence the Whitney equisingularity of the pair $\{F^{-1}(0), 0 \times \mathbb{R}\}$ also holds for them.

To obtain the equisingularity of $F^{-1}(0)$, when $F(x, y, z, t) = f(x, y, z) + ty^8$, we can show that y^8 is in the integral closure of the ideal $m_3 \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle$. These calculations can be found in [10]. In the real analytic case, the integral closure of an ideal I is defined as the set of all elements h such that for all analytic curve $\phi: \mathbb{R}, 0 \to \mathbb{R}^n, 0$, $h \circ \phi \in (\phi^* I)C_{n+1}$ (see Gaffney, [6]).

Gaffney also shows that if the smooth points of each component of $F^{-1}(0)$ are dense on that component, then a necessary condition for the Whitney equisingularity of the pair $\{F^{-1}(0), 0 \times \mathbb{R}\}$ is that $\frac{\partial F}{\partial t}$ is in the integral closure of the ideal generated by $\left\{x_i \frac{\partial F}{\partial x_j}{}_{i,j=1,\ldots,n}\right\}$.

Using this result, we can show that the real analogue of the Briançon-Speder family $F(x, y, z, t) = f(x, y, z) + ty^6 z$ is also non equisingular. In fact, we see that $y^6 z$ does not belong to the integral closure of the ideal $m_3 \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle$, since the above condition fails for the curve $\psi: \mathbb{R} \to \mathbb{R}^4$, defined by $\psi(\lambda) = (\lambda^5, \lambda^5, \lambda^8, -5\lambda^2)$.

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