Hokkaido Mathematical Journal Vol. 26 (1997) p. 63–88

Axisymmetric solutions and singular parabolic equations in the theory of viscosity solutions

(Dedicated to Professor Kôji Kubota on his sixtieth birthday)

Masaki OHNUMA

(Received November 16, 1995)

Abstract. We extend the theory of viscosity solutions for singular parabolic equations including, for example, axisymmetrized level set equation for mean curvature flow equation. We establish a comparison principle for viscosity solutions of singular degenerate parabolic equations including such an equation. We discuss the relation between axisymmetric viscosity solutions of original level set equation for mean curvature flow equation and the viscosity solution of axisymmetrized one.

Key words: singular degenerate parabolic equations, viscosity solutions, comparison principle, axisymmetric solutions.

1. Introduction

We are concerned with a degenerate parabolic equation of form:

$$u_t + F_0(\nabla_{x,r}u, \nabla^2_{x,r}u) - \frac{\nu u_r}{r^\beta} = 0 \text{ in } Q = (0,T)\times\Omega\times(0,R), \tag{1.1}$$

$$-u_r = 0 \text{ on } S = (0,T)\times\Omega\times\{0\}, \tag{1.2}$$

where Ω is a domain in $\boldsymbol{R}^m$, T, R and ν are positive numbers and β is a positive parameter. Here $u_t = \partial u/\partial t, \nabla_x u$ and $u_r = \partial u/\partial r$ denote the time derivative of u, the gradient of u in space variables x and the space derivative of u in r, respectively. We denote by $\nabla_{x,r}u = (\nabla_x u, u_r)$ and $\nabla^2_{x,r}u$ the gradient of u and the Hessian of u in space variables (x,r), respectively. The function $F_0 = F_0(p,X)$ is not continuous on $p = 0$. As explained later in section 3, the equation (1.1) has many examples. One of them is of the form:

$$u_t - |\nabla_{x,r}u|\mathrm{div}_{x,r}\left(\frac{\nabla_{x,r}u}{|\nabla_{x,r}u|}\right) - \frac{n-m-1}{r}u_r = 0 \tag{1.3}$$

which is introduced as axisymmetrized level set equation for mean curvature flow equation, where for C^1 function $f_i : Q \to \boldsymbol{R}$ $(i = 1,\dots,m+1)$ the

---

1991 Mathematics Subject Classification : 35K65, 35B05.

divergence of $f = (f_1, \ldots, f_{m+1})$ is defined by $\mathrm{div}_{x,r} f = \partial f_1/\partial x_1 + \cdots + \partial f_m/\partial x_m + \partial f_{m+1}/\partial r$ and $n - m - 1 > 0$. Many authors study the level set equation for mean curvature flow equation. Axisymmetric solutions are expected to satisfy (1.3). Actually, the equation (1.3) with $n = 3$, $m = 1$ is derived from the original level set equation and studied by Chen-Giga-Hitaka-Honma [3] in view of numerical analysis. Axisymmetric solutions of the original level set equation are studied by Altschuler-Angenent-Giga [1].

Our main goal is to extend the theory of viscosity solutions so that it is directly applicable to (1.1) having singularity at $r = 0$. In fact, we establish a notion of viscosity solutions for (1.1) and (1.2) to get the comparison principle. The equation (1.1) appears to have no meaning at $r = 0$. However, multiplying (1.1) by r^β and letting r tend to zero yield $-u_r = 0$. We are tempting to think that (1.1) and (1.2) is a boundary value problem with $-u_r = 0$ in viscosity sense. However, it turns out that this observation is not enough to obtain a suitable notion of viscosity solutions. At $r = 0$ we always require $-u_r = 0$ in the viscosity sense.

As we pointed out, many examples of the equation (1.1) are derived by restricting the equation

$$U_t + \widetilde{F}(DU, D^2U) = 0 \quad \text{in} \quad (0,T) \times \Omega' \times \Omega'' \tag{1.4}$$

on the space of axisymmetric functions, where Ω', Ω'' are domains in $\boldsymbol{R}^m$, $\boldsymbol{R}^{n-m}$, respectively. Here DU and D^2U denote the gradient of U and the Hessian of U in the all space variables.

To check that our notion of viscosity solutions is appropriate, we prove that an axisymmetric sub(super)solution of (1.4) is a sub(super)solution of (1.1) and (1.2). Unfortunately, we are unable to prove the converse. However, if the solution of (1.4) with axisymmetric data does exist, we argue as follows. A solution of (1.1) and (1.2) with the same data must be an axisymmetric solution of (1.4) (provided that (1.4) has a comparison principle). We thus observe that a solution of (1.1) and (1.2) is actually an axisymmetric solution of (1.4) in this situation (Remark 5.7).

The advantage of our theory is that we are able to handle $\psi : Q \to \boldsymbol{R}$ satisfying

$$\psi_t - |\nabla_{x,r}\psi| \mathrm{div}_{x,r}\left(\frac{\nabla_{x,r}\psi}{|\nabla_{x,r}\psi|}\right) - \frac{\nu\psi_r}{r} = 0$$

even if ν is not an integer. This leads a possibility that the level set method would be applicable for "quenching problem" $v:(0,T)\times\boldsymbol{R}\rightarrow[0,\infty)$ satisfying

$$v_t-\frac{v_{xx}}{1+v_x^2}+\frac{\nu}{v}=0,$$

where the level set of $\psi=\psi(t,x,r)$ is given as $r=v(t,x)$. We shall discuss this problem in our forthcoming paper [10].

A feature of the equation (1.1) is having singularity in space variables $r=0$. Recently Siconolfi [12] and Ishii-Ramaswamy [7] study equations of first order having a singularity in space variables. The interesting aspect of their problems is that uniqueness of solutions does not hold in usual viscosity (solutions) sense.

The paper is organized as follows. In section 2 we give a notion of solutions for (1.1) and (1.2). In section 3 we give some examples of (1.1). In section 4 we shall establish a comparison principle of the solutions for (1.1) and (1.2) on a bounded domain. In section 5 we shall discuss the relation between solutions of (1.1) and (1.2) and axisymmetric solutions of (1.4) provided that (1.4) satisfies suitable conditions.

In our forthcoming paper we shall prove the existence of solutions of the Cauchy problem of (1.1) and (1.2). This paper gives a first step to consider the quenching problem in the level set method. We shall discuss the problem of beyond quenching applying the level set method in our forthcoming paper.

This work is a part of doctoral dissertation.

Acknowledgment The author wishes to express his thanks to Professor Yoshikazu Giga for his suggestion to consider this problem and for his encouragement. The author is grateful to Professor Shun'ichi Goto for his helpful discussion. The author is also grateful to Professor Kôji Kubota for his expositive remarks. This work was done while the author was a JSPS fellow for Japanese Junior Scientists. The work of the author was partly supported by the Japan Ministry of Education, Science and Culture.

2. Definition of viscosity solutions

Let Ω be a bounded domain in $\boldsymbol{R}^m$. Let T, R and ν be positive numbers and let β be a positive parameter. We consider a degenerate parabolic equation of the form:

$$u_t+F_0(\nabla_{x,r}u,\nabla_{x,r}^2u)-\frac{\nu u_r}{r^\beta}=0\ \text{ in }\ Q=(0,T)\times\Omega\times(0,R),\quad(2.1)$$

$$-u_r = 0 \quad \text{on} \quad S = (0,T) \times \Omega \times \{0\}. \tag{2.2}$$

Here F_0 is assumed to satisfy

$$\begin{aligned} &F_0 : (\boldsymbol{R}^{m+1} \backslash \{0\}) \times \boldsymbol{S}^{m+1} \longrightarrow \boldsymbol{R} \text{ is continuous,} \\ &\text{where } \boldsymbol{S}^{m+1} \text{ denotes the space of real symmetric} \\ &\text{matrices with order } m+1, \end{aligned} \tag{$F_0 1$}$$

$$\begin{aligned} &F_0 \text{ is degenerate elliptic, } i.e., \\ &F_0(p, X+Y) \leq F_0(p, X) \quad \text{for all} \quad Y \geq 0, \quad p \neq 0. \end{aligned} \tag{$F_0 2$}$$

$$-\infty < F_{0*}(0, O) = {F_0}^*(0, O) < +\infty, \tag{$F_0 3$}$$

where F_{0*} and ${F_0}^*$ are the lower and upper semicontinuous relaxation (envelope) of F_0 on $\boldsymbol{R}^{m+1} \times \boldsymbol{S}^{m+1}$, respectively, *i.e.*,

$$\begin{aligned} &F_{0*}(p, X) \\ &\quad = \lim_{\varepsilon \downarrow 0} \inf\{F(q, Y); q \in \boldsymbol{R}^{m+1} \backslash \{0\}, |p-q| < \varepsilon, |X-Y| < \varepsilon\} \end{aligned}$$

and ${F_0}^* = -(-F_0)_*$. Here $|X|$ denotes the operator norm. Note that if F_0 satisfies $(F_0 2)$ then so do F_{0*} and ${F_0}^*$, respectively, even if $p = 0$.

The equation (2.1) appears to have no meaning at $r = 0$. However, we multiply both sides of (2.1) by r^β and let r tend to zero so that we obtain (2.2). In the equation (2.2) the "minus" sign is important to consider these problems in the viscosity (solutions) sense (see definition 2.1). We generalize (2.1) to

$$u_t + F\left(\nabla_{x,r} u, \nabla^2_{x,r} u, \frac{u_r}{r^\beta}\right) = 0 \quad \text{in} \quad Q = (0,T) \times \Omega \times (0,R). \tag{2.1$'$}$$

We set $\Sigma = Q \cup S$. We give a definition of viscosity solutions of (2.1$'$) and (2.2) in Σ. Let F be a function from $(\boldsymbol{R}^{m+1} \backslash \{0\}) \times \boldsymbol{S}^{m+1} \times \boldsymbol{R}$ to $\boldsymbol{R}$.

Definition 2.1 Let $u : \Sigma \to \boldsymbol{R} \cup \{\pm\infty\}$.

(i) A function u is a viscosity subsolution of (2.1$'$) and (2.2) in Σ, if $u^* < +\infty$ in $\overline{Q}$ and for all $\varphi \in C^2(\Sigma)$ and local maximum points $(\hat{t}, \hat{x}, \hat{r})$ of $u^* - \varphi$,

$$\begin{cases} \varphi_t + F_*\left(\nabla_{x,r}\varphi, \nabla^2_{x,r}\varphi, \dfrac{\varphi_r}{r^\beta}\right) \leqq 0 & \text{at} \quad (\hat{t}, \hat{x}, \hat{r}) \quad \text{if} \quad \hat{r} > 0, \\ -\varphi_r \leqq 0 & \text{at} \quad (\hat{t}, \hat{x}, 0). \end{cases}$$

(ii) A function u is a viscosity supersolution of (2.1′) and (2.2) in Σ, if $u_* > -\infty$ in $\overline{Q}$ and for all $\varphi \in C^2(\Sigma)$ and local minimum points $(\hat{t}, \hat{x}, \hat{r})$ of $u_* - \varphi$,

$$\begin{cases} \varphi_t + F^*\left(\nabla_{x,r}\varphi, \nabla^2_{x,r}\varphi, \dfrac{\varphi_r}{r^\beta}\right) & \geqq 0 \quad \text{at} \quad (\hat{t}, \hat{x}, \hat{r}) \quad \text{if} \quad \hat{r} > 0, \\ -\varphi_r & \geqq 0 \quad \text{at} \quad (\hat{t}, \hat{x}, 0). \end{cases}$$

(iii) If u is a viscosity subsolution and a viscosity supersolution of (2.1′) and (2.2) in Σ, then we call u a viscosity solution of (2.1′) and (2.2) in Σ, where F_* and F^* are the lower and upper semicontinuous relaxation(envelope) of F on $\boldsymbol{R}^{m+1} \times \boldsymbol{S}^{m+1} \times \boldsymbol{R}$, respectively, *i.e.*,

$$F_*(p, X, a) = \liminf_{\varepsilon \downarrow 0} \{F(q, Y, b);\ q \in \boldsymbol{R}^{m+1} \setminus \{0\}, |p - q| < \varepsilon, |X - Y| < \varepsilon, |a - b| < \varepsilon\}$$

and $F^* = -(-F)_*$.

As usual we give another definition of viscosity solutions which is equivalent to Definition 2.1.

Definition 2.2 Let $u : \Sigma \to \boldsymbol{R} \cup \{\pm\infty\}$.

(i) A function u is called a viscosity subsolution of (2.1′) and (2.2) in Σ, if $u^* < +\infty$ in $\overline{Q}$ and

$$\tau + F_*\left(p, X, \frac{\rho}{r^\beta}\right) \leqq 0 \quad \text{for all} \quad (\tau, p, X) \in \mathcal{P}^{2,+}_\Sigma u^*(t, x, r),\ (t, x, r) \in Q$$

and

$$-\rho \leqq 0 \quad \text{for all} \quad (\tau, p, X) \in \mathcal{P}^{2,+}_\Sigma u^*(t, x, 0),\ (t, x, 0) \in S,$$

where $p = (p', \rho) \in \boldsymbol{R}^m \times \boldsymbol{R}$.

(ii) A function u is called a viscosity supersolution of (2.1′) and (2.2) in Σ, if $u_* > -\infty$ in $\overline{Q}$ and

$$\tau + F^*\left(p, X, \frac{\rho}{r^\beta}\right) \geqq 0 \quad \text{for all} \quad (\tau, p, X) \in \mathcal{P}^{2,-}_\Sigma u_*(t, x, r),\ (t, x, r) \in Q$$

and

$$-\rho \geqq 0 \quad \text{for all} \quad (\tau, p, X) \in \mathcal{P}^{2,-}_\Sigma u_*(t, x, 0),\ (t, x, 0) \in S.$$

Here $\mathcal{P}^{2,+}_\Sigma$ denotes the parabolic super 2-jet in Σ, *i.e.*, $\mathcal{P}^{2,+}_\Sigma u(t, x, r)$ is the set of $(\tau, p, X) \in \boldsymbol{R} \times \boldsymbol{R}^{m+1} \times \boldsymbol{S}^{m+1}$ with $p = (p', \rho)$ such that

$$\begin{aligned} u(\underline{t}, y, s) \leqq\ & u(t, x, r) + \tau(\underline{t} - t) + \left\langle \begin{pmatrix} p' \\ \rho \end{pmatrix}, \begin{pmatrix} y - x \\ s - r \end{pmatrix} \right\rangle \\ & + \frac{1}{2}\left\langle X \begin{pmatrix} y - x \\ s - r \end{pmatrix}, \begin{pmatrix} y - x \\ s - r \end{pmatrix} \right\rangle \\ & + o(|\underline{t} - t| + |y - x|^2 + |s - r|^2) \\ & \text{as} \quad (\underline{t}, y, s) \to (t, x, r) \quad \text{for all} \quad (\underline{t}, y, s) \in \Sigma \end{aligned}$$

and $\mathcal{P}_{\Sigma}^{2,-} u = -\mathcal{P}_{\Sigma}^{2,+}(-u)$, where $< \ , \ >$ denotes the Euclidean inner product.

(iii) If u is a viscosity subsolution and a viscosity supresolution of (2.1′) and (2.2) in Σ, then we call u a viscosity solution of (2.1′) and (2.2) in Σ.

3. Examples of equation

We give some examples of equation (2.1). A typical example comes from the level set equation for mean curvature flow equation :

$$U_t - |DU| \mathrm{div}\left(\frac{DU}{|DU|}\right) = 0 \quad \text{in} \quad (0, T) \times \widetilde{\Omega}, \tag{3.1}$$

where $\widetilde{\Omega}$ is a bounded domain in $\boldsymbol{R}^n$ and $D = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$. We consider an x'-axisymmetric solution $u(t, x', r) = U(t, x)$, where $x = (x', x'') \in \boldsymbol{R}^m \times \boldsymbol{R}^{n-m}$, $|x''| = r$ and $n > m + 1$. Then we can transform (3.1) to

$$u_t - |\nabla_{x,r} u| \mathrm{div}_{x,r}\left(\frac{\nabla_{x,r} u}{|\nabla_{x,r} u|}\right) - \frac{n - m - 1}{r} u_r = 0 \quad \text{in} \ (0, T) \times \Omega' \times (0, R) \tag{3.2}$$

for some bounded domain Ω' in $\boldsymbol{R}^m$ and $R > 0$. Here $\nabla_{x,r} = (\nabla_{x'}, \partial/\partial r) = (\partial/\partial x_1, \ldots, \partial/\partial x_m, \partial/\partial r)$ and for functions $f_i : \boldsymbol{R}^m \times \boldsymbol{R} \to \boldsymbol{R}$ $(i = 1, \ldots, m+1)$ we define $\mathrm{div}_{x,r} f = \sum_{i=1}^{m} \partial f_i/\partial x_i + \partial f_{m+1}/\partial r$ with $f = (f_1, \ldots, f_{m+1})$. Of course even if $n = m+1$, we can transform (3.1) to (3.2), but it is natural to consider (3.2) under the condition $n > m + 1$. This equation (3.2) with $n = 3, m = 1$ is used to calculate (numerically) motion of x_1-axisymmetric surfaces by mean curvature in [3] by the level set method. Evolutions of such surfaces are well studied analytically in [1].

We generalize (3.2) to

$$u_t - |\nabla_{x,r} u| \mathrm{div}_{x,r}\Big(\frac{\nabla_{x,r} u}{|\nabla_{x,r} u|}\Big) - \frac{\nu}{r} u_r = 0, \tag{3.3}$$

where ν is a positive constant not necessarily integer. This equation has the form (2.1) with F_0 satisfying ($F_0$1)–($F_0$3). In this situation it is impossible to transform to an original equation like (3.1). However, using our definition of viscosity solutions, we can directly study the equation (3.3).

An interesting example comes from the quenching problem introduced by Kawarada [9]. We consider a little bit its generalization. For a given continuous function $\phi : \boldsymbol{R} \to [0, \infty)$ we consider

$$v_t - \phi(v_x) v_{xx} + \frac{1}{v^\beta} = 0 \quad \text{in} \quad (0,T) \times (-L, L), \tag{3.4}$$

where β is a positive parameter and L is a positive constant. In [9] essentially, the case $\beta = 1$ and $\phi =$ Constant > 0 is considered. We only study positive solutions of (3.4). We shall formally transform (3.4) to the equation as a special form of (2.1).

For each time we regard the graph of solutions of (3.4) as a curve in the plane. First we shall derive a curve evolution equation from (3.4). The curve evolution equation should be described by an upward normal vector and a curvature. For a given smooth curve the unit upward normal vector $\vec{n}$ and the curvature κ are of form:

$$\begin{aligned} \vec{n} &= \Big(- \frac{v_x}{(1+v_x^2)^{1/2}}, \frac{1}{(1+v_x^2)^{1/2}} \Big) = (n_1, n_2), \\ \kappa &= \frac{v_{xx}}{(1+v_x^2)^{3/2}}. \end{aligned}$$

In general, an upward velocity of a curve is v_t, so we obtain the velocity V of upward normal vector direction of form:

$$V = \frac{v_t}{(1+v_x^2)^{1/2}}.$$

When $v(t,x)$ satisfies the equation (3.4), a curve $r = v(t,x)$ moves by

$$V = \frac{\kappa}{n_2^2} \phi\Big(- \frac{n_1}{n_2} \Big) - \frac{n_2}{r^\beta}. \tag{3.5}$$

We apply the level set method. Indeed, we introduce an auxiliary function ψ which describes the curve as a zero-level set; *i.e.*, $\{(t,x,r); \psi(t,x,r) = 0\}$

on the curve. Let us calculate the level set equation for (3.5). Assume that $\psi < 0$ above the curve and $\psi > 0$ below the curve. Then we have

$$V = \frac{\psi_t}{|\nabla_{x,r}\psi|}, \quad \vec{n} = -\frac{\nabla_{x,r}\psi}{|\nabla_{x,r}\psi|}, \quad \kappa = \mathrm{div}_{x,r}\left(\frac{\nabla_{x,r}\psi}{|\nabla_{x,r}\psi|}\right)$$

and we obtain a level set equation of the form:

$$\psi_t - \left(\frac{|\nabla_{x,r}\psi|}{\psi_r}\right)^2 |\nabla_{x,r}\psi| \mathrm{div}_{x,r}\left(\frac{\nabla_{x,r}\psi}{|\nabla_{x,r}\psi|}\right)\phi\left(-\frac{\psi_x}{\psi_r}\right) - \frac{\psi_r}{r^\beta} = 0. \tag{3.6}$$

We thus have transformed (3.4) to (3.6). To consider (3.6) in our viscosity sense we impose the condition (2.2) for (3.6) as for (2.1).

In the equation (3.4) when v goes to zero, we call that v quenches. When v quenches, we cannot consider the equation. But using our method even if v quenches, we have the solution for the level set equation (3.6) with (2.2). By an analysis of the solution of (3.6) with (2.2), it is helpful to study the solution of (3.4), especially, the behavior of solutions beyond quenching.

4. Comparison theorem

Let Ω be a bounded domain in $\boldsymbol{R}^m$ and let T and R be positive numbers. Let β be a positive parameter. We consider a singular degenerate parabolic equation of the form:

$$u_t + F\left(\nabla_{x,r}u, \nabla^2_{x,r}u, \frac{u_r}{r^\beta}\right) = 0 \ \text{ in } \ Q = (0,T)\times\Omega\times(0,R), \tag{4.1}$$

$$-u_r = 0 \ \text{ on } \ S = (0,T)\times\Omega\times\{0\}. \tag{4.2}$$

We list assumptions of $F = F(p, X, a)$.

$F : (\boldsymbol{R}^{m+1}\backslash\{0\}) \times \boldsymbol{S}^{m+1} \times \boldsymbol{R} \to \boldsymbol{R}$ is continuous, where $\boldsymbol{S}^{m+1}$ denotes the space of real symmetric matrices with order $m+1$. (F1)

(i) $F(p, X+Y, a) \leqq F(p, X, a)$ for all $Y \geqq O$, $p \neq 0$;

we usually call that F is degenerate elliptic. (F2)

(ii) $F(p, X, a+b) \leqq F(p, X, a)$ for all $b \geqq 0$, $p \neq 0$.

By definition it is easy to show that if F fulfills (F2) so do F_* and F^*, respectively, even if $p = 0$.

$$-\infty < F_*(0, O, a) = F^*(0, O, a) < +\infty \qquad \text{for all constant } a \in \boldsymbol{R}. \tag{F3}$$

These assumption is automatically satisfied if F fulfills

$$F(p, X, a) = F_0(p, X) - \nu a, \quad \nu \text{ is a positive constant} \tag{F4}$$

and F_0 satisfies $(F_0 1)$, $(F_0 2)$ and $(F_0 3)$. Let D be a domain in the Euclidean space. For $Q = (0, T) \times D$, the set $\partial_p Q = (\{0\} \times D) \cup ([0, T] \times \partial D)$ is called the parabolic boundary of Q. We are now in position to state our main comparison theorem.

Theorem 4.1 *Suppose that Ω is a bounded domain in $\boldsymbol{R}^m$ and that F satisfies* (F1)–(F3). *Let u and v be viscosity sub- and super-solution of* (4.1) *and* (4.2), *respectively. If $u^* \leqq v_*$ on $\partial_p Q \backslash S$, then $u^* \leqq v_*$ in $Q \cup S$.*

Remark 4.2.
(i) We write (3.3) of the form (2.1′). Then F satisfies (F1)–(F3).
(ii) Let ϕ be appeared in (3.4). Suppose that $\lim_{|\sigma| \to \infty} \phi(\sigma)(1 + \sigma^2)$ exists. If we write (3.6) of the form (2.1′), then F satisfies all assumptions of (F1)–(F3).
(iii) The continuity condition (F3) might be removed if we modify the definition of viscosity solutions as in [8]. However, we do not pursue this direction in this paper.

We shall prove Theorem 4.1 by preparing several propositions. The basic strategy of the proof of Theorem 4.1 is contradiction. Roughly speaking we assume $u^*(t, x, r) - v_*(\underline{t}, y, s) > 0$ and (t, x, r) is close to $(\underline{t}, y, s)$. For a such point we shall find a parabolic super 2-jet of

$$w(t, x, r, \underline{t}, y, s) = u^*(t, x, r) - v_*(\underline{t}, y, s)$$

at $(t, x, r, \underline{t}, y, s)$. We should find a nice parabolic super 2-jet of w to get a contradiction. For this purpose we introduce a test function $\Phi(t, x, r, \underline{t}, y, s)$ and study the maximum of $w - \Phi$.

Let ε, μ, δ and γ be positive constants. Here and hereafter fix μ as an even integer satisfying $\mu \geqq \max\{\beta+1, 4\}$. We shall use

$$\begin{aligned}\Phi(t,x,r,\underline{t},y,s) \; = \; & \frac{|x-y|^4}{4\varepsilon} + \frac{|r-s|^\mu}{\mu\varepsilon} + \frac{(t-\underline{t})^2}{2\varepsilon} \\ & + \delta(R-r) + \delta(R-s) + \frac{\gamma}{T-t} + \frac{\gamma}{T-\underline{t}}\end{aligned}$$

as a test function.

Proposition 4.3 *Suppose that w is upper semicontinuous (in short u.s.c) in $\overline{Q} \times \overline{Q}$, $w < +\infty$ in $\overline{Q} \times \overline{Q}$ and that*

$$\alpha = \sup\{w(t,x,r,t,x,r); (t,x,r) \in \overline{Q}, 0 \leqq t < T\} > 0. \tag{4.3}$$

Set $\Psi(t,x,r,\underline{t},y,s) = w(t,x,r,\underline{t},y,s) - \Phi(t,x,r,\underline{t},y,s)$, then there are positive constants δ_0 and γ_0 such that

$$\sup_{\overline{Q}\times\overline{Q}} \Psi(t,x,r,\underline{t},y,s) > \frac{\alpha}{2} \tag{4.4}$$

holds for all $0 < \delta < \delta_0$, $0 < \gamma < \gamma_0$, $\varepsilon > 0$.

Proof. Since w is bounded in $\overline{Q} \times \overline{Q}$ and (4.3) holds, we easily see that there is a point $(t_0, x_0, r_0) \in \overline{Q}(t_0 < T)$, such that $w(t_0,x_0,r_0,t_0,x_0,r_0) > 3\alpha/4$. Choose δ and γ are sufficiently small so that $2\delta(R-r_0)+2\gamma/(T-t_0) < \alpha/4$. We now observe that $\Psi(t_0,x_0,r_0,t_0,x_0,r_0) > \alpha/2$. □

Let $(\hat{t},\hat{x},\hat{r},\hat{\underline{t}},\hat{y},\hat{s}) \in \overline{Q} \times \overline{Q}$ with $\hat{t}, \hat{\underline{t}} < T$ be a maximum point of Ψ, *i.e.*,

$$\sup_{\overline{Q}\times\overline{Q}} \Psi(t,x,r,\underline{t},y,s) = \Psi(\hat{t},\hat{x},\hat{r},\hat{\underline{t}},\hat{y},\hat{s}).$$

Proposition 4.4 *Let δ_0 and γ_0 be as in Proposition 4.3. Suppose that w is u.s.c in $\overline{Q} \times \overline{Q}$ and that Ψ attains its maximum at $(\hat{t},\hat{x},\hat{r},\hat{\underline{t}},\hat{y},\hat{s})$. Assume that there is a modulus function m (i.e., $m : [0,\infty) \to [0,\infty)$ is continuous, nondecreasing and $m(0) = 0$) such that*

$$w(t,x,r,\underline{t},y,s) \leqq m(|x-y| + |r-s| + |t-\underline{t}|) \quad on \quad \partial_p^2(Q \times Q)\backslash W,$$

where $\partial_p^2(Q \times Q) = (\partial_p Q \times \overline{Q}) \cup (\overline{Q} \times \partial_p Q)$ and $W = (S \times Q) \cup (Q \times S)$. Then there is $\varepsilon_0 > 0$ such that Ψ attains its maximum over $\overline{Q} \times \overline{Q}$ at a point $(\hat{t},\hat{x},\hat{r},\hat{\underline{t}},\hat{y},\hat{s}) \in (0,T) \times \Omega \times [0,R) \times (0,T) \times \Omega \times [0,R)$ for all $0 < \varepsilon < \varepsilon_0, 0 < \delta < \delta_0$ and $0 < \gamma < \gamma_0$.

Proof. From Proposition 4.3 it follows $\Psi(\hat{t},\hat{x},\hat{r},\hat{\underline{t}},\hat{y},\hat{s}) > 0$ for $0<\delta<\delta_0$, $0<\gamma<\gamma_0$, $\varepsilon>0$. This yields

$$w(\hat{t},\hat{x},\hat{r},\hat{\underline{t}},\hat{y},\hat{s}) \geqq \frac{|\hat{x}-\hat{y}|^4}{4\varepsilon}+\frac{|\hat{r}-\hat{s}|^\mu}{\mu\varepsilon}+\frac{(\hat{t}-\hat{\underline{t}})^2}{2\varepsilon} + \delta(R-\hat{r})+\delta(R-\hat{s})+\frac{\gamma}{T-\hat{t}}+\frac{\gamma}{T-\hat{\underline{t}}},$$

then we obtain

$$w(\hat{t},\hat{x},\hat{r},\hat{\underline{t}},\hat{y},\hat{s}) \geqq \frac{|\hat{x}-\hat{y}|^4}{4\varepsilon},\quad \frac{|\hat{r}-\hat{s}|^\mu}{\mu\varepsilon},\quad \frac{(\hat{t}-\hat{\underline{t}})^2}{2\varepsilon}.$$

Since Q is bounded and w is u.s.c, there is a positive constant M such that

$$w(t,x,r,\underline{t},y,s) \leqq M \quad \text{in} \quad \overline{Q}\times\overline{Q}.$$

We now observe that

$$|\hat{x}-\hat{y}|\to 0,\quad |\hat{r}-\hat{s}|\to 0 \quad \text{and} \quad |\hat{t}-\hat{\underline{t}}|\to 0 \quad \text{as} \quad \varepsilon\to 0; \tag{4.5}$$

these are uniform in $0<\delta<\delta_0$, $0<\gamma<\gamma_0$. Suppose that the conclusion of proposition were false. By the properties of function $\gamma/(T-t)+\gamma/(T-\underline{t})$ we see $\hat{t}$, $\hat{\underline{t}}<T$. There would exist a sequence $\{\varepsilon_j\}$ with $\varepsilon_j\to 0, \{\delta_j\}\subset(0,\delta_0), \{\gamma_j\}\subset(0,\gamma_0)$ such that $\partial_p^2(Q\times Q)\backslash W$ contains a maximum point $(\hat{t}_j,\hat{x}_j,\hat{r}_j,\hat{\underline{t}}_j,\hat{y}_j,\hat{s}_j)$ of Ψ for the value $\varepsilon=\varepsilon_j$, $\delta=\delta_j$, $\gamma=\gamma_j$. By (4.3) and the assumption, we see

$$\begin{aligned}\frac{\alpha}{2} &\leqq \Psi(\hat{t}_j,\hat{x}_j,\hat{r}_j,\hat{\underline{t}}_j,\hat{y}_j,\hat{s}_j) \leqq \omega(\hat{t}_j,\hat{x}_j,\hat{r}_j,\hat{\underline{t}}_j,\hat{y}_j,\hat{s}_j)\\ &\leqq m(|\hat{x}_j-\hat{y}_j|+|\hat{r}_j-\hat{s}_j|+|\hat{t}_j-\hat{\underline{t}}_j|).\end{aligned}$$

Since (4.5) holds as $\varepsilon_j\to 0$, we get a contradiction. □

Proof of Theorem 4.1. We may assume that u and v are upper semicontinuous and lower semicontinuous, respectively, so that

$$w(t,x,r,\underline{t},y,s)=u(t,x,r)-v(\underline{t},y,s)$$

is upper semicontinuous. We would like to get a contradiction by assuming that

$$\alpha=\sup\{w(t,x,r,t,x,r);(t,x,r)\in\overline{Q},\ 0\leq t<T\}>0.$$

Let $(\hat{t},\hat{x},\hat{r},\hat{\underline{t}},\hat{y},\hat{s})$ be a maximum point of Ψ in $\overline{Q}\times\overline{Q}$, *i.e.*,

$$\sup_{\overline{Q}\times\overline{Q}}\Psi(t,x,r,\underline{t},y,s)=\Psi(\hat{t},\hat{x},\hat{r},\hat{\underline{t}},\hat{y},\hat{s}).$$

Note that $u\leqq v$ on $\partial_pQ\backslash S$ implies the existence of modulus m satisfying

$$w(t,x,r,\underline{t},y,s)\leqq m(|x-y|+|r-s|+|t-\underline{t}|)\ \text{ on }\ \partial_p^2(Q\times Q)\backslash W$$

since $\partial_pQ\backslash S$ is compact. Thus, we see all conclusions of Propositions 4.3 and 4.4 would hold. Proposition 4.4 says that Ψ attains its maximum over $\overline{Q}\times\overline{Q}$ at $(\hat{t},\hat{x},\hat{r},\hat{\underline{t}},\hat{y},\hat{s})\in(0,T)\times\Omega\times[0,R)\times(0,T)\times\Omega\times[0,R)$ for small ε,δ and γ. We set $\xi=(x,r)$ and $\eta=(y,s)$. Then we observe that

$$w(t,\xi,\underline{t},\eta)\leqq w(\hat{t},\hat{\xi},\hat{\underline{t}},\hat{\eta})+\Phi(t,\xi,\underline{t},\eta)-\Phi(\hat{t},\hat{\xi},\hat{\underline{t}},\hat{\eta}),$$

where $\hat{\xi}=(\hat{x},\hat{r})$ and $\hat{\eta}=(\hat{y},\hat{s})$. Expand Φ at $(\hat{t},\hat{\xi},\hat{\underline{t}},\hat{\eta})$, then we obtain

$$\left(\begin{pmatrix}\hat{\Phi}_{t,\xi}\\ \hat{\Phi}_{\underline{t},\eta}\end{pmatrix},A\right)\in J^{2,+}w(\hat{t},\hat{\xi},\hat{\underline{t}},\hat{\eta})$$

with

$$\hat{\Phi}_{t,\xi}=\begin{pmatrix}\hat{\Phi}_t\\ \hat{\Phi}_\xi\end{pmatrix},\ \hat{\Phi}_{\underline{t},\eta}=\begin{pmatrix}\hat{\Phi}_{\underline{t}}\\ \hat{\Phi}_\eta\end{pmatrix}\ \text{and}\ A=\nabla^2\hat{\Phi}\ \in\boldsymbol{S}^{2m+4},$$

where $\hat{\Phi}_t=\Phi_t(\hat{t},\hat{\xi},\hat{\underline{t}},\hat{\eta}),\hat{\Phi}_\xi=\nabla_\xi\Phi(\hat{t},\hat{\xi},\hat{\underline{t}},\hat{\eta}),\hat{\Phi}_{\xi\xi}=\nabla_\xi^2\Phi(\hat{t},\hat{\xi},\hat{\underline{t}},\hat{\eta})$ and so on. Direct calculations yield

$$\begin{aligned}
\hat{\Phi}_t&=\gamma/(T-\hat{t})^2,\ \hat{\Phi}_{\underline{t}}=\gamma/(T-\hat{\underline{t}})^2,\\
\hat{\Phi}_\xi&=\begin{pmatrix}\hat{\Phi}_x\\ \hat{\Phi}_r\end{pmatrix}=\frac{1}{\varepsilon}\begin{pmatrix}|\hat{x}-\hat{y}|^2(\hat{x}-\hat{y})\\ (\hat{r}-\hat{s})^{\mu-1}-\varepsilon\delta\end{pmatrix},\\
\hat{\Phi}_\eta&=\begin{pmatrix}\hat{\Phi}_y\\ \hat{\Phi}_s\end{pmatrix}=\frac{1}{\varepsilon}\begin{pmatrix}-|\hat{x}-\hat{y}|^2(\hat{x}-\hat{y})\\ -(\hat{r}-\hat{s})^{\mu-1}-\varepsilon\delta\end{pmatrix},\\
\hat{\Phi}_{\xi\xi}&=\frac{1}{\varepsilon}\begin{pmatrix}A'&O\\ O&(\mu-1)(\hat{r}-\hat{s})^{\mu-2}\end{pmatrix}\in\boldsymbol{S}^{m+1}
\end{aligned}$$

with $A'=|\hat{x}-\hat{y}|^2I_m+2(\hat{x}-\hat{y})\otimes(\hat{x}-\hat{y})\in\boldsymbol{S}^m$ and $\hat{\Phi}_{\xi\xi}=-\hat{\Phi}_{\xi\eta}=-\hat{\Phi}_{\eta\xi}=\hat{\Phi}_{\eta\eta}\in\boldsymbol{S}^{m+1}$, where I_m is identity matrix with order m and $\otimes$ denotes the tensor product. Since $\hat{r}$ is close to $\hat{s}$, we may assume $|\hat{r}-\hat{s}|\leqq 1$ and since $(\hat{x}-\hat{y})\otimes(\hat{x}-\hat{y})\leqq|\hat{x}-\hat{y}|^2I_m$, then we observe

$$\hat{\Phi}_{\xi\xi}\leqq\frac{\mu-1}{\varepsilon}|\hat{\xi}-\hat{\eta}|^2I_{m+1}.$$

Here the assumption $\mu \geqq 4$ is invoked. By Crandall-Ishii's Lemma (c.f. [4], [5]) we observe that for any positive constant λ there are $Z_1,\ Z_2 \in \boldsymbol{S}^{m+2}$, such that

$$\left(\begin{pmatrix} \hat{\Phi}_t \\ \hat{\Phi}_\xi \end{pmatrix}, Z_1 \right) \in \overline{J^{2,+}}u(\hat{t},\hat{\xi}), \ \left(\begin{pmatrix} -\hat{\Phi}_{\underline{t}} \\ -\hat{\Phi}_\eta \end{pmatrix}, -Z_2 \right) \in \overline{J^{2,-}}v(\hat{\underline{t}},\hat{\eta})$$

and

$$-\Big(\frac{1}{\lambda}+|A|\Big)I_{2m+4} \leqq \begin{pmatrix} Z_1 & O \\ O & Z_2 \end{pmatrix} \leqq A+\lambda A^2. \tag{4.6}$$

Since $Z_1,\ Z_2 \in \boldsymbol{S}^{m+2}$, we see

$$Z_1 = \begin{pmatrix} * & * \\ * & X \end{pmatrix},\ Z_2 = \begin{pmatrix} * & * \\ * & Y \end{pmatrix} \quad \text{for some} \quad X,\ Y \in \boldsymbol{S}^{m+1}.$$

By [11, Corollary 3.6] we obtain

$$(\hat{\Phi}_t, \hat{\Phi}_\xi, X) \in \overline{\mathcal{P}^{2,+}}u(\hat{t},\hat{\xi}), \quad (-\hat{\Phi}_{\underline{t}}, -\hat{\Phi}_\eta, -Y) \in \overline{\mathcal{P}^{2,-}}v(\hat{\underline{t}},\hat{\eta}).$$

We see $\hat{r} > 0$. Indeed, since u is a subsolution of (4.1) and (4.2), u should satisfy $-\Phi_r(\hat{t},\hat{x},0,\hat{\underline{t}},\hat{y},\hat{s}) \leqq 0$ if $\hat{r} = 0$ (see Definition 2.2). But $-\Phi_r(\hat{t},\hat{x},0,\hat{\underline{t}},\hat{y},\hat{s}) = -(-\hat{s})^{\mu-1}/\varepsilon+\delta > 0$, which is against the definition of subsolutions. In the same way we know $\hat{s} > 0$. Therefore, we obtain

$$0 \geqq \frac{\gamma}{(T-\hat{t})^2} + \frac{\gamma}{(T-\hat{\underline{t}})^2} + F_*\Big(\hat{\Phi}_\xi, X, \frac{\hat{\Phi}_r}{\hat{r}^\beta}\Big) - F^*\Big(-\hat{\Phi}_\eta, -Y, -\frac{\hat{\Phi}_s}{\hat{s}^\beta}\Big).$$

Since $\hat{\Phi}_r/\hat{r}^\beta \leqq (\hat{r}-\hat{s})^{\mu-1}/\varepsilon\hat{r}^\beta$, $\hat{\Phi}_s/\hat{s}^\beta \leqq -(\hat{r}-\hat{s})^{\mu-1}/\varepsilon\hat{s}^\beta$ and (F2) (ii) holds, we see

$$0 \geqq \frac{2\gamma}{T^2} + F_*\Big(\hat{\Phi}_\xi, X, \frac{(\hat{r}-\hat{s})^{\mu-1}}{\varepsilon\hat{r}^\beta}\Big) - F^*\Big(-\hat{\Phi}_\eta, -Y, \frac{(\hat{r}-\hat{s})^{\mu-1}}{\varepsilon\hat{s}^\beta}\Big).$$

Moreover, since μ is even, we have $(\hat{r}-\hat{s})^{\mu-1}/\varepsilon\hat{r}^\beta \leqq (\hat{r}-\hat{s})^{\mu-1}/\varepsilon\hat{s}^\beta$. Using this and (F2) (ii), we get two key inequalities ;

$$0 \geqq \frac{2\gamma}{T^2} + F_*\Big(\hat{\Phi}_\xi, X, \frac{(\hat{r}-\hat{s})^{\mu-1}}{\varepsilon\hat{r}^\beta}\Big) - F^*\Big(-\hat{\Phi}_\eta, -Y, \frac{(\hat{r}-\hat{s})^{\mu-1}}{\varepsilon\hat{r}^\beta}\Big), \tag{4.7}$$

$$0 \geqq \frac{2\gamma}{T^2} + F_*\Big(\hat{\Phi}_\xi, X, \frac{(\hat{r}-\hat{s})^{\mu-1}}{\varepsilon\hat{s}^\beta}\Big) - F^*\Big(-\hat{\Phi}_\eta, -Y, \frac{(\hat{r}-\hat{s})^{\mu-1}}{\varepsilon\hat{s}^\beta}\Big). \tag{4.8}$$

Since $\overline{Q}$ is compact and the matrix inequality (4.6) holds, we may assume that $(\hat{t}, \hat{x}, \hat{r}, \hat{\underline{t}}, \hat{y}, \hat{s}) \to (t_0, x_0, r_0, \underline{t}_0, y_0, s_0)$ and $X \to X_0, Y \to Y_0$ as $\delta \to 0$ for some $t_0,\ \underline{t}_0 \in [0,T]$, $x_0, y_0 \in \overline{\Omega}$, $r_0, s_0 \in [0,R]$ and $X_0, Y_0 \in \boldsymbol{S}^{m+1}$ by taking a subsequence. Although $\lim_{\delta\to 0} \hat{\Phi}_\xi = -\lim_{\delta\to 0} \hat{\Phi}_\eta$ exist, $(\hat{r}-\hat{s})^{\mu-1}/\varepsilon\hat{r}^\beta$ and $(\hat{r}-\hat{s})^{\mu-1}/\varepsilon\hat{s}^\beta$ may go infinity as $\delta \to 0$. We have to divide cases depending whether or not $(\hat{r}-\hat{s})^{\mu-1}/\varepsilon\hat{r}^\beta$ and $(\hat{r}-\hat{s})^{\mu-1}/\varepsilon\hat{s}^\beta$ go infinity. Moreover, our argument depends whether or not $\hat{\Phi}_\xi$ goes to zero as $\delta \to 0$ since $F = F(p, X, a)$ is discontinuous on $p = 0$. We divide cases as follows:

Case I: $\hat{r} \to 0$ and $\hat{s} \to 0$ as $\delta \to 0$.

Case I*a*: There are sequences $\{\hat{r}_j\}$ and $\{\hat{s}_j\}$ such that $\hat{s}_j/\hat{r}_j \to +\infty$ as $j \to \infty$.

(i) $|x_0 - y_0| = 0$ and $r_0 - s_0 = 0$.

(ii) $|x_0 - y_0| \neq 0$ or $r_0 - s_0 \neq 0$.

Case I*b*: There are sequences $\{\hat{r}_j\}$ and $\{\hat{s}_j\}$ such that $\hat{s}_j/\hat{r}_j \to 0$ as $j \to \infty$.

(i) $|x_0 - y_0| = 0$ and $r_0 - s_0 = 0$.

(ii) $|x_0 - y_0| \neq 0$ or $r_0 - s_0 \neq 0$.

Case I*c*: There are sequences $\{\hat{r}_j\}$ and $\{\hat{s}_j\}$ such that $\hat{s}_j/\hat{r}_j \to c$ as $j \to \infty$ for some positive constant c.

(i) $|x_0 - y_0| = 0$ and $r_0 - s_0 = 0$.

(ii) $|x_0 - y_0| \neq 0$ or $r_0 - s_0 \neq 0$.

Case II: $\hat{r} \to r_0 \neq 0$ or $\hat{s} \to s_0 \neq 0$ as $\delta \to 0$.

(i) $|x_0 - y_0| = 0$ and $r_0 - s_0 = 0$.

(ii) $|x_0 - y_0| \neq 0$ or $r_0 - s_0 \neq 0$.

We shall discuss the inequality (4.7) or (4.8).

Case I*a*. We set $\hat{s}_j/\hat{r}_j = \nu_j$. Since $\mu \geqq \beta + 1$, then we observe

$$\begin{aligned}\lim_{j\to\infty} \frac{(\hat{r}_j - \hat{s}_j)^{\mu-1}}{\varepsilon \hat{s}_j^\beta} &= \lim_{j\to\infty} \frac{\hat{s}_j^{\mu-1}(1/\nu_j - 1)^{\mu-1}}{\varepsilon \hat{s}_j^\beta} \\ &= \lim_{j\to\infty} \frac{1}{\varepsilon}\hat{s}_j^{\mu-1-\beta}(1/\nu_j - 1)^{\mu-1} = c_1\end{aligned}$$

for some bounded constant c_1. We study (4.8). Letting $\delta \downarrow 0$ we obtain

$$0 \geqq \frac{2\gamma}{T^2} + F_*\Big(\frac{1}{\varepsilon}(|x_0 - y_0|^2(x_0 - y_0), (r_0 - s_0)^{\mu-1}), X_0, c_1\Big)$$

$$- F^*\left(\frac{1}{\varepsilon}(|x_0 - y_0|^2(x_0 - y_0), (r_0 - s_0)^{\mu-1}), -Y_0, c_1\right).$$

Case I*a* (i): $|x_0 - y_0| = 0$ and $r_0 - s_0 = 0$.

By the matrix inequality (4.6) we obtain $X_0, Y_0 \leqq O$. Indeed, multiplying (4.6) by $(0, \underline{p}, 0, \underline{p}) \in \boldsymbol{R} \times \boldsymbol{R}^{m+1} \times \boldsymbol{R} \times \boldsymbol{R}^{m+1}$ from the left and by ${}^t(0, \underline{p}, 0, \underline{p})$ from the right, we derive from the right inequality of (4.6) that $X_0, Y_0 \leqq O$ since $x_0 = y_0$ and $r_0 = s_0$. Since (F2) (i) and (F3) hold, we observe

$$\begin{aligned} 0 &\geqq \frac{2\gamma}{T^2} + F_*((0,0), X_0, c_1) - F^*((0,0), -Y_0, c_1) \\ &\geqq \frac{2\gamma}{T^2} + F_*((0,0), O, c_1) - F^*((0,0), O, c_1) \\ &= \frac{2\gamma}{T^2}. \end{aligned}$$

Thus we get a contradiction.

Case I*a* (ii): $|x_0 - y_0| \neq 0$ or $r_0 - s_0 \neq 0$.

We only prove in the case $|x_0 - y_0| = 0$ and $r_0 - s_0 \neq 0$ since other cases can be proved similarly. By the matrix inequality (4.6) we obtain $X_0 + Y_0 \leqq O$ as in Case I*a* (i). Since (F2) (i) and (F3) hold, we see

$$\begin{aligned} 0 &\geqq \frac{2\gamma}{T^2} + F\left(\left(0, \frac{1}{\varepsilon}(r_0 - s_0)^{\mu-1}\right), X_0, c_1\right) \\ &\qquad - F\left(\left(0, \frac{1}{\varepsilon}(r_0 - s_0)^{\mu-1}\right), -Y_0, c_1\right) \\ &\geqq \frac{2\gamma}{T^2} + F\left(\left(0, \frac{1}{\varepsilon}(r_0 - s_0)^{\mu-1}\right), X_0, c_1\right) \\ &\qquad - F\left(\left(0, \frac{1}{\varepsilon}(r_0 - s_0)^{\mu-1}\right), X_0, c_1\right) \\ &\geqq \frac{2\gamma}{T^2}. \end{aligned}$$

We get a contradiction again.

Case I*b*. We set $\hat{s}_j/\hat{r}_j = \nu_j$. Since $\mu \geqq \beta + 1$, then we observe

$$\begin{aligned} \lim_{j\to\infty} \frac{(\hat{r}_j - \hat{s}_j)^{\mu-1}}{\varepsilon \hat{r}_j^{\beta}} &= \lim_{j\to\infty} \frac{\hat{r}_j^{\mu-1}(1-\nu_j)^{\mu-1}}{\varepsilon \hat{r}_j^{\beta}} \\ &= \lim_{j\to\infty} \frac{1}{\varepsilon}\hat{r}_j^{\mu-1-\beta}(1-\nu_j)^{\mu-1} = c_2 \end{aligned}$$

for some bounded constant c_2. We study (4.7). We can get a contradiction as in Case Ia.
Case Ic. We set $\hat{s}_j/\hat{r}_j = \nu_j$. Since $\mu \geqq \beta + 1$, then we observe

$$\begin{aligned} \lim_{j\to\infty} \frac{(\hat{r}_j - \hat{s}_j)^{\mu-1}}{\varepsilon \hat{r}_j^{\beta}} &= \lim_{j\to\infty} \frac{\hat{r}_j^{\mu-1}(1-\nu_j)^{\mu-1}}{\varepsilon \hat{r}_j^{\beta}} \\ &= \lim_{j\to\infty} \frac{1}{\varepsilon} \hat{r}_j^{\mu-1-\beta}(1-\nu_j)^{\mu-1} = c_3 \end{aligned}$$

for some bounded constant c_3. We study (4.7). Similarly we obtain a contradiction as in Case Ia.
Case II. Here we only prove in the case $\hat{r} \to r_0 \neq 0$ as $\delta \to 0$. The other case is verified in the same way. We study (4.7). Letting $\delta \to 0$ we observe that

$$\begin{aligned} 0 \geqq \frac{2\gamma}{T^2} &+ F_*\left(\frac{1}{\varepsilon}(|x_0 - y_0|^2(x_0 - y_0), (r_0 - s_0)^{\mu-1}), X_0, \frac{(r_0 - s_0)^{\mu-1}}{\varepsilon r_0^{\beta}}\right) \\ &- F^*\left(\frac{1}{\varepsilon}(|x_0 - y_0|^2(x_0 - y_0), (r_0 - s_0)^{\mu-1}), -Y_0, \frac{(r_0 - s_0)^{\mu-1}}{\varepsilon r_0^{\beta}}\right). \end{aligned}$$

Case II (i): $\quad |x_0 - y_0| = 0 \quad$ and $\quad r_0 - s_0 = 0.$

By the matrix inequality (4.6) we obtain $X_0, Y_0 \leqq O$. Since (F2) (i) and (F3) hold we observe that

$$\begin{aligned} 0 &\geqq \frac{2\gamma}{T^2} + F_*((0,0), X_0, 0) - F^*((0,0), -Y_0, 0) \\ &\geqq \frac{2\gamma}{T^2} + F_*((0,0), O, 0) - F^*((0,0), O, 0) \\ &= \frac{2\gamma}{T^2}. \end{aligned}$$

Thus we get a contradiction.
Case II (ii) is proved as in Case Ia (ii).

The proof of Theorem 4.1 is now completed. □

We remark a comparison theorem can be extended when Ω is not necessarily bounded at least for (2.1) and (2.2) provided that F_0 satisfies ($F_0$1)–($F_0$3). Although the basic idea of the proof is similarly, it needs some extra work as in [8]. We do not treat unbounded Ω in this paper.

5. Consistency

In section 3 we gave the level set equation for mean curvature flow equation (3.1) and the x'-axisymmetrized equation (3.2). We shall discuss the relation between a solution of (3.1) and a solution of (3.2). We consider the relation in general form. We generalized (3.1) of the form:

$$U_t + \widetilde{F}(DU, D^2U) = 0 \quad \text{in} \quad (0,T) \times \widetilde{\Omega}, \tag{5.1}$$

where $\widetilde{\Omega}$ is a bounded domain in $\boldsymbol{R}^n$, $D = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$ and D^2U denotes the Hessian of U in $\boldsymbol{S}^n$. Here we assume

$$\widetilde{F} : (\boldsymbol{R}^n \backslash \{0\}) \times \boldsymbol{S}^n \to \boldsymbol{R} \text{ is continuous,} \tag{$\widetilde{F}1$}$$

$$\begin{aligned} &\widetilde{F} \text{ is degenerate elliptic, } i.e., \\ &\widetilde{F}(p, X+Y) \leqq \widetilde{F}(p, X) \quad \text{for all} \quad Y \geqq O,\ p \neq 0, \end{aligned} \tag{$\widetilde{F}2$}$$

$$-\infty < \widetilde{F}_*(0, O) = \widetilde{F}^*(0, O) < +\infty. \tag{$\widetilde{F}3$}$$

Note that by definition we see that if $\widetilde{F}$ fulfills ($\widetilde{F}2$), so do $\widetilde{F}_*$ and $\widetilde{F}^*$, respectively, even if $p = 0$.

We generalize (3.2) of the form:

$$u_t + F\left(\nabla_{x,r} u, \nabla^2_{x,r} u, \frac{u_r}{r}\right) = 0 \quad \text{in} \quad Q = (0,T) \times \Omega' \times (0,R), \tag{5.2}$$

$$-u_r = 0 \quad \text{on} \quad S = (0,T) \times \Omega' \times \{0\}, \tag{5.3}$$

where Ω' is a bounded domain in $\boldsymbol{R}^m$, $\nabla_{x,r} = (\partial/\partial x_1, \ldots, \partial/\partial x_m, \partial/\partial r)$ and $\nabla^2_{x,r} u$ denotes the Hessian of u in $\boldsymbol{S}^{m+1}$. We assume that

$$\begin{aligned} &U \ \text{ is an } x'\text{-axisymmetric function, } i.e., \\ &\text{there is a function } \ u : Q = (0,T) \times \Omega' \times (0,R) \to \boldsymbol{R} \\ &\text{such that } \ U(t,x) = u(t, x', r), \end{aligned} \tag{5.4}$$

where $x = (x', x'') \in \boldsymbol{R}^m \times \boldsymbol{R}^{n-m}$ and $|x''| = r$. We define a set of matrices $ASO(n,m)$ as follows:

$$ASO(n,m) = \left\{ \Theta \in M_n; \Theta = \begin{pmatrix} I_m & O \\ O & \Theta'' \end{pmatrix}, \Theta'' \in SO(n-m) \right\},$$

where M_n denotes the set of $n \times n$ matrices and $SO(n-m)$ denotes the group of $(n-m) \times (n-m)$ rotation matrices. Through out this section we assume that

$$\begin{aligned}&\widetilde{F} \text{ is rotationally invariant in variables } x'', i.e.,\\ &\widetilde{F}(\Theta p, \Theta X {}^t\Theta) = \widetilde{F}(p, X) \text{ for all } p \in \boldsymbol{R}^n \backslash \{0\},\\ &X \in \boldsymbol{S}^n \text{ and } \Theta \in ASO(n,m).\end{aligned} \tag{5.5}$$

Proposition 5.1 *Let $U \in C^2((0,T) \times \widetilde{\Omega})$. Assume that* (5.4) *and* (5.5) *hold. Suppose that $DU \neq 0$. Then*

$$\widetilde{F}(DU, D^2U) = \widetilde{F}\left(\begin{pmatrix} \nabla_{x,r}u \\ 0 \end{pmatrix}, \begin{pmatrix} \nabla^2_{x,r}u & O \\ O & \frac{u_r}{r} I_{n-(m+1)} \end{pmatrix} \right).$$

This is easy to prove so we omit the proof.

Remark 5.2. Note that if $\widetilde{F}$ fulfills (5.5), so do $\widetilde{F}_*$ and $\widetilde{F}^*$, respectively, even if $p = 0$. So Proposition 5.1 holds for $\widetilde{F}_*$ and $\widetilde{F}^*$, respectively, without assuming $DU \neq 0$.

We define

$$F(p, X, a) = \widetilde{F}\left(\begin{pmatrix} p \\ 0 \end{pmatrix}, \begin{pmatrix} X & O \\ O & aI_{n-(m+1)} \end{pmatrix} \right) \tag{5.6}$$

for all $p \in \boldsymbol{R}^{m+1} \backslash \{0\}$, $X \in \boldsymbol{S}^{m+1}$ and $a \in \boldsymbol{R}$.

Remark 5.3. By definition we see that (5.6) holds for F_* and $\widetilde{F}_*$ even if $p = 0$. Also, (5.6) holds for F^* and $\widetilde{F}^*$ for all $p \in \boldsymbol{R}^{m+1}$.

Lemma 5.4 *If $\widetilde{F}$ satisfies $(\widetilde{F}1)$ and $(\widetilde{F}2)$, then F satisfies $(F1)$ and $(F2)$. Moreover, if $(\widetilde{F}3)$ and $(F4)$ hold then F satisfies $(F3)$.*

Proof. It is easy to see that F satisfies (F1). We shall prove that (F2) holds for F. Let $p \in \boldsymbol{R}^{m+1} \backslash \{0\}$, X, $Y \in \boldsymbol{S}^{m+1}$ and $a \in \boldsymbol{R}$. For $X \geqq Y$ we shall show

$$F(p, X, a) \leqq F(p, Y, a).$$

By the definition (5.6) and the degenerate elliptic condition $(\widetilde{F}2)$ we observe that

$$F(p, X, a) = \widetilde{F}\left(\begin{pmatrix} p \\ 0 \end{pmatrix}, \begin{pmatrix} X & O \\ O & aI_{n-(m+1)} \end{pmatrix} \right)$$

$$
\begin{aligned}
&\leqq \widetilde{F}\left(\begin{pmatrix} p \\ 0 \end{pmatrix}, \begin{pmatrix} Y & O \\ O & aI_{n-(m+1)} \end{pmatrix}\right) \\
&= F(p, Y, a).
\end{aligned}
$$

For $a \geqq b$ we can similarly prove $F(p, X, a) \leqq F(p, X, b)$.

It remains to show that F satisfies (F3). By ($\widetilde{F}3$) and Remark 5.3 we easily see $-\infty < F_*(0, O, 0) = F^*(0, O, 0) < +\infty$. Moreover, from (F4) we get $-\infty < F_{0*}(0, O) = F_0{}^*(0, O) < +\infty$. Finally, we obtain $-\infty < F_*(0, O, a) = F_{0*}(0, O) - \nu a = F_0{}^*(0, O) - \nu a = F^*(0, O, a) < +\infty$ for all $a \in \boldsymbol{R}$, where ν is a positive constant appeared in (F4). $\square$

Next we want to discuss the relation between a solution of (5.1) and a solution of (5.2) and (5.3).

Theorem 5.5 *Let U be a viscosity subsolution of* (5.1). *Suppose that* (5.4) *and* (5.5) *hold and that $\widetilde{F}$ satisfies* ($\widetilde{F}1$) *and* ($\widetilde{F}2$). *Assume that*

$$
\varlimsup_{a \to +\infty} \widetilde{F}_*\left(\begin{pmatrix} p' \\ 0 \end{pmatrix}, \begin{pmatrix} X & O \\ O & -aI_{n-m} \end{pmatrix}\right) = +\infty, \tag{$\widetilde{F}4$}
$$

where $(p', 0) \in \boldsymbol{R}^m \times \boldsymbol{R}^{n-m}$, $X \in \boldsymbol{S}^m$, $a > 0$ and $n - m - 1 > 0$. Then u is a viscosity subsolution of (5.2) *and* (5.3).

Proof. Here we write $u^* = u$ and $U^* = U$. Let $\varphi \in C^2(\Sigma)$ and $(\hat{t}, \hat{x}', \hat{r}) \in \Sigma$ satisfy

$$
\max_{\Sigma}(u - \varphi) = (u - \varphi)(\hat{t}, \hat{x}', \hat{r}),
$$

where $\Sigma = Q \cup S$.

Case (i) $\hat{r} > 0$. We set $\phi(t, x) = \varphi(t, x', r)$ with $x = (x', x'')$ and $|x''| = r$. Then $\phi \in C^2((0, T) \times \Omega' \times (\Omega'' \backslash \{0\}))$ satisfies

$$
\max_{(0,T)\times\widetilde{\Omega},\ |x''| \leq R} (U - \phi) = (U - \phi)(\hat{t}, \hat{x})
$$

with $\hat{x} = (\hat{x}', \hat{x}'')$ and $|\hat{x}''| = (\hat{x}_{m+1}^2 + \cdots + \hat{x}_n^2)^{1/2} = \hat{r}$, where $\widetilde{\Omega} = \Omega' \times \Omega'' \in \boldsymbol{R}^m \times \boldsymbol{R}^{n-m}$. Note that ϕ is C^2 around $(\hat{t}, \hat{x})$ since $\hat{r} \neq 0$. Since U is a viscosity subsolution of (5.1), we know

$$
\phi_t + \widetilde{F}_*(D\phi, D^2\phi) \leqq 0 \quad \text{at} \quad (\hat{t}, \hat{x}).
$$

By the definition of $\widetilde{F}_*$ the rotational invariance (5.5) holds for $\widetilde{F}_*$ even if $p = 0$ (Remark 5.2). Also (5.6) holds for $\widetilde{F}_*$ and F_* (Remark 5.3). By

Proposition 5.1 we observe that

$$\begin{aligned}\widetilde{F}_*(D\phi, D^2\phi) &= \widetilde{F}_*\left(\begin{pmatrix} \nabla_{x,r}\varphi \\ 0 \end{pmatrix}, \begin{pmatrix} \nabla^2_{x,r}\varphi & 0 \\ 0 & \frac{\varphi_r}{r} I_{n-(m+1)} \end{pmatrix} \right) \\ &= F_*\left(\nabla_{x,r}\varphi, \nabla^2_{x,r}\varphi, \frac{\varphi_r}{r} \right).\end{aligned}$$

Therefore, we get

$$\varphi_t + F_*\left(\nabla_{x,r}\varphi, \nabla^2_{x,r}\varphi, \frac{\varphi_r}{r} \right) \leqq 0 \quad \text{at} \quad (\hat{t}, \hat{x}', \hat{r}).$$

Case (ii) $\hat{r} = 0$. We shall show

$$-\varphi_r \leqq 0 \qquad \text{at} \quad (\hat{t}, \hat{x}', 0).$$

To show this we argue by contradiction. We suppose that u is not a viscosity subsolution of (5.2) and (5.3). Then there would exist

$$\left(\tau, \begin{pmatrix} p' \\ -\zeta \end{pmatrix}, \begin{pmatrix} X & * \\ * & * \end{pmatrix} \right) \in \mathcal{P}^{2,+} u(\hat{t}, \hat{x}', 0) \tag{5.7}$$

such that $\zeta > 0$, where $\tau, \zeta \in \boldsymbol{R}$, $p' \in \boldsymbol{R}^m$ and $X \in \boldsymbol{S}^m$. We shall construct an element of parabolic super 2-jet of $U(\hat{t}, \hat{x}', 0)$. From (5.7) we observe that

$$\begin{aligned}& U(t, x', x'') - U(\hat{t}, \hat{x}', 0) \\ &\quad = u(t, x', r) - u(\hat{t}, \hat{x}', 0) \\ &\quad \leqq \tau(t - \hat{t}) + \left\langle \begin{pmatrix} p' \\ -\zeta \end{pmatrix}, \begin{pmatrix} x' - \hat{x}' \\ r \end{pmatrix} \right\rangle \\ &\qquad + \frac{1}{2} \left\langle \begin{pmatrix} X & * \\ * & * \end{pmatrix} \begin{pmatrix} x' - \hat{x}' \\ r \end{pmatrix}, \begin{pmatrix} x' - \hat{x}' \\ r \end{pmatrix} \right\rangle \\ &\qquad\qquad + o(|x' - \hat{x}'|^2 + r^2 + |t - \hat{t}|) \\ &\quad = \tau(t - \hat{t}) + < p', x' - \hat{x}' > - \zeta |x''| + o(|x''|) \\ &\qquad + \frac{1}{2} \langle X(x' - \hat{x}'), (x' - \hat{x}') \rangle + o(|x' - \hat{x}'|^2 + |x''|^2 + |t - \hat{t}|) \\ &\qquad\qquad \text{as} \quad (t, x', x'') \to (\hat{t}, \hat{x}', 0).\end{aligned}$$

For all $\varepsilon > 0$ we can choose $\delta_1 > 0$ such that if $|x''| < \delta_1$ then $o(|x''|) < \varepsilon |x''|$. Therefore, we get

$$U(t, x', x'') - U(t, \hat{x}', 0)$$

$$\leqq \tau(t-\hat{t})+ < p', x'-\hat{x}' > -(\zeta-\varepsilon)|x''|$$
$$+\frac{1}{2}\langle X(x'-\hat{x}'), x'-\hat{x}'\rangle + o(|x'-\hat{x}'|^2+|x''|^2+|t-\hat{t}|).$$

Hence, we take $\varepsilon = \zeta/2$ to get

$$\begin{aligned} &U(t,x',x'')-U(t,\hat{x}',0)\\ &\leqq \tau(t,-\hat{t})+ < p', x'-\hat{x}' > -\frac{\zeta}{2}|x''|\\ &\quad +\frac{1}{2}\langle X(x'-\hat{x}'),(x'-\hat{x}')\rangle + o(|x'-\hat{x}'|^2+|x''|^2+|t-\hat{t}|)\\ &\qquad\qquad \text{as} \quad (t,x',x'')\to(\hat{t},\hat{x}',0). \end{aligned}$$

Moreover, for all $a>0$ we can choose $\delta_2>0$ such that if $|x''|<\delta_2$ then $\frac{\zeta}{2}|x''| \geqq a|x''|^2$. Therefore, if $|x''|<\min\{\delta_1,\delta_2\}$, we observe that

$$\begin{aligned} &U(t,x',x'')-U(t,\hat{x}',0)\\ &\leqq \tau(t-\hat{t})+ < p', x'-\hat{x}' > -a|x''|^2\\ &\quad +\frac{1}{2}\langle X(x'-\hat{x}'),(x'-\hat{x}')\rangle + o(|x'-\hat{x}'|^2+|x''|^2+|t-\hat{t}|)\\ &= \tau(t-\hat{t})+ < p', x'-\hat{x}' >\\ &\quad +\frac{1}{2}\langle X(x'-\hat{x}'),(x'-\hat{x}')\rangle + \frac{1}{2}\langle -2aI_{n-m}x'', x''\rangle\\ &\qquad\qquad + o(|x'-\hat{x}'|^2+|x''|^2+|t-\hat{t}|)\\ &= \tau(t-\hat{t}) + \left\langle \begin{pmatrix} p' \\ 0 \end{pmatrix}, \begin{pmatrix} x'-\hat{x}' \\ x'' \end{pmatrix} \right\rangle\\ &\quad +\frac{1}{2}\left\langle \begin{pmatrix} X & O \\ O & -2aI_{n-m} \end{pmatrix}\begin{pmatrix} x'-\hat{x}' \\ x'' \end{pmatrix}, \begin{pmatrix} x'-\hat{x}' \\ x'' \end{pmatrix} \right\rangle\\ &\qquad\qquad + o(|x'-\hat{x}'|^2+|x''|^2+|t-\hat{t}|). \end{aligned}$$

Thus we obtained an element of parabolic super 2-jet of $U(\hat{t},\hat{x}',0)$;

$$\left(\tau, \begin{pmatrix} p' \\ 0 \end{pmatrix}, \begin{pmatrix} X & O \\ O & -2aI_{n-m} \end{pmatrix}\right) \in \mathcal{P}^{2,+}U(\hat{t},\hat{x}',0).$$

Since $(\widetilde{F}4)$ holds, there is a sequence $a_j\to\infty$ such that

$$\tau + \widetilde{F}_*\left(\begin{pmatrix} p' \\ 0 \end{pmatrix}, \begin{pmatrix} X & O \\ O & -2a_jI_{n-m} \end{pmatrix}\right) \to +\infty \quad \text{as } j\to+\infty.$$

This shows that U is not a viscosity subsolution of (5.1), which contradicts the assumption. $\square$

Similar assertion holds for the supersolution case replaced by a condition

$$\varliminf_{a\to+\infty} \widetilde{F}^*\left(\begin{pmatrix} p' \\ 0 \end{pmatrix}, \begin{pmatrix} X & O \\ O & aI_{n-m} \end{pmatrix}\right) = -\infty \tag{$\widetilde{F}5$}$$

instead of ($\widetilde{F}4$).

Remark 5.6. The conditions ($\widetilde{F}4$) and ($\widetilde{F}5$) say that $\widetilde{F}$ has some " strict" parabolicity. In fact, all first order equations are excluded by ($\widetilde{F}4$), ($\widetilde{F}5$). We shall observe that the conditions ($\widetilde{F}4$) and ($\widetilde{F}5$) are not too restrictive. Indeed, there are many useful examples. Here we only check the condition ($\widetilde{F}4$) because ($\widetilde{F}5$) can be checked in the same way.

(i) The level set equation (3.1) for mean curvature flow is given by taking

$$\widetilde{F}(p, Y) = -\mathrm{trace}\left(I_n - \frac{p\otimes p}{|p|^2}\right)Y \tag{5.8}$$

in (5.1). Note that this $\widetilde{F}$ satisfies (5.5), ($\widetilde{F}1$)–($\widetilde{F}3$). We know that

$$\widetilde{F}_*(0, Y) = -\sum_{i=2}^{n} \lambda_i(Y), \quad \widetilde{F}^*(0, Y) = -\sum_{i=1}^{n-1} \lambda_i(Y), \tag{5.9}$$

where $\lambda_i(Y)$ is eigenvalue of Y and $\lambda_1(Y) \leqq \lambda_2(Y) \leqq \cdots \leqq \lambda_n(Y)$. We shall give the proof of (5.9) for completeness. By definition we obtain

$$\widetilde{F}_*(0, Y) = \inf_{|q|=1} \{-\mathrm{trace}(I_n - q\otimes q)Y\}.$$

We may assume that Y is a diagonal matrix. Since $Y \in \boldsymbol{S}^n$, there is an orthogonal matrix P such that PY^tP is a diagonal matrix. Then we see

$$\begin{aligned} &\inf_{|q|=1}\left\{-\mathrm{trace}\Big(P(I_n - q\otimes q)Y^tP\Big)\right\} \\ &\quad = \inf_{|Pq|=1}\left\{-\mathrm{trace}\Big((I_n - Pq\otimes Pq)PY^tP\Big)\right\} \end{aligned}$$

since $|Pq| = |q| = 1$. Thus we may assume that Y is a diagonal matrix.

Take q as a unit eigenvector of the eigenvalue $\lambda_1(Y)$, then we get

$$\inf_{|q|=1} \{-\mathrm{trace}(I_n - q \otimes q)Y\} \leqq -\sum_{i=2}^{n} \lambda_i(Y).$$

On the other hand, the opposite inequality is easy since $\lambda_1(Y) \leqq \lambda_i(Y)$ for all i $(2 \leqq i \leqq n)$. Similarly we can prove the latter equality of (5.9). We shall check this $\widetilde{F}$ satisfies $(\widetilde{F}4)$. We recall the condition $n > m+1$ which is natural to consider axisymmetric functions.

Case 1: $p' = 0$. By the former equality of (5.9) we get

$$\widetilde{F}_*\left(0, \begin{pmatrix} X & O \\ O & -aI_{n-m} \end{pmatrix}\right) = -\left(-a(n-m-1) + \sum_{i=1}^{m} \lambda_i(X)\right).$$

for sufficiently large $a > 0$.

Case 2: $p' \neq 0$; *i.e.*, $p_i \neq 0$ for some i $(1 \leq i \leq m)$. A direct calculation yields

$$\begin{aligned}
&\widetilde{F}\left(\begin{pmatrix} p' \\ 0 \end{pmatrix}, \begin{pmatrix} X & O \\ O & -aI_{n-m} \end{pmatrix}\right) \\
&= -\mathrm{trace}\left(I_n - \frac{(p',0) \otimes (p',0)}{|p'|^2}\right)\begin{pmatrix} X & O \\ O & -aI_{n-m} \end{pmatrix} \\
&= -\mathrm{trace}\left(I_m - \frac{p' \otimes p'}{|p'|^2}\right)X - \mathrm{trace}(-aI_{n-m}) \\
&= \Big(\text{Constant independent of } a\Big) + a(n-m).
\end{aligned}$$

Thus in both cases $\widetilde{F}$ satisfies $(\widetilde{F}4)$.

(ii) More generally we discuss an anisotropic curvature flow equation introduced by Angenent-Gurtin [2]. Its level set equation is given by taking

$$\widetilde{F}(p, Y) = -\mathrm{trace}\, A(\overline{p})Y$$

in (5.1), where $A(\overline{p})$ is a given matrix in $\boldsymbol{S}^n$, $A(\overline{p})(= A(p/|p|))$ is continuous on $\boldsymbol{R}^n \backslash \{0\}$. Although this $\widetilde{F}$ satisfies $(\widetilde{F}1)$–$(\widetilde{F}3)$, we do not know that $\widetilde{F}$ fulfills (5.5). For a derivation of the anisotropic curvature flow equation we refer to the nice book [6] by Gurtin. By definition we see

$$\widetilde{F}_*(0,Y) = \inf_{|q|=1}(-\operatorname{trace} A(q)Y), \quad \widetilde{F}^*(0,Y) = \sup_{|q|=1}(-\operatorname{trace} A(q)Y).$$

Since $A(\overline{p}) \in \boldsymbol{S}^n$ we can write

$$A(\overline{p}) = \begin{pmatrix} A_1(\overline{p}) & A_2(\overline{p}) \\ {}^t A_2(\overline{p}) & A_3(\overline{p}) \end{pmatrix},$$

where $A_1(\overline{p}) \in \boldsymbol{S}^m$, $A_2(\overline{p})$ is an $m \times (n-m)$ matrix and $A_3(\overline{p}) \in \boldsymbol{S}^{n-m}$. Here we assume $\operatorname{trace} A_3(\overline{p}) > 0$. We remark for $\widetilde{F}$ of (5.8). This $\widetilde{F}$ is given by $A_1(\overline{p}) = I_m - p' \otimes p'/|p|^2$, $A_2(\overline{p}) = -p'{}^t p''/|p|^2$ and $A_3(\overline{p}) = I_{n-m} - p'' \otimes p''/|p|^2$ with $p = (p', p'') \in \boldsymbol{R}^m \times \boldsymbol{R}^{n-m}$. Then $\operatorname{trace} A_3(\overline{p}) \geqq n - m - 1 > 0$.

Case 1: $p' = 0$. By the definition of $\widetilde{F}_*$ we observe

$$\begin{aligned} &\widetilde{F}_*\left(0, \begin{pmatrix} X & O \\ O & -aI_{n-m} \end{pmatrix}\right) \\ &= \inf_{|q|=1}\left(-\operatorname{trace} A(q)\begin{pmatrix} X & O \\ O & -aI_{n-m} \end{pmatrix}\right) \\ &\geqq -\inf_{|q|=1} \operatorname{trace} A_1(q)X + a \inf_{|q|=1} \operatorname{trace} A_3(q). \end{aligned}$$

Note that $\inf\{\operatorname{trace} A_1(q); |q| = 1\} < +\infty$ since $A(q)$ is continuous on S^{n-1} and that if $\operatorname{trace} A_3(q) > 0$ $(q \in S^{n-1})$ then $\inf\{\operatorname{trace} A_3(q); |q| = 1\} > 0$, where S^{n-1} denotes the unit sphere in $\boldsymbol{R}^n$.

Case 2: $p' \neq 0$. A direct calculation yields

$$\begin{aligned} \widetilde{F}\left(\begin{pmatrix} p' \\ 0 \end{pmatrix}, \begin{pmatrix} X & O \\ O & -aI_{n-m} \end{pmatrix}\right) &= -\operatorname{trace} A(\overline{p})\begin{pmatrix} X & O \\ O & -aI_{n-m} \end{pmatrix} \\ &= -\operatorname{trace} A_1(\overline{p})X + a\operatorname{trace} A_3(\overline{p}). \end{aligned}$$

In both cases $\widetilde{F}$ satisfies $(\widetilde{F}4)$.

Remark 5.7. Unfortunately, we are unable to prove the converse of Theorem 5.5. However, if the solution of the initial-boundary value problem for (5.1) with x'-axisymmetric data does exist then the solution of (5.2) and (5.3) with the same data must be the x'-axisymmetric solution of (5.1) provided that (5.1) has a comparison principle. Let u be a viscosity solution of (5.2) and (5.3) with data g. Suppose that (5.1) has a solution with the same data g. Here we denote by $\widetilde{U}(t,x',x'')$ the solution. From comparison principle for (5.1) we see $\widetilde{U}$ is x'-axisymmetric. Indeed, the rotationally invariant condition (5.5) holds for $\widetilde{F}$, x'-axisymmetrized solution of $\widetilde{U}$ should consist $\widetilde{U}$ using comparison theorem to these solutions. Therefore, we can represent $\widetilde{U}(t,x',x'') = \tilde{u}(t,x',r)$ with some x'-axisymmetric function $\tilde{u}$. Applying Theorem 5.5 we show $\tilde{u}$ is a solution of (5.2) and (5.3) with data g. Since we have the comparison theorem for (5.2) and (5.3) we obtain $u = \tilde{u}$. Thus we show $u(t,x',r) = \widetilde{U}(t,x',x'')$. We thus observe that a solution of (5.2) and (5.3) is actually an axisymmetric solution of (5.1) in this situation.

References

[1] Altschuler S.J., Angenent S.B. and Giga Y., *Mean curvature flow through singularities for surfaces of rotation.* to appear in J. Geometric Analysis.

[2] Angenent S.B. and Gurtin M.E., *Anisotropic motion of a phase interface: Well-posedness of the initial-value problem and Qualitative properties of the interface.* J. reine angew. Math. **446** (1994), 1–47.

[3] Chen Y.-G, Giga Y., Hitaka T. and Honma M., *A stable difference scheme for computing motion of level surfaces by the mean curvature.* Proc. of the third GARC symposium on pure and applied mathematics, (ed. D. Kim et al.), Korea (1994), 1–19.

[4] Crandall M.G. and Ishii H., *The maximum principle for semicontinuous functions.* Differential and Integral Equations **3** (1990), 1001–1014.

[5] Crandall M.G., Ishii H. and Lions P.-L., *User's guide to viscosity solutions of second order partial differential equations.* Bull. Amer. Math. Soc. **27** (1992), 1–67.

[6] Gurtin M.E., *Thermomechanics of Evolving Phase Boundaries in the Plane.* Oxford Univ. Press, Oxford, UK, 1993.

[7] Ishii H. and Ramaswamy M., *Uniqueness results for a class of Hamilton-Jacobi equations with singular coefficients.* preprint.

[8] Ishii H. and Souganidis P.E., *Generalized motion of noncompact hypersurfaces with velocity having arbitrary growth on the curvature tensor.* Tôhoku Math. J. **47** (1995), 227–250.

[9] Kawarada H., *On solutions of initial-boundary problem for* $u_t = u_{xx} + 1/(1-u)$. Publ. RIMS, Kyoto Univ. **10** (1975), 729–736.

[10] Ohnuma M., in preparation.

[11] Ohnuma M. and Sato K., *Singular degenerate parabolic equations with applications to the p-Laplace diffusion equation.* preprint.

[12] Siconolfi A., *A first order Hamilton-Jacobi equation with singularity and the evolution of level sets.* Comm. in P.D.E. **20** (1995), 277–307.

Masaki Ohnuma
Department of Mathematics
Faculty of Science
Hokkaido University
Sapporo 060, Japan
E-mail: m-onuma@math.hokudai.ac.jp.