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On the scattering theory for the cubic nonlinear Schrödinger and Hartree type equations in one space dimension

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Abstract. We study the scattering problem and asymptotics for large time of solutions to the Cauchy problem for the nonlinear Schrödinger and Hartree type equations with subcritical nonlinearities

$$\begin{cases} iu_t + \frac{1}{2}u_{xx} = f(|u|^2)u, & (t,x) \in \mathbf{R}^2 \\ u(0,x) = u_0(x), & x \in \mathbf{R}, \end{cases}$$

where the nonlinear interaction term is $f(|u|^2) = V * |u|^2$, $V(x) = \lambda|x|^{-\delta}$, $\lambda \in \mathbf{R}$, $0 < \delta < 1$ in the Hartree type case, or $f(|u|^2) = \lambda|t|^{1-\delta}|u|^2$ in the case of the cubic nonlinear Schrödinger equation. We suppose that the initial data $e^{\beta|x|}u_0 \in L^2$, $\beta > 0$ with sufficiently small norm $\epsilon = \|e^{\beta|x|}u_0\|_{L^2}$. Then we prove the sharp decay estimate $\|u(t)\|_{L^p} \leq C\epsilon t^{\frac{1}{p}-\frac{1}{2}}$, for all $t \geq 1$ and for every $2 \leq p \leq \infty$. Furthermore we show that for $\frac{1}{2} < \delta < 1$ there exists a unique final state $\hat{u}_+ \in L^2$ such that for all $t \geq 1$

$$\|u(t) - \exp\Big(-\frac{it^{1-\delta}}{1-\delta}f(|\hat{u}_+|^2)\Big(\frac{x}{t}\Big)\Big)U(t)u_+\|_{L^2} = O(t^{1-2\delta})$$

and uniformly with respect to x

$$u(t,x) = \frac{1}{(it)^{\frac{1}{2}}}\hat{u}_+\Big(\frac{x}{t}\Big)\exp\Big(\frac{ix^2}{2t} - \frac{it^{1-\delta}}{1-\delta}f(|\hat{u}_+|^2)\Big(\frac{x}{t}\Big)\Big) + O(t^{1/2-2\delta}),$$

where $\hat{\phi}$ denotes the Fourier transform of ϕ. Our results show that the regularity condition on the initial data which was assumed in the previous paper [9] is not needed. Also a smoothing effect for the solutions in an analytic function space is discussed.

Key words: nonlinear Schrödinger, scattering, subcritical case.

1. Introduction

We study the asymptotic behavior for large time of solutions to the Cauchy problem

$$\begin{cases} i\partial_t u = -\frac{1}{2}\partial_x^2 u + f(|u|^2)u, & (t,x) \in \mathbf{R}^2, \\ u(0,x) = u_0(x), & x \in \mathbf{R}, \end{cases} \tag{1.1}$$

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for the Hartree type equation with a long range potential, when

$$f(|u|^2) = V * |u|^2 = \int V(x-y)|u|^2(y)dy, \tag{1.2}$$

$$V(x) = \lambda|x|^{-\delta}, \quad \lambda \in \mathbf{R}, \quad 0 < \delta < 1$$

and for the cubic nonlinear Schrödinger equation with time growth condition, when

$$f(|u|^2) = \lambda|t|^{1-\delta}|u|^2, \quad 0 < \delta < 1. \tag{1.3}$$

In the previous paper [9] we proved that if the initial function u_0 has an analytic continuation on the strip $S(\beta) = \{z = x + iy; -\infty < x < \infty, -\beta < y < \beta\}$ and satisfies some exponential decay condition with respect to x, then the solution of (1.1) with (1.2) or (1.3) exists and satisfies the sharp decay estimate $\|u(t)\|_{L^p} \le C\epsilon t^{\frac{1}{p}-\frac{1}{2}}$, for all $t \ge 1$ and for every $2 \le p \le \infty$. Furthermore we showed that for $\frac{1}{2} < \delta < 1$ there exists a unique final state $u_+ \in L^2$ such that for all $t \ge 1$

$$\|u(t) - \exp\Big(-\frac{it^{1-\delta}}{1-\delta} f(|\hat{u}_+|^2)\Big(\frac{x}{t}\Big)\Big)U(t)u_+\|_{L^2} = O(t^{1-2\delta}),$$

where $\hat{\phi}$ denotes the Fourier transform of ϕ.

Our purpose in this paper is to remove the regularity condition on the data which was assumed in the previous paper [9]. We also discuss a smoothing effect for the solutions in an analytic function space. Smoothing effect for nonlinear Schrödinger equation was studied in [12] in the framework of the usual Sobolev space and in [13] in the framework of analytic function spaces.

Scattering problem for the Hartree type equations with short range potentials was studied in [5, 15] for the space dimensions greater than 2. Existence of modified scattering states and modified wave operators for the Hartree equation was shown in [8] and [3], respectively. The Hartree equation (i.e., (1.1)–(1.2) with $\delta = 1$) is considered as a critical case in the scattering problem. For the nonlinear Schrödinger equations with power nonlinearities there are many works (see, e.g., [1, 3, 4, 7, 8, 13, 14, 17, 18, 19]). In the super-critical case ($\delta > 1$) the scattering problem was studied in [4, 6, 14, 19]. For the critical case ($\delta = 1$) in [8, 17] the modified wave operators were constructed and in [3, 18] the existence of the modified scattering states was proved. However there are a few works on the scattering problem for sub-critical case ($0 < \delta < 1$). In this paper we consider two types of sub-critical cases. If $\delta \in (0,1)$ then we prove in Theorem 1.1 below the sharp time decay estimates of the solutions and large time asymptotics (1.4). For the case $\delta \in (1/2,1)$ we will construct in Theorem 1.2 the modified scattering states and write the phase function more precisely in the asymptotic formula (1.4). Finally in Corollary 1.3 we describe the smoothing effect in an analytic function space.

In what follows we consider the positive time t only since for the negative one the results are analogous. Before stating our results we give some notations and function spaces. We let $\partial_x = \partial/\partial x$ and $\mathcal{F}\phi$ or $\hat{\phi}$ be the Fourier transform of ϕ defined by $\mathcal{F}\phi(\chi) = \frac{1}{(2\pi)^{1/2}} \int e^{-ix\chi}\phi(x)dx$ and $\mathcal{F}^{-1}\phi(x)$ or $\check{\phi}(x)$ be the inverse Fourier transform of ϕ, i.e. $\mathcal{F}^{-1}\phi(x) = \frac{1}{(2\pi)^{1/2}} \int e^{ix\chi}\phi(\chi)d\chi$. We introduce some function spaces. As usual the Lebesgue space is $L^p = \{\phi \in \mathcal{S}'; \|\phi\|_p < \infty\}$, where $\|\phi\|_p = (\int |\phi(x)|^p dx)^{1/p}$ if $1 \le p < \infty$ and $\|\phi\|_\infty = \text{ess.}\sup\{|\phi(x)|; x \in \mathbf{R}\}$ if $p = \infty$. For simplicity we let $\|\phi\| = \|\phi\|_2$. Weighted Sobolev space $H^{m,s} = \{\phi \in \mathcal{S}'; \|\phi\|_{m,s} = \|(1+|x|^2)^{s/2}(1-\Delta)^{m/2}\phi\| < \infty\}$, $m, s \in \mathbf{R}$ and the homogeneous Sobolev space $\dot{H}^{m,s} = \{\phi \in \mathcal{S}'; \||x|^s(-\Delta)^{m/2}\phi\| < \infty\}$ with the seminorm $\|\phi\|_{\dot{H}^{m,s}} = \||x|^s(-\Delta)^{m/2}\phi\|$. Also we define the analytic function space $\mathcal{H}^s_\sigma = \{\phi \in L^2; \|(1+|\chi|^2)^{s/2}e^{\sigma|\chi|}\hat{\phi}(\chi)\| < \infty\}$, $s \in \mathbf{R}$ with the norm $\|\phi\|_{\mathcal{H}^s_\sigma} = \|(1+|\chi|^2)^{s/2}e^{\sigma|\chi|}\hat{\phi}(\chi)\|$, which can be expressed in x-representation in terms of the analyticity in the strip $-\sigma \le \operatorname{Im} z \le \sigma$ via the following norm $\|\phi(\cdot + i\sigma)\|_{H^{s,0}} + \|\phi(\cdot - i\sigma)\|_{H^{s,0}}$. Indeed we have the inequality $\|\phi\|_{\mathcal{H}^s_\sigma} \le \|\phi(\cdot + i\sigma)\|_{H^{s,0}} + \|\phi(\cdot - i\sigma)\|_{H^{s,0}} \le 2\|\phi\|_{\mathcal{H}^s_\sigma}$ (see, e.g., [7, 16]). We let $(\psi, \varphi) = \int \psi(x) \cdot \overline{\varphi}(x)dx$. By $C(I;E)$ we denote the space of continuous functions from an interval I to a Banach space E.

The free Schrödinger evolution group $U(t) = e^{it\Delta/2}$ gives us the solution of the Cauchy problem for the linear Schrödinger equation ((1.1) with $f = 0$). It can be represented explicitly in the following manner

$$U(t)\phi = \frac{1}{(2\pi it)^{1/2}} \int e^{i(x-y)^2/2t}\phi(y)dy = \mathcal{F}^{-1}e^{-it|\chi|^2/2}\mathcal{F}\phi.$$

Note that $U(t) = M(t)D(t)\mathcal{F}M(t)$, where $M = M(t) = \exp(\frac{ix^2}{2t})$ and $D(t)$ is the dilation operator defined by $(D(t)\psi)(x) = \frac{1}{(it)^{1/2}}\psi(\frac{x}{t})$. Then since $D(t)^{-1} = iD(\frac{1}{t})$ we have $U(-t) = \overline{M}\mathcal{F}^{-1}D(t)^{-1}\overline{M} = \overline{M}i\mathcal{F}^{-1}D(\frac{1}{t})\overline{M}$, where $\overline{M} = M(-t) = \exp(-\frac{ix^2}{2t})$.

Different positive constants might be denoted by the same letter C.

We now state our results in this paper.

Theorem 1.1 *Let $\delta \in (0,1)$. We assume that $(1+|x|^2)^{5/4}e^{2\beta|x|}u_0 \in L^2$, $\beta > 0$ and the value $\epsilon = \|(1+|x|^2)^{5/4}e^{2\beta|x|}u_0\|$ is sufficiently small. Then*

(1) *there exists a unique global solution $u \in C(\mathbf{R}; L^2)$ of the equation (1.1) with (1.2) or (1.3) satisfying*

$$\|u(t)\| = \|u_0\|, \quad \|(1+|x|^2)^{5/4}e^{\sigma(t)|x|}U(-t)u(t)\| \le C\epsilon \exp(C\epsilon t^{1-\delta}),$$

where $\sigma = \sigma(t) = \beta + \beta(1+t)^{-\gamma}$, $\gamma \in (0, \frac{\delta}{2}]$. Moreover the following decay estimate

$$\|u(t)\|_p \le C\epsilon t^{\frac{1}{p}-\frac{1}{2}}$$

is valid for all $t \ge 1$, where $2 \le p \le \infty$;

(2) *there exists a unique final state u_+ such that $e^{\beta|x|}u_+ \in L^2$ and the following asymptotics*

$$u(t,x) = \frac{1}{(it)^{\frac{1}{2}}}\hat{u}_+\Big(\frac{x}{t}\Big)\exp\Big(\frac{ix^2}{2t} - \frac{it^{1-\delta}}{1-\delta}f(|\hat{u}_+|^2)\Big(\frac{x}{t}\Big) + O(1+t^{1-2\delta})\Big) + O(t^{-\frac{1}{2}-\delta}) \qquad (1.4)$$

is true for $t \to \infty$ uniformly with respect to $x \in \mathbf{R}$.

For the values $\delta \in (\frac{1}{2}, 1)$ we obtain the existence of the modified scattering states.

Theorem 1.2 *Let $\delta \in (\frac{1}{2}, 1)$ and u be the solution of (1.1) obtained in Theorem 1.1. Then there exists a unique final state $u_+ \in L^2$ such that $e^{\beta|x|}u_+ \in L^2$ and the following asymptotics*

$$u(t,x) = \frac{1}{(it)^{\frac{1}{2}}}\hat{u}_+\Big(\frac{x}{t}\Big)\exp\Big(\frac{ix^2}{2t} - \frac{it^{1-\delta}}{1-\delta}f(|\hat{u}_+|^2)\Big(\frac{x}{t}\Big)\Big) + O(t^{-\frac{1}{2}+1-2\delta})$$

is valid for $t \to \infty$ uniformly with respect to $x \in \mathbf{R}$ and the estimate

$$\Big\|u(t) - \exp\Big(-\frac{it^{1-\delta}}{1-\delta}f(|\hat{u}_+|^2)\Big(\frac{x}{t}\Big)\Big)U(t)u_+\Big\| \le Ct^{1-2\delta}$$

is true for all $t \geq 1$.

Corollary 1.3 *Let* u *be the solution of* (1.1) *obtained in Theorem* 1.1. *Then this solution satisfies the analytical smoothing effect, i.e.* $u(t,x)$ *has an analytic continuation to the strip* $\{z = x+iy; -R < x < R, -t\sigma(t) < y < t\sigma(t)\}$ *on the complex plane and satisfies the following estimate*

$$\begin{aligned}\int_{-R}^{R} |u(t,x-it\sigma(t))|^2 dx + \int_{-R}^{R} |u(t,x+it\sigma(t))|^2 dx \\ \leq C\epsilon^2 \exp(C\epsilon t^{1-\delta} + CR\beta).\end{aligned}$$

for any $R > 0$, *where* $\sigma = \beta + \beta(1+t)^{-\gamma}$, $0 < \gamma \leq \delta/2$.

This corollary shows that all singularities of the initial data in the interval $(-R,R)$ go to infinity at once and so the solution becomes real analytic with respect to $x \in (-R,R)$.

2. Local existence to the Cauchy problem (1.1)

We introduce the function space

$$\begin{aligned}X_T = \Big\{\varphi \in C([0,T];L^2); \|\varphi\|_{X_T}^2 \equiv \sup_{0\leq t\leq T} \|U(-t)\varphi(t)\|_Y^2 \\ + 2\int_0^T |\sigma'(t)| \|\sqrt{|x|}U(-t)\varphi(t)\|_Y^2 dt < \infty\Big\},\end{aligned}$$

where $\|\varphi(t)\|_Y = \|E(t)\varphi(t)\|$, $E(t,x) = (1+|x|^2)^{5/4}e^{\sigma(t)|x|}$ and $\sigma(t) = \beta + \beta(1+t)^{-\gamma}$, $0 < \gamma \leq \delta/2$. We let $X_{T,\rho}$ be the closed ball in X_T with a center at the origin and a radius ρ. In this section we will prove the following local existence theorem.

Theorem 2.1 *Suppose that the initial data* u_0 *satisfies the condition of Theorem* 1.1. *Then there exists a time* $T > 1$ *and a unique solution* $u \in C([0,T];L^2)$ *such that* $\|u\|_{X_T} \leq 2\epsilon$.

Proof. We consider the linearized Cauchy problem (1.1)

$$\begin{cases} i\partial_t u = -\dfrac{1}{2}\partial_x^2 u + f(|v|^2)v, & (t,x) \in \mathbf{R}^2, \\ u(0,x) = u_0(x), & x \in \mathbf{R}, \end{cases} \tag{2.1}$$

where $v \in X_{T,\rho}$. Multiplying both sides of (2.1) by $E(t)U(-t)$ with $E(t,x) =$

$(1+|x|^2)^{5/4}e^{\sigma(t)|x|}$ we obtain

$$i\partial_t E(t)U(-t)u - i\sigma'(t)|x|E(t)U(-t)u = E(t)U(-t)f(|v|^2)v,$$

whence multiplying both sides by $E(t)U(-t)u(t)$ integrating with respect to x and taking the imaginary part we get

$$\frac{1}{2}\frac{d}{dt}\|U(-t)u(t)\|_Y^2 + |\sigma'(t)|\|\sqrt{|x|}U(-t)u(t)\|_Y^2 \\ \le \|U(-t)f(|v|^2)v\|_Y\|U(-t)u(t)\|_Y. \tag{2.2}$$

Since $U(t)E(t)U(-t) = M\mathcal{F}E(t,\chi t)\mathcal{F}^{-1}\bar{M}$ by Hölder's and Sobolev's inequalities we obtain

$$\begin{aligned}\|U(-t)f(|v|^2)v\|_Y &= \|(1-t^2\partial_x^2)^{5/4}f(|\bar{M}v|^2)\bar{M}v\|_{\mathcal{H}_{\sigma(t)t}} \\ &\le Ct^{-\delta}\|(1-t^2\partial_x^2)^{5/4}\bar{M}v\|^3_{\mathcal{H}_{\sigma(t)t}} \\ &= Ct^{-\delta}\|U(-t)v\|_Y^3.\end{aligned}$$

Hence we have by (2.2)

$$\|U(-t)u(t)\|_Y \le \epsilon + C\rho^3T^{1-\delta}. \tag{2.3}$$

Substituting (2.3) to the right hand side of (2.2) we get

$$\|u\|_{X_T}^2 \le \epsilon^2 + C\rho^3T^{1-\delta}(\epsilon + C\rho^3T^{1-\delta})$$

which implies

$$\|u\|_{X_T} \le 2\epsilon, \tag{2.4}$$

if we take ρ sufficiently small such that $C\rho^3T^{1-\delta}(\epsilon + C\rho^3T^{1-\delta}) \le \epsilon^2$. In the same way we prove the estimate

$$\|u_1 - u_2\|_{X_T} \le \frac{1}{2}\|v_1 - v_2\|_{X_T}, \tag{2.5}$$

where u_j, $j = 1, 2$ are the corresponding solutions of the Cauchy problems

$$\begin{cases} i\partial_t u_j = -\frac{1}{2}\partial_x^2 u_j + f(|v_j|^2)v_j, \\ u_j(0,x) = u_0(x). \end{cases}$$

We have the desired result from (2.4) and (2.5). Theorem 2.1 is proved. □

3. Proof of Theorems

We define a new function $v(t,\chi) = D(t)^{-1}e^{\frac{-ix^2}{2t}}u = \mathcal{F}e^{\frac{ix^2}{2t}}U(-t)u(t)$, $\chi = x/t$, then as in [11] we see that v satisfies

$$iv_t + \frac{1}{2t^2}v_{\chi\chi} = t^{-\delta}f(|v|^2)v. \tag{3.1}$$

Decay in time of the right hand side of the equation is not sufficient to get the existence of global solutions with the sharp decay estimates. In order to remove the nonlinear term we introduce a phase function g such that

$$\begin{cases} g_t = t^{-\delta}f(|v|^2) + \dfrac{1}{2t^2}(g_\chi)^2, & t > 1, \\ g(1) = 0. \end{cases} \tag{3.2}$$

The function g is well defined by Theorem 2.1 since

$$\begin{aligned} \|v(t)\|_{\mathcal{H}^{5/2}_{\sigma(t)}} &= \|(1+|x|^2)^{5/4}e^{\sigma(t)|x|}e^{\frac{ix^2}{2t}}U(-t)u(t)\| \\ &= \|(1+|x|^2)^{5/4}e^{\sigma(t)|x|}U(-t)u(t)\| \\ &= \|U(-t)u(t)\|_Y \le \|u\|_{X_T} \le 2\epsilon \end{aligned}$$

for all $0 \le t \le T$, where $T > 1$. We put $w = ve^{ig}$, then w is also well defined and exists locally in time. Furthermore if we multiply (3.1) by e^{ig} and use (3.2) we easily see that w satisfies the Cauchy problem

$$\begin{cases} w_t = \dfrac{1}{t^2}w_\chi g_\chi + \dfrac{i}{2t^2}w_{\chi\chi} + \dfrac{1}{2t^2}wg_{\chi\chi}, & t > 1, \\ w(1) = v(1) = \mathcal{F}e^{\frac{ix^2}{2}}U(-1)u(1). \end{cases}$$

Thus we removed the nonlinear term with the insufficient time decay but instead we now encounter the derivative loss. This is the reason why we need an analytic function space. Note that analytic function spaces were used to solve some nonlinear evolution equations with nonlinearities involving the derivatives of unknown function (see, e.g., [2, 7, 16]).

Therefore we consider the system of equations

$$\begin{cases} w_t = \dfrac{1}{t^2}w_\chi g_\chi + \dfrac{i}{2t^2}w_{\chi\chi} + \dfrac{1}{2t^2}wg_{\chi\chi}, & t > 1, \\ g_t = t^{-\delta}f(|w|^2) + \dfrac{1}{2t^2}(g_\chi)^2, & t > 1, \\ g(1) = 0, \quad w(1) = v(1) = \mathcal{F}e^{\frac{ix^2}{2}}U(-1)u(1). \end{cases} \tag{3.3}$$

In order to obtain the desired result we prove the global in time existence of solutions to (3.3) under the condition that $\|v(1)\|_{\mathcal{H}^{5/2}_{\sigma(1)}}$ is sufficiently small. And by virtue of Theorem 2.1 the norm $\|v(1)\|_{\mathcal{H}^{5/2}_{\sigma(1)}}$ is sufficiently small provided that the initial data u_0 are sufficiently small.

Theorem 3.1 *Suppose that the initial data $v(1)$ is such that the value $2\epsilon = \|v(1)\|_{\mathcal{H}^{5/2}_{\sigma(1)}}$ is sufficiently small.*
Then there exists a unique solution $w \in C([1,\infty),\mathcal{H}^1_\beta)$, $g \in C([1,\infty),L^\infty)$, $g_\chi \in C([1,\infty),\mathcal{H}^2_\beta)$ of the Cauchy problem (3.3) satisfying the following estimates

$$\|w\|_{\mathcal{H}^{5/2}_\sigma} \le 3\epsilon, \quad t^{\delta-1}(\|g\|_\infty + \|g_\chi\|_{\mathcal{H}^2_\beta}) \le 3\epsilon, \quad t^{\delta-1-\gamma}\|g_\chi\|_{\mathcal{H}^{5/2}_\sigma} \le 3\epsilon,$$

where $\sigma = \beta + \beta(1+t)^{-\gamma}$, $0 < \gamma \le \frac{\delta}{2}$.

Remark. In the case of the power nonlinearity (1.3) we can replace the a-priori estimate $t^{\delta-1}(\|g\|_\infty + \|g_x\|_{\mathcal{H}^2_\beta}) \le 3\epsilon$ by $t^{\delta-1}\|g\|_{\mathcal{H}^3_\beta}$ in the theorem.

Proof. Let us consider the linearized version of (3.1)

$$\begin{cases} w_t = \dfrac{1}{t^2}\tilde{w}_\chi \tilde{g}_\chi + \dfrac{i}{2t^2} w_{\chi\chi} + \dfrac{1}{2t^2}\tilde{w}\tilde{g}_{\chi\chi}, \\ g_t = t^{-\delta} f(|\tilde{w}|^2) + \dfrac{1}{2t^2}(\tilde{g}_\chi)^2, \\ g(1) = 0, \quad w(1) = v(1) = \mathcal{F}e^{\frac{ix^2}{2}}U(-1)u(1). \end{cases} \tag{3.4}$$

We introduce the function space

$$Z = \left\{ \begin{pmatrix} \varphi^{(1)} \\ \varphi^{(2)} \end{pmatrix}; \varphi^{(1)} \in C([1,\infty); L^2), \right.$$
$$\left. \varphi^{(2)} \in C([1,\infty); L^\infty \cap \dot{H}^{1,0}), \quad \left\| \begin{pmatrix} \varphi^{(1)} \\ \varphi^{(2)} \end{pmatrix} \right\|_Z < \infty \right\},$$

where

$$\begin{aligned} \left\| \begin{pmatrix} \varphi^{(1)} \\ \varphi^{(2)} \end{pmatrix} \right\|_Z^2 &= \sup_{1 \le t < \infty} (\|\mathcal{F}^{-1}\varphi^{(1)}(t)\|_Y^2 + t^{2(\delta-1-\gamma)}\|\mathcal{F}^{-1}\varphi^{(2)}_\chi(t)\|_Y^2 \\ &\quad + t^{2(\delta-1)}(\|\varphi^{(2)}(t)\|_\infty^2 + \|\varphi^{(2)}_\chi(t)\|^2_{\mathcal{H}^2_\beta}) \\ &\quad + 2\int_1^\infty \sigma'(t)\|\sqrt{|x|}\mathcal{F}^{-1}\varphi^{(1)}(t)\|_Y^2 dt \end{aligned}$$

$$+2\int_1^\infty \sigma'(t)\|\sqrt{|x|}\mathcal{F}^{-1}\varphi_\chi^{(2)}(t)\|_Y^2 dt.$$

We denote by Z_ρ the closed ball in Z with a center at the origin and a radius ρ. We define the mapping $\mathcal{A}$ by

$$\begin{pmatrix} w \\ g \end{pmatrix} = \mathcal{A}\begin{pmatrix} \tilde{w} \\ \tilde{g} \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} \tilde{w} \\ \tilde{g} \end{pmatrix} \in Z_\rho,$$

and w, g are the solutions of the Cauchy problem (3.4). Denote $h = t^{\delta-1-\gamma}g_\chi$, $\tilde{h} = t^{\delta-1-\gamma}\tilde{g}_\chi$ then from the system (3.4) we get

$$\begin{cases} w_t = \dfrac{1}{2t^2}(2t^{1+\gamma-\delta}\tilde{h}\tilde{w}_\chi + iw_{\chi\chi} + t^{1+\gamma-\delta}\tilde{w}\tilde{h}_\chi), \\ h_t = \dfrac{1}{t^{1+\gamma}}\partial_\chi f(|\tilde{w}|^2) + \dfrac{1}{t^{1+\delta-\gamma}}\tilde{h}\tilde{h}_\chi - \dfrac{1+\gamma-\delta}{t}\tilde{h}, \\ h(1) = 0, \quad w(1) = v(1). \end{cases} \tag{3.5}$$

Taking the Fourier transform of (3.5), multiplying the resulting system by $E^2(t,x)\overline{\check{w}(t,x)}$, $E^2(t,x)\overline{\check{h}(t,x)}$ respectively, integrating with respect to space variable and taking the real part of the result we obtain

$$\begin{cases} \dfrac{d}{dt}\|w\|^2_{\mathcal{H}^{5/2}_\sigma} - 2\sigma'(t)\|\sqrt{|x|}E\check{w}\|^2 = 2\mathrm{Re}(E\hat{w}, E\mathcal{F}^{-1}\tilde{G}_1), \\ \dfrac{d}{dt}\|h\|^2_{\mathcal{H}^{5/2}_\sigma} - 2\sigma'(t)\|\sqrt{|x|}E\check{h}\|^2 = 2\mathrm{Re}(E\check{h}, E\mathcal{F}^{-1}\tilde{G}_2) \\ \qquad\qquad - 2\dfrac{1+\gamma-\delta}{t}\|h\|^2_{\mathcal{H}^{5/2}_\sigma}, \end{cases} \tag{3.6}$$

where

$$\begin{cases} \tilde{G}_1 = \dfrac{1}{2t^{1+\delta-\gamma}}(2\tilde{h}\tilde{w}_\chi + \tilde{w}\tilde{h}_\chi) \\ \tilde{G}_2 = \dfrac{1}{t^{1+\gamma}}\partial_\chi f(|\tilde{w}|^2) + \dfrac{1}{t^{1+\delta-\gamma}}\tilde{h}\tilde{h}_\chi. \end{cases}$$

By the Schwarz inequality we get

$$\begin{aligned} &|\mathrm{Re}(E\check{w}, E\mathcal{F}^{-1}\tilde{G}_1)| \\ &\quad\le \|w\|_{\mathcal{H}^3_\sigma}\|\tilde{G}_1\|_{\mathcal{H}^2_\sigma} \\ &\quad\le Ct^{-(1+\delta-\gamma)}\|\tilde{w}\|_{\mathcal{H}^3_\sigma}(\|\tilde{h}(t,\cdot+i\sigma)\tilde{w}_\chi(t,\cdot+i\sigma)\|_{2,0} \\ &\qquad + \|\tilde{h}(t,\cdot-i\sigma)\tilde{w}_\chi(t,\cdot-i\sigma)\|_{2,0} + \|\tilde{h}_\chi(t,\cdot+i\sigma)\tilde{w}(t,\cdot+i\sigma)\|_{2,0} \\ &\qquad + \|\tilde{h}_\chi(t,\cdot-i\sigma)\tilde{w}(t,\cdot-i\sigma)\|_{2,0}) \end{aligned}$$

$$\leq Ct^{-(1+\delta-\gamma)}\|\tilde{w}\|_{\mathcal{H}^3_\sigma}(\|\tilde{h}\|_{\mathcal{H}^2_\sigma}\|\tilde{w}\|_{\mathcal{H}^3_\sigma}+\|\tilde{h}\|_{\mathcal{H}^3_\sigma}\|\tilde{w}\|_{\mathcal{H}^2_\sigma})$$
$$\leq C\rho t^{-(1+\delta-\gamma)}(\|\tilde{w}\|^2_{\mathcal{H}^3_\sigma}+\|\tilde{h}\|^2_{\mathcal{H}^3_\sigma})$$

and since $|w|^2 = w\overline{w}$ has an analytic continuation as $w(t,z)\overline{w(t,\bar{z})}$ for the complex values of the independent variable z in a strip $-\sigma < \operatorname{Im} z < \sigma$ we obtain

$$\begin{aligned}
&|\operatorname{Re}(E\check{h}, E\mathcal{F}^{-1}\tilde{G}_2)| \\
&\quad\leq \|h\|_{\mathcal{H}^3_\sigma}\|\tilde{G}_2\|_{\mathcal{H}^2_\sigma} \\
&\quad\leq C\|h\|_{\mathcal{H}^3_\sigma}(t^{-1-\gamma}(\|\tilde{w}(t,\cdot+i\sigma)\overline{\tilde{w}(t,\cdot-i\sigma)}\|_{2+\delta,0} \\
&\qquad+\|\tilde{w}(t,\cdot-i\sigma)\overline{\tilde{w}(t,\cdot+i\sigma)}\|_{2+\delta,0}) \\
&\qquad+t^{-(1+\delta-\gamma)}(\|\tilde{h}^2(t,\cdot+i\sigma)\|_{3,0}+\|\tilde{h}^2(t,\cdot-i\sigma)\|_{3,0})) \\
&\quad\leq C\|h\|_{\mathcal{H}^3_\sigma}(t^{-1-\gamma}\|\tilde{w}\|_{\mathcal{H}^3_\sigma}\|\tilde{w}\|_{\mathcal{H}^2_\sigma}+t^{-(1+\delta-\gamma)}\|\tilde{h}\|_{\mathcal{H}^3_\sigma}\|\tilde{h}\|_{\mathcal{H}^2_\sigma}) \\
&\quad\leq C\rho t^{-1-\gamma}(\|\tilde{w}\|^2_{\mathcal{H}^3_\sigma}+\|\tilde{h}\|^2_{\mathcal{H}^3_\sigma}).
\end{aligned}$$

Thus for the value $J = \|w\|^2_{\mathcal{H}^{5/2}_\sigma}+\|h\|^2_{\mathcal{H}^{5/2}_\sigma}$ from the system (3.6) we get

$$\frac{d}{dt}J^2+2|\sigma'(t)|(\|w\|^2_{\mathcal{H}^3_\sigma}+\|h\|^2_{\mathcal{H}^3_\sigma})\leq Ct^{-1-\gamma}\rho(\|\tilde{w}\|^2_{\mathcal{H}^3_\sigma}+\|\tilde{h}\|^2_{\mathcal{H}^3_\sigma}).$$

Integrating with respect to t, we get

$$\begin{aligned}
&J^2(t)+2\int_1^t|\sigma'(s)|(\|w(s)\|^2_{\mathcal{H}^3_\sigma}+\|h(s)\|^2_{\mathcal{H}^3_\sigma})ds \\
&\quad\leq J^2(1)+C\rho\int_1^t|\sigma'(s)|(\|\tilde{w}(s)\|^2_{\mathcal{H}^3_\sigma}+\|\tilde{h}(s)\|^2_{\mathcal{H}^3_\sigma})ds \\
&\quad\leq J^2(1)+C\rho^3\leq(2\epsilon)^2+\epsilon^2,
\end{aligned}\tag{3.7}$$

if we take ρ such that $C\rho^3\leq\epsilon^2$. And for the L^∞ norm of g we get

$$\begin{aligned}
\|g\|_\infty &= \left\|\int_1^t g_t dt\right\|_\infty \leq \int_1^t t^{-\delta}\|f(|\tilde{w}|^2)\|_\infty dt+\int_1^t\|(\tilde{g}_\chi)^2\|_\infty\frac{dt}{t^2} \\
&\leq Ct^{1-\delta}(\|\tilde{w}\|^2_{1,0}+\|\tilde{g}_\chi\|^2_{1,0})\leq Ct^{1-\delta}\rho^2.
\end{aligned}$$

Hence

$$t^{\delta-1}\|g\|_\infty\leq\epsilon^2,\tag{3.8}$$

if we take ρ satisfying $C\rho^2\leq\epsilon^2$. Similarly we have

$$t^{\delta-1}\|g_\chi\|_{\mathcal{H}^2_\beta}\leq\epsilon^2.\tag{3.9}$$

From (3.7)–(3.9) the estimate

$$\left\| \begin{pmatrix} w \\ g \end{pmatrix} \right\|_{Z_T} = \left\| \mathcal{A} \begin{pmatrix} \tilde{w} \\ \tilde{g} \end{pmatrix} \right\|_{Z} \leq 3\epsilon$$

follows, which shows that $\mathcal{A}$ is a mapping from $Z_{3\epsilon}$ into itself. In the same way we can show that $\mathcal{A}$ is a contraction mapping from $Z_{3\epsilon}$ into itself. Therefore there exists a unique solution of the system (3.1) in Z. From (3.1) and Sobolev's inequality we easily have the continuity in time since

$$\|h(t) - h(s)\|_{\mathcal{H}^2_\beta} \leq C\epsilon^2|t-s|, \quad \|w(t) - w(s)\|_{\mathcal{H}^1_\beta} \leq C\epsilon^2|t-s|.$$

This completes the proof of Theorem 3.1. □

We are now in a position to prove Theorems 1.1–1.2.

Proof of Theorem 1.1. From the second equation of the system (3.3) we have

$$(E\check{g})_t - 2\sigma'(t)|x|E\check{g} = E\mathcal{F}^{-1}\left(t^{-\delta}f(|w|^2) + \frac{1}{2t^2}(g_\chi)^2\right),$$

whence integrating with respect to t we get

$$\begin{aligned} &E\check{g}(t) - E\check{g}(1) \\ &\quad = \int_1^t \left(2\sigma'(s)|x|E\check{g}(s) + s^{-\delta}E\mathcal{F}^{-1}f(|w|^2) + \frac{1}{2s^2}E\mathcal{F}^{-1}(g_\chi)^2\right) ds. \end{aligned}$$

Hence by Sobolev's inequality and Theorem 3.1

$$\begin{aligned} \|\mathcal{F}E\check{g}(t)\|_\infty \leq\ & \|\mathcal{F}E\check{g}(1)\|_\infty + C\|g_\chi\|_{\mathcal{H}^1_\sigma} \\ & + \int_1^t \Big(s^{-\delta}\|\mathcal{F}E\mathcal{F}^{-1}f(|w|^2)\|_\infty \\ & \qquad + \frac{1}{2s^2}\|\mathcal{F}E\mathcal{F}^{-1}(g_\chi)^2\|_\infty\Big) ds \\ \leq\ & \epsilon + C\epsilon^2 t^{1-\delta}. \end{aligned} \tag{3.10}$$

Using the identity $\mathcal{F}MU(-t)u(t) = w(t)\exp(-ig(t))$ and (3.10) we have

$$\begin{aligned} &\|(1+|x|^2)^{5/4}e^{\sigma(t)|x|}U(-t)u(t)\| \\ &\quad = \|\mathcal{F}MU(-t)u(t)\|_{\mathcal{H}^{5/2}_\sigma} \\ &\quad = \|w(t)\exp(-ig(t))\|_{\mathcal{H}^{5/2}_\sigma} \leq C\epsilon\exp(C\epsilon t^{1-\delta}). \end{aligned} \tag{3.11}$$

From (3.10), (3.11), Theorem 2.1 and Theorem 3.1 we find that there exists a unique global solution u of (1.1) such that $\|u\|_{X_T} < \infty$

$$\|u(t)\| = \|u_0\| \quad \text{and}$$
$$\|(1+|x|^2)^{5/4} e^{\sigma(t)|x|} U(-t)u(t)\| \le C\epsilon \exp(C\epsilon t^{1-\delta}) \tag{3.12}$$

for any $t \in [0,T]$. In view of the definitions of w and g we have

$$\begin{aligned} u(t) &= M(t)D(t)w(t)\exp(-ig) \\ &= \frac{1}{\sqrt{it}} M(t) w\Big(t, \frac{x}{t}\Big) \exp\Big(-ig\Big(t, \frac{x}{t}\Big)\Big). \end{aligned}$$

Whence we easily get

$$\begin{aligned} \|u(t)\|_p &\le Ct^{-1/2}\Big\|w\Big(t, \frac{\cdot}{t}\Big)\Big\|_p \le Ct^{-1/2}\Big(\int |w(t, \frac{x}{t})|^p dx\Big)^{1/p} \\ &= Ct^{1/p-1/2}\Big(\int |w(t,y)|^p dy\Big)^{1/p} \\ &= Ct^{1/p-1/2}\|w\|_p \le C\epsilon t^{1/p-1/2} \end{aligned} \tag{3.13}$$

for all $2 \le p \le \infty$. Inequalities (3.12) and (3.13) yield the first part of Theorem 1.1.

Further we have

$$\begin{aligned} \|w(t)-w(s)\|_{\mathcal{H}^0_\beta} &\le \int_s^t \|w_\tau(\tau)\|_{\mathcal{H}^0_\beta} d\tau \\ &\le C\int_s^t (\|g_\chi w_\chi\|_{\mathcal{H}^0_\beta} + \|w_{\chi\chi}\|_{\mathcal{H}^0_\beta} + \|w g_{\chi\chi}\|_{\mathcal{H}^0_\beta})\frac{d\tau}{\tau^2} \\ &\le C\epsilon \int_s^t \frac{d\tau}{\tau^{1+\delta}} \le C\epsilon s^{-\delta} \end{aligned} \tag{3.14}$$

for all $t > s > 1$ since by Theorem 3.1 we see that $\|g\|_\infty + \|g_\chi\|_{\mathcal{H}^1_\beta} \le C\epsilon^2 t^{1-\delta}$. Therefore there exists a unique limit $W_+ \in \mathcal{H}^0_\beta$ such that $\lim_{t\to\infty} w(t) = W_+$ in $\mathcal{H}^0_\beta$ and thus we get

$$\begin{aligned} u(t,x) &= \frac{1}{\sqrt{it}} M(t) w\Big(t, \frac{x}{t}\Big) e^{-ig(t,\frac{x}{t})} \\ &= \frac{1}{\sqrt{it}} M(t) W_+\Big(\frac{x}{t}\Big) e^{-ig(t,\frac{x}{t})} + O(\epsilon t^{-\frac{1}{2}-\delta}) \end{aligned}$$

uniformly with respect to $x \in \mathbf{R}$ since for all $2 \le p \le \infty$ we have

$$\begin{aligned}
&\left\| u(t) - \frac{1}{\sqrt{it}} M(t) W_+\left(\frac{\cdot}{t}\right) e^{-ig(t,\frac{\cdot}{t})} \right\|_p \\
&\le C t^{-1/2} \left\| w\left(t, \frac{\cdot}{t}\right) - W_+\left(\frac{\cdot}{t}\right) \right\|_p \\
&\le C t^{1/p-1/2} \| w(t) - W_+ \|_p \le C t^{1/p-1/2} \| w(t) - W_+ \|_{1/2-1/p,0} \\
&\le C\epsilon t^{1/p-1/2-\delta}.
\end{aligned}$$

For the phase g we obtain

$$g(t) = \int_1^t f(|w|^2) \frac{d\tau}{\tau^\delta} + \int_1^t (g_\chi)^2 \frac{d\tau}{2\tau^2} = \int_1^t f(|w|^2) \frac{d\tau}{\tau^\delta} + O(t^{1-2\delta})$$

uniformly with respect to $x \in \mathbf{R}$. Then we write the identity

$$\begin{aligned}
\int_1^t f(|w|^2) \frac{d\tau}{\tau^\delta} &= f(|W_+|^2) \frac{(t^{1-\delta} - 1)}{1-\delta} + \Psi(t) \\
&\quad + (f(|w|^2) - f(|W_+|^2)) \frac{(t^{1-\delta} - 1)}{1-\delta},
\end{aligned}$$

where $\Psi = \int_1^t (f(|w(\tau)|^2) - f(|w(t)|^2)) \frac{d\tau}{\tau^\delta}$. Since $\| f(|w(t)|^2) - f(|w(\tau)|^2) \|_\infty \le C\epsilon \| w(t) - w(\tau) \|_{1,0} \le C\epsilon^2 \tau^{-\delta}$ we get $\int_1^t f(|w|^2) \frac{d\tau}{\tau^\delta} = \frac{t^{1-\delta}}{1-\delta} f(|W_+|^2) + O(1 + t^{1-2\delta})$. From these estimates the second result of Theorem 1.1 follows with $\hat{u}_+ = W_+$. □

Proof of Theorem 1.2. Denote

$$\Phi(t) = \int_1^t f(|w(\tau)|^2) \frac{d\tau}{\tau^\delta} - f(|w(t)|^2) \frac{t^{1-\delta} - 1}{1-\delta} + \int_1^t (g_\chi(\tau))^2 \frac{d\tau}{2\tau^2}.$$

Then we have

$$\begin{aligned}
\Phi(t) - \Phi(s) &= \int_s^t (f(|w(\tau)|^2) - f(|w(t)|^2)) \frac{d\tau}{\tau^\delta} \\
&\quad - (f(|w(t)|^2) - f(|w(s)|^2)) \frac{s^{1-\delta} - 1}{1-\delta} \\
&\quad + \int_s^t (g_\chi(\tau))^2 \frac{d\tau}{2\tau^2}, \qquad (3.15)
\end{aligned}$$

where $1 < s < t$. We apply Theorem 3.1 and (3.14) to (3.15) to get $\| \Phi(t) - \Phi(s) \|_{\mathcal{H}^0_\beta} \le C\epsilon s^{1-2\delta}$ for $1 < s < t$. This implies that there exists a unique limit $\Phi_+ = \lim_{t\to\infty} \Phi(t) \in \mathcal{H}^0_\beta$ such that

$$\|\Phi(t) - \Phi_+\|_{\mathcal{H}^0_\beta} \le C\epsilon t^{1-2\delta} \tag{3.16}$$

since we now consider the case $\frac{1}{2} < \delta < 1$.

Furthermore $\Phi(t) = g(t) - \frac{t^{1-\delta}-1}{1-\delta} f(|w(t)|^2)$ so we have by virtue of (3.14) and (3.16)

$$\|g(t) - \frac{t^{1-\delta}-1}{1-\delta} f(|W_+|^2) - \Phi_+\|_\infty \le C\epsilon t^{1-2\delta}. \tag{3.17}$$

We now put $\hat{u}_+ = W_+ \exp(-i\Phi_+ + \frac{i}{1-\delta} f(|W_+|^2)) \in \mathcal{H}^0_\beta$. Therefore we obtain the asymptotics for $t\to\infty$ uniformly with respect to $x \in \mathbf{R}$

$$u(t,x) = \frac{1}{\sqrt{it}} \hat{u}_+\Big(\frac{x}{t}\Big) \exp\Big(\frac{ix^2}{2t} - \frac{it^{1-\delta}}{1-\delta} f(|\hat{u}_+|^2)\Big(\frac{x}{t}\Big)\Big) + O(t^{1/2-2\delta}).$$

Via (3.17), and Theorem 3.1 we have

$$\begin{aligned}
&\Big\|\mathcal{F}MU(-t)u(t) - \hat{u}_+ \exp\Big(-i\frac{t^{1-\delta}}{1-\delta} f(|\hat{u}_+|^2)\Big)\Big\| \\
&= \Big\|w(t)\exp(-ig(t)) - W_+ \exp\Big(-i\frac{t^{1-\delta}-1}{1-\delta} f(|\hat{W}_+|^2) - i\Phi_+\Big)\Big\| \\
&\le \|w(t) - W_+\| + \|W_+\| \Big\|g(t) - f(|W_+|^2)\frac{t^{1-\delta}-1}{1-\delta} - \Phi_+\Big\|_\infty \\
&\le C\epsilon t^{1-2\delta},
\end{aligned}$$

whence we get

$$\begin{aligned}
&\Big\|u(t) - \exp\Big(-i\frac{t^{1-\delta}}{1-\delta} f(|\hat{u}_+|^2)(\frac{x}{t})\Big)U(t)u_+\Big\| \\
&= \Big\|u(t) - M(t)D(t)\exp\Big(-i\frac{t^{1-\delta}}{1-\delta} f(|\hat{u}_+|^2)\Big)\mathcal{F}M(t)u_+\Big\| \\
&\le \Big\|M(t)D(t)\Big(\mathcal{F}M(t)U(-t)u(t) - \hat{u}_+ \exp\Big(-i\frac{t^{1-\delta}}{1-\delta} f(|\hat{u}_+|^2)\Big)\Big)\Big\| \\
&\quad + \Big\|M(t)D(t)\exp\Big(-i\frac{t^{1-\delta}}{1-\delta} f(|\hat{u}_+|^2)\Big)\mathcal{F}(M(t)-1)u_+\Big\| \\
&\le Ct^{1-2\delta} + C\|\mathcal{F}(M(t)-1)u_+\| \le Ct^{1-2\delta} + Ct^{-1}\|x^2 u_+\| \\
&\le Ct^{1-2\delta}
\end{aligned}$$

since $\|x^2u_+\| = \|\partial_\chi^2\hat{u}_+\| = \|\partial_\chi^2(W_+e^{i\Phi_+ + \frac{i}{1-\delta}f(|W_+|^2)})\| \le C\epsilon$. This completes the proof of Theorem 1.2. □

Proof of Corollary 1.3. We have $U(t)\varphi(x)U(-t) = M(t)\varphi(it\partial_x)M(-t)$. Hence by Theorem 1.1

$$\begin{aligned}\|M(-t)u(t)\|_{\mathcal{H}^0_{t\sigma(t)}} &= \|U(t)e^{\sigma(t)|x|}U(-t)u(t)\| \\ &\le C\epsilon\exp(C\epsilon t^{1-\delta}).\end{aligned} \tag{3.18}$$

Estimate (3.18) yields

$$\begin{aligned}&\int_{-R}^{R}|u(t,x-it\sigma(t))|^2dx + \int_{-R}^{R}|u(t,x+it\sigma(t))|^2dx \\ &\le e^{CR\sigma(t)}\biggl(\int_{-R}^{R}|e^{x\sigma(t)}u(t,x-it\sigma(t))|^2dx \\ &\qquad + \int_{-R}^{R}|e^{-x\sigma(t)}u(t,x+it\sigma(t))|^2dx\biggr) \\ &\le e^{CR\sigma(t)}\biggl(\Bigl\|e^{-\frac{i(\cdot+it\sigma(t))^2}{2t}}u(t,\cdot+it\sigma(t))\Bigr\|^2 \\ &\qquad + \Bigl\|e^{-\frac{i(\cdot-it\sigma(t))^2}{2t}}u(t,\cdot-it\sigma(t))\Bigr\|^2\biggr) \\ &\le Ce^{CR\sigma(t)}\|M(-t)u(t)\|^2_{\mathcal{H}^0_{t\sigma(t)}} \\ &\le C\epsilon^2\exp(C\epsilon t^{1-\delta} + CR\beta).\end{aligned}$$

This completes the proof of the result. □

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References

[1] Cazenave T. and Weissler F.B., *The Cauchy problem for the critical nonlinear Schrödinger equation in H^s*. Nonlinear Analysis. Theory Method Appl. **14** (1990), 807–836.

[2] D'Ancona P. and Spagnolo S., *Global solvability for the degenerate Kirchhoff equation with real analytic data.* Invent. Math. **108** (1992), 253–277.

[3] Ginibre J. and Ozawa T., *Long range scattering for nonlinear Schrödinger and Hartree equations in space dimension $n \ge 2$*. Commun. Math. Phys. **151** (1993), 619–645.

[4] Ginibre J. and Velo G., *On a class of nonlinear Schrödinger equations. I.* The Cauchy problem, general case; II Scattering theory, general case. J. Funct. Anal. **32** (1979), 1–71.

[5] Ginibre J. and Velo G., *On a class of nonlinear Schrödinger equations with non-local interactions.* Math. Z. **170** (1980), 109–136.

[6] Ginibre J. and Velo G., *Scattering theory in the energy space of a class of nonlinear Schrödinger equations.* J. Math. Pures Appl. **64** (1985), 363–401.

[7] Hayashi N., *Global existence of small analytic solutions to nonlinear Schödinger equations.* Duke Math. J. **60** (1990), 717–727.

[8] Hayashi N. and Naumkin P.I., *Asymptotic behavior in time of solutions to nonlinear Schrödinger and the Hartree equations.* Amer. J. Math. (to appear).

[9] Hayashi N. and Naumkin P.I., *Scattering theory and asymptotics for large time of solutions to the Hartree type equations with long range potential.* preprint (1997).

[10] Hayashi N., Naumkin P.I. and Ozawa T., *Scattering theory for the Hartree equation.* SIAM J. Math. Anal. (to appear).

[11] Hayashi N. and Ozawa T., *Scattering theory in the weighted $L^2(\mathbf{R}^n)$ spaces for some Schrödinger equations.* Ann. IHP (Phys. Théor.) **48**, (1988), 17–37.

[12] Hayashi N. and Ozawa T., *Smoothing effect for some Schrödinger equations.* J. Funct. Anal. **85**, (1989), 307–348.

[13] Hayashi N. and Saitoh S., *Analyticity and global existence of small solutions to some nonlinear Schrödinger equations.* Commun. Math. Phys. **129**, (1990), 27–42.

[14] Hayashi N. and Tsutsumi Y., *Remarks on the scattering problem for nonlinear Schrödinger equations.* Differential Equations and Mathematical Physics, vol. 1285, Springer-Verlag, 1986, pp. 162–168, Lecture Note in Mathematics. Edited by I.W. Knowles and Y. Saito,

[15] Hayashi N. and Tsutsumi Y., *Scattering theory for Hartree type equations.* Ann. IHP (Phys. Théor) **46** (1987), 187–213.

[16] Kato T. and Masuda K., *Nonlinear evolution equations and analyticity I.* Ann. Inst. Henri Poincaré Analyse non linéaire **3** (1986), 455–467.

[17] Naumkin P.I., *Asymptotics for large time for nonlinear Schrödinger equation (in Russian).* Izvestia RAS ser. Math. (to appear).

[18] Ozawa T., *Long range scattering for nonlinear Schrödinger equations in one space dimension.* Commun. Math. Phys. **139** (1991), 479–493.

[19] Tsutsumi Y., *Scattering problem for nonlinear Schrödinger equations.* Ann. IHP (Phys. Théor.) **43** (1985), 321–347.

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