Hokkaido Mathematical Journal Vol. 27 (1998) p. 633–650

Stability of optical caustics with r-corners

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(Received August 25, 1997)

Abstract. In this papar, we investigate the stability of the optical caustic generated by a light source hypersurface with an r-*corner* in a smooth manifold under a fixed Hamiltonian system. Main results are the stability of optical caustics under the perturbation of hypersurfaces and a realization of a caustic as a stable optical caustic generated by some hypersurfaces.

Key words: lagrangian singularity, caustic, singularity.

1. Introduction

In [5] K. Jänich explained the wavefront propagation mechanism on a manifold which is completely described by a positive and positively homogeneous Hamiltonian function on the cotangent bundle and investigated the local gradient models given by the ray length function. He considered the case when the initial wave front is a smooth hypersurface without boundary. This case is corresponding to the theory of Lagrangian singularities (cf., [1]).

In this paper we consider the case when the initial wave front is a hypersurface with an r-*corner* (§1). The rays incident to conormal directions from each edges of the hypersurface gives a *regular* r-*cubic configuration* (cf., Section 3) at a point in the cotangent bundle, which is a generalized notion of Lagrangian submanifolds. The *optical caustic with an* r-*corner* generated by the hypersurface is given as the caustic of the regular r-cubic configuration. The notion of regular r-cubic configurations in complex analytic category has been introduced in [3], [4] and the real version has been developed in [8]. In [8] we have shown that any regular r-cubic configuration (at least locally) has a generating family which is a kind of families of functions. We also have shown that the stability of regular r-cubic configuration corresponds to the stable generating family.

In this paper we consider the following problems, extending of the investigations by K. Jänich [5] and G. Wassermann [10]: For a fixed Hamiltonian

1991 Mathematics Subject Classification : 58C28.

function on the cotangent bundle;

(1) Is the stability of the optical caustic with the r-corner under perturbations of the hypersurface equivalent to the stability of a generating family of the corresponding regular r-cubic configuration?

(2) For a given function germs, when does there exist a light source hypersurface germ with an r-corner which satisfy the following conditions (a) (b)?: a) A generating family of the corresponding regular r-cubic configuration is an unfolding of the given function germ, b) The optical caustic with the r-corner generated by the hypersurface is stable.

The answer of (1) is 'Yes'. This means that the classification stable optical caustics with r-corners is reduced to the classification of stable generating family under the *reticular R^+-equivalence* (cf., Section 3).

We give a partial answer to the probrem (2). The answer of (2) gives us a method to decide when the caustic defined by a function germ in the classification list can be realized as a stable optical caustic with an r-corner for a fixed Hamiltonian function.

In the investigations by K. Jänich and G. Wassermann the R-L-equivalence was used as the equivalence relation of function germs. Instead, we use the reticular R$^+$-equivalence as the equivalence relations because the R$^+$-equivalence among generating families is naturally used in the theory of Lagrangian singularities (cf., [1]) and the reticular R$^+$-equivalence relation is its extension.

In §1, we introduce the basic notations and setting of this paper. In §2, we give a brief summary of the theory of regular r-cubic configurations. In §3, we recall the deformation theory of function germs with respect to the reticular R$^+$-equivalence. The main Theorem 5.2, which solves the first probrem in Introduction, is formulated and proved in §4. In the last section we solve the second probrem under some conditions on the Hamiltonian function.

2. Preliminaries

Fix non-negative integers r and k. Let M be an $m(= r + k + 1)$-dimensional differentiable manifold and $H : T^*M\backslash 0 \to \mathbf{R}$ be a C^∞-function, called a *Hamiltonian function*. We suppose that H is everywhere positive and positively homogeneous of degree one, that is $H(\lambda\xi) = \lambda H(\xi)$ for all $\lambda > 0$ and $\xi \in T^*M\backslash 0$. Let X_H denote the corresponding Hamiltonian vector field on $T^*M\backslash 0$. Then X_H is given locally by the Hamiltonian equations:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i},$$

where (q,p) are local canonical coordinates of T^*M.

We set $E = H^{-1}(1)$ and consider the following canonical projections $\pi_M : T^*M \to M$, $\pi_E : \mathbf{R} \times E \to E$, $\pi_{\mathbf{R}} : \mathbf{R} \times E \to \mathbf{R}$. We denote by E_q the fiber of the spherical cotangent bundle $\pi|_E$ at $q \in M$.

Let $q_0 \in M$, $t_0 \geq 0$, $\xi_0 \in E_{q_0}$ and η_0 the image of the phase flow of X_H at (t_0, ξ_0). Since the phase flow of X_H preserves values of H, the local phase flow $\Psi : (\mathbf{R} \times T^*M\backslash 0, (t_0, \xi_0)) \to (T^*M\backslash 0, \eta_0)$ of X_H induces the map $\Phi : (\mathbf{R} \times E, (t_0, \xi_0)) \longrightarrow (\mathbf{R} \times E, (t_0, \eta_0))$ given by $\Phi(t,\xi) = (t, \Psi(t,\xi))$.

We set $\exp = \pi_M \circ \Phi : (\mathbf{R} \times E, (t_0, \xi_0)) \to (M, u_0)$, $\exp_{q_0} = \exp|_{\mathbf{R}\times E_{q_0}}$, $\exp^- = \pi_M \circ \Phi^{-1} : (\mathbf{R} \times E, (t_0, \eta_0)) \to (M, q_0)$, $\exp^-_{u_0} = \exp^-|_{\mathbf{R}\times E_{u_0}}$, $\phi_1 = (\pi_M, \exp) : (\mathbf{R} \times E, (t_0, \xi_0)) \to (M \times M, (q_0, u_0))$, $\phi_2 = (\exp^-, \pi_M) : (\mathbf{R} \times E, (t_0, \eta_0)) \to (M \times M, (q_0, u_0))$, where $u_0 = \pi_M(\eta_0)$. Then the following diagram is commutative:

$$\begin{array}{ccccccc}
 & (\mathbf{R} \times E, (t_0, \xi_0)) & & \xrightarrow{\Phi} & & (\mathbf{R} \times E, (t_0, \eta_0)) & \\
 & \swarrow \exp & & \phi_1 \searrow \quad \swarrow \phi_2 & & \exp^- \searrow & \\
(M, u_0) & & \xleftarrow{\pi_2} & (M \times M, (q_0, u_0)) & \xrightarrow{\pi_1} & & (M, q_0)
\end{array}$$

By [5, 2.2] we have the following proposition

Proposition 2.1 *If $\exp_{q_0}$ is regular then ϕ_1 and ϕ_2 are diffeomorphisms.*

If $\exp_{q_0}$ is regular, then we define the function germ

$$\tau = \pi_{\mathbf{R}} \circ \phi_1^{-1} = \pi_{\mathbf{R}} \circ \phi_2^{-1} : (M \times M, (q_0, u_0)) \to (\mathbf{R}, t_0).$$

We call τ the *ray length function* associated with the regular point (t_0, ξ_0) of $\exp_{q_0}$. Set $\xi = \pi_E \circ \phi_1^{-1} : (M \times M, (q_0, u_0)) \to (E, \xi_0)$, $\eta = \pi_E \circ \phi_2^{-1} : (M \times M, (q_0, u_0)) \to (E, \eta_0)$. By [5, Lemma 2] we have

$$d_q\tau(q,u) = -\xi(q,u), \; d_u\tau(q,u) = \eta(q,u) \qquad \text{for} \;\; (q,u) \in (M \times M, (q_0, u_0))$$

Let $\mathbf{H}^r = \{(x_1, \ldots, x_r) \in \mathbf{R}^r \mid x_1 \geq 0, \ldots, x_r \geq 0\}$ be an r-corner. Let V^0 be the hypersurface germ in (M, q_0) satisfying $\xi_0|_{T_{q_0}V^0} = 0$ with an r-corner defined as the image of an immersion $\iota : (\mathbf{H}^r \times \mathbf{R}^k, 0) \to (M, q_0)$.

We parameterize V^0 by ι.

From now on, we fix an $m(= r + k + 1)$-dimensional manifold M, a Hamiltonian function $H : T^*M\backslash 0 \to \mathbf{R}$, $q_0 \in M$, $\xi_0 \in E_{q_0}$ and $t_0 \geq 0$. We suppose that (t_0, ξ_0) be a regular point of the ray length function τ of $\exp_{q_0}$ and put $\eta_0 = \pi_E \circ \Phi(t_0, \xi_0)$, $u_0 = \pi_M(\eta_0)$.

3. Regular r-cubic configuration associated with a light source hypersurface with an r-corner

We now give a brief summary of the theory of regular r-cubic configurations which has been developed in [8].

Set $\Lambda_\sigma^0 = \{(q,p) \in (T^*\mathbf{R}^m, 0) \mid q_\sigma = p_{I_r-\sigma} = q_{r+1} = \cdots = q_m = 0, q_{I_r-\sigma} \geq 0\}$ for $\sigma \subset I_r = \{1, \ldots, r\}$, where (q,p) are canonical coordinates of $(T^*\mathbf{R}^m, 0)$

Definition 3.1 Let $\eta \in T^*M\backslash 0$ and Λ_σ be a lagrangian submanifold of $(T^*M\backslash 0, \eta)$ for $\sigma \subset I_r$. We call $\{\Lambda_\sigma\}_{\sigma \subset I_r}$ a *regular r-cubic configuration* if there exists a symplectomorphism germ $S : (T^*\mathbf{R}^m, 0) \to (T^*M\backslash 0, \eta)$ such that $\Lambda_\sigma = S(\Lambda_\sigma^0)$ for $\sigma \subset I_r$. The *caustic* of $\{\Lambda_\sigma\}_{\sigma \subset I_r}$ is defined by the union of the critical values of $\pi|_{\Lambda_\sigma}$ for $\sigma \subset I_r$ and $\pi(\Lambda_\sigma \cap \Lambda_\tau)$ for $\sigma \neq \tau \subset I_r$.

Equivalence relations of regular r-cubic configurations: Let $\eta_1, \eta_2 \in T^*M\backslash 0$ and $\{\Lambda_\sigma^i\}_{\sigma \subset I_r}$ be a regular r-cubic configuration in $(T^*M\backslash 0, \eta_i)$ for $i = 1, 2$. We say that $\{\Lambda_\sigma^1\}_{\sigma \subset I_r}$ and $\{\Lambda_\sigma^2\}_{\sigma \subset I_r}$ are *lagrangian equivalent* if there exists a lagrangian equivalence $\Theta : (T^*M\backslash 0, \eta_1) \to (T^*M\backslash 0, \eta_2)$ such that $\Theta(\Lambda_\sigma^1) = (\Lambda_\sigma^2)$ for $\sigma \subset I_r$.

Generating families: Let $\mathcal{E}(r; s)$ be the set of smooth function germs on $(\mathbf{H}^r \times \mathbf{R}^s, 0)$ and let $\mathrm{m}(r; s) = \{f \in \mathcal{E}(r; s) \mid f(0) = 0\}$ be its maximal ideal. Let $\eta \in T^*M\backslash 0$. We say that $F(x, y, u) \in \mathrm{m}(r; s + m)$, where $x \in \mathbf{H}^r$, $y \in \mathbf{R}^s$ and $u \in \mathbf{R}^m$, is a *generating family* of a regular r-cubic configuration $\{\Lambda_\sigma\}_{\sigma \subset I_r}$ in $(T^*M\backslash 0, \eta)$ if F is *non-degenerate*, that is

$$\operatorname{rank} \begin{pmatrix} \dfrac{\partial^2 F}{\partial x \partial y} & \dfrac{\partial^2 F}{\partial x \partial u} \\ \dfrac{\partial^2 F}{\partial y \partial y} & \dfrac{\partial^2 F}{\partial y \partial u} \end{pmatrix}_0 = r + s,$$

and $F|_{x_\sigma = 0}$ is generating family of Λ_σ under an identification of $(M, \pi(\eta))$

and $(\mathbf{R}^m, 0)$, that is

$$\Lambda_\sigma = \{d_u F(x,y,u) \in (T^*M \backslash 0, \eta) \mid \\ x_\sigma = d_{x_{I_r - \sigma}} F(x,y,u) = d_y F(x,y,u) = 0\}$$

for suitable coordinates $(u_1, \ldots, u_m)$ of $(M, \pi(\eta))$.

Equivalence relations of generating families: We denote $\mathcal{B}(r;l)$ the set of diffeomorphism germs on $(\mathbf{H}^r \times \mathbf{R}^l, 0)$ which preserve $(\mathbf{H}^r \cap \{x_\sigma = 0\}) \times \mathbf{R}^l$ for all $\sigma \subset I_r$. We say that function germs $f, g \in \mathrm{m}(r;s)$ are *reticular R-equivalent* if there exists $\phi \in \mathcal{B}(r;s)$ such that $g = f \circ \phi$. We say that function germs $F(x,y,v), G(x,y,v) \in \mathrm{m}(r; s+n)$, where $x \in \mathbf{H}^r$, $y \in \mathbf{R}^s$ and $v \in \mathbf{R}^n$, are *reticular R^+-equivalent* (as n-dimensional unfoldings) if there exist $\Phi \in \mathcal{B}(r; s+n)$ and $\alpha \in \mathrm{m}(n)$ satisfying the following:

(1) $\Phi = (\phi, \psi)$, where $\phi : (\mathbf{H}^r \times \mathbf{R}^{s+n}, 0) \to (\mathbf{H}^r \times \mathbf{R}^s, 0)$ and $\psi : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$.

(2) $G(x,y,v) = F(\phi(x,y,v), \psi(v)) + \alpha(v)$ for $(x,y,v) \in (\mathbf{H}^r \times \mathbf{R}^{s+n}, 0)$.

We say that function germs $F(x, y_1, \ldots, y_{s_1}, v) \in \mathrm{m}(r; s_1 + n)$ and $G(x, y_1, \ldots, y_{s_2}, v) \in \mathrm{m}(r; s_2 + n)$ are *stably reticular R^+-equivalent* if F and G are reticular R^+-equivalent after the addition of non-degenerate quadratic forms in the variables y.

Theorem 3.2 [8] *Two regular r-cubic configurations defined at some points in $T^*M \backslash 0$ are lagrangian equivalent if and only if their generating families are stably reticular R^+-equivalent.*

For each $\sigma \subset I_r$ we define L^0_σ by the set of conormal vectors of $V^0_\sigma := V^0 \cap \{x_\sigma = 0\}$ in (E, ξ_0) as the initial rays incident from V^0_σ. Then we regard the set Λ_σ the image of covectors in L^0_σ by $\pi_E \circ \Phi$ around time t_0, that is

$$\Lambda_\sigma = \{\pi_E \circ \Phi(t, \xi) \in (E, \eta_0) \mid (t, \xi) \in (\mathbf{R}, t_0) \times L^0_\sigma\}.$$

Then we regard Λ_σ as the set of rays incident from V^0_σ at time t_0. We also regard the union of Λ_σ for all $\sigma \subset I_r$ as the set of rays incident from the hypersurface V^0 at time t_0.

We now prove that, if $\exp_{q_0}$ is regular, then a hypersurface germ in (M, q_0) with an r-corner normally oriented by ξ_0, defines a regular r-cubic configuration in $(T^*M \backslash 0, \eta_0)$.

Proposition 3.3 *Let V^0 be the hypersurface germ with an r-corner in*

(M,q_0) satisfying $\xi_0|_{T_{q_0}V^0} = 0$ defined as the image of an immersion ι : $(\mathbf{H}^r \times \mathbf{R}^k, 0) \to (M, q_0)$. Let Λ_σ be the set of rays incident from $V^0_\sigma := V^0 \cap \{x_\sigma = 0\}$ at time t_0 for $\sigma \subset I_r$. Then $F := \tau \circ (\iota \times id_u) - t_0 \in \mathrm{m}(r; k+m)$ is a generating family of the regular r-cubic configuration $\{\Lambda_\sigma\}_{\sigma \subset I_r}$ in $(T^*M \backslash 0, \eta_0)$.

In this case we call $\{\Lambda_\sigma\}_{\sigma\subset I_r}$ the *regular r-cubic configuration associated with V^0 at time t_0* and we call the caustic of $\{\Lambda_\sigma\}_{\sigma\subset I_r}$ the *optical caustic with the r-corner associated with V^0 at time t_0*.

Proof. By [5, p. 171 Sublemma] we have

$$\begin{pmatrix} d_u d_x F \\ d_u F \end{pmatrix} : T_{u_0}M \to T^*_{q_0}V^0 \oplus \mathbf{R}$$

is an isomorphism. This means that

$$\mathrm{rank}\begin{pmatrix} \dfrac{\partial^2 F}{\partial x \partial u} \\ \dfrac{\partial^2 F}{\partial y \partial u} \end{pmatrix}_0 = r + k.$$

Hence F is non-degenerate.

Let $\sigma \subset I_r$ and $\eta_u \in (E, \eta_0)$. Then $\eta_u \in \Lambda_\sigma$ if and only if $\eta_u = \pi_E \circ \Phi(t, \xi_q)$ for some $\xi_q \in E_q$ and $t \in (\mathbf{R}, t_0)$ satisfying $q \in V^0_\sigma$ and $\xi_q|_{T_q V^0_\sigma} = 0$, if and only if $\eta_u = d_u\tau(q,u)$ for some $q \in V^0_\sigma$ and $u \in (M, u_0)$ satisfying $d_q\tau(q,u)\,|_{T_qV^0_\sigma} = 0$, and this holds if and only if $\eta_u = d_uF(x,y,u)$ for some $(x,y,u) \in (\mathbf{H}^r \times \mathbf{R}^{k+m}, 0)$ satisfying $x_\sigma = 0$ and $d_{x_{I_r - \sigma}}F(x,y,u) = d_yF(x,y,u) = 0$. Hence $F|_{x_\sigma = 0}$ is a generating family of Λ_σ. □

4. Stability of unfoldings

In order to investigate stabilities of an optical caustic with an r-corner, we require some results of the singularity theory of function germs with respect to *reticular R^+-equivalence.* Basic techniques we used in this paper depend heavily on the results in this section, however the all arguments are almost parallel along the ordinary theory of the right-equivalence (cf., [2], [9]), so that we omit the detail.

We denote by $J^l(r+k, 1)$ the set of l-jets at 0 of germs in $\mathrm{m}(r;k)$ and by $\pi_l : \mathrm{m}(r;k) \to J^l(r+k,1)$ the natural projection. We denote by $j^l f(0)$

the l-jet of $f \in \mathrm{m}(r;k)$.

Lemma 4.1 *Let $f \in \mathrm{m}(r;k)$ and $O^l_{rR}(j^l f(0))$ be the submanifold of $J^l(r+k,1)$ consist of the image by π_l of the orbit of reticular R-equivalence of f. Put $z = j^l f(0)$. Then*

$$T_z(O^l_{rR}(z)) = \pi_l\Bigg(\left\langle x_1\frac{\partial f}{\partial x_1},\dots,x_r\frac{\partial f}{\partial x_r}\right\rangle_{\mathcal{E}(r;k)} + \mathrm{m}(r;k)\left\langle\frac{\partial f}{\partial y_1},\dots,\frac{\partial f}{\partial y_k}\right\rangle\Bigg).$$

We say that a function germ $f \in \mathrm{m}(r;k)$ is *reticular R-l-determined* if all function germ which has same l-jet of f is reticular R-equivalent to f.

Lemma 4.2 *If $f \in \mathrm{m}(r;k)$ and if*

$$\mathrm{m}(r;k)^{l+1} \subset \mathrm{m}(r;k)\Bigg(\left\langle x_1\frac{\partial f}{\partial x_1},\dots,x_r\frac{\partial f}{\partial x_r}\right\rangle + \mathrm{m}(r;k)\left\langle\frac{\partial f}{\partial y_1},\dots,\frac{\partial f}{\partial y_k}\right\rangle\Bigg) + \mathrm{m}(r;k)^{l+2},$$

then f is reticular R-l-determined. Conversely if $f \in \mathrm{m}(r;k)$ is reticular R-l-determined, then

$$\mathrm{m}(r;k)^{l+1} \subset \left\langle x_1\frac{\partial f}{\partial x_1},\dots,x_r\frac{\partial f}{\partial x_r}\right\rangle_{\mathcal{E}(r;k)} + \mathrm{m}(r;k)\left\langle\frac{\partial f}{\partial y_1},\dots,\frac{\partial f}{\partial y_k}\right\rangle.$$

For each $f(x,y) \in \mathrm{m}(r;k)^2$ we define the *corank* of f by the corank of the matrix $(\frac{\partial^2 f}{\partial y^2}(0))$.

Lemma 4.3 (Splitting lemma) *Let $f \in \mathrm{m}(r;k)^2$ and l be the corank of f. Then there exist a function germ $f_0 \in \mathrm{m}(r;l)^2$ and a non-degenerate quadratic form $Q(y_{l+1},\dots,y_k)$ such that $f_0|_{x=0} \in m(0;l)^3$ and f is reticular R-equivalent to $f_0(x_1,\dots,x_r,y_1,\dots,y_l) + Q(y_{l+1},\dots,y_k)$.*

Let $F \in \mathrm{m}(r;k+n_1)$, $G \in \mathrm{m}(r;k+n_2)$ be unfoldings of $f \in \mathrm{m}(r;k)$. We say that F *is reticular R^+-f-induced from* G if there exist smooth map germs $\phi : (\mathbf{H}^r \times \mathbf{R}^{k+n_2},0) \to (\mathbf{H}^r \times \mathbf{R}^k,0)$, $\psi : (\mathbf{R}^{n_2},0) \to (\mathbf{R}^{n_1},0)$ and $\alpha \in \mathrm{m}(0;n_2)$ satisfying the following conditions:

(1) $\phi((\mathbf{H}^r \cap \{x_\sigma = 0\}) \times \mathbf{R}^{k+n_2}) \subset (\mathbf{H}^r \cap \{x_\sigma = 0\}) \times \mathbf{R}^k$ for $\sigma \subset I_r$.

(2) $G(x,y,v) = F(\phi(x,y,v),\psi(v)) + \alpha(v)$ for $x \in \mathbf{H}^r$, $y \in \mathbf{R}^k$ and

$v \in \mathbf{R}^{n_2}$.

Definition 4.4 Here we define several notions of stabilities of unfoldings. Let $f \in \mathrm{m}(r;k)$ and $F \in \mathrm{m}(r;k+n)$ be an unfolding of f.

We define a smooth map germ

$$j_1^l F : (\mathbf{R}^{r+k+n}, 0) \longrightarrow (J^l(r+k,1), j^l f(0))$$

as follows: Let $\tilde{F} : U \to \mathbf{R}$ be a representative of F. For each $(x,y,u) \in U$, We define $F_{(x,y,u)} \in \mathrm{m}(r;k)$ by $F_{(x,y,u)}(x',y') = F(x+x', y+y', u) - F(x,y,u)$. Now define $j_1^l F(x,y,u)$ as the l-jet of $F_{(x,y,u)}$. $j_1^l F$ depends only on the germ at 0 of F. We say that F is *reticular R^+-l-transversal* if $j_1^l F|_{x=0}$ is transversal to $O^l_{rR}(j^l f(0))$. It is easy to check that F is reticular R^+-l-transversal if and only if

$$\mathcal{E}(r;k) = \left\langle x_1 \frac{\partial f}{\partial x_1}, \ldots, x_r \frac{\partial f}{\partial x_r}, \frac{\partial f}{\partial y_1}, \ldots, \frac{\partial f}{\partial y_k} \right\rangle_{\mathcal{E}(r;k)} + V_F + \mathrm{m}(r;k)^{l+1},$$

where $V_F = L_{\mathbf{R}}\langle 1, \frac{\partial F}{\partial u_1}|_{u=0}, \ldots, \frac{\partial F}{\partial u_n}|_{u=0} \rangle$.

We say that F is *reticular R^+-stable* if the following condition holds: For any neighborhood U of 0 in $\mathbf{R}^{r+k+n}$ and any representative $\tilde{F} \in C^\infty(U, \mathbf{R})$ of F, there exists a neighborhood $N_{\tilde{F}}$ of $\tilde{F}$ such that for any element $\tilde{G} \in N_{\tilde{F}}$ the germ $\tilde{G}|_{\mathbf{H}^r \times \mathbf{R}^{k+n}}$ at $(0, y_0, u_0')$ is reticular R^+-equivalent to F for some $(0, y_0, u_0') \in U$.

We say that F is *reticular R^+-versal* if F is reticular R^+-f-induced from all unfolding of f.

We say that F is *reticular R^+-infinitesimal versal* if

$$\mathcal{E}(r;k) = \left\langle x_1 \frac{\partial f}{\partial x_1}, \ldots, x_r \frac{\partial f}{\partial x_r}, \frac{\partial f}{\partial y_1}, \ldots, \frac{\partial f}{\partial y_k} \right\rangle_{\mathcal{E}(r;k)} + V_F.$$

Theorem 4.5 *Let $F \in \mathrm{m}(r;k+n)$ be an unfolding of $f \in \mathrm{m}(r;k)$. Then the following are equivalent.*

(1) *F is reticular R^+-stable.*

(2) *F is reticular R^+-versal.*

(3) *F is reticular R^+-infinitesimal versal.*

For $f \in \mathrm{m}(r;k)$, we define the *reticular R-codimension of f* by the

R-dimension of the vector space

$$\mathcal{E}(r;k)/\left\langle x_1\frac{\partial f}{\partial x_1},\ldots,x_r\frac{\partial f}{\partial x_r},\frac{\partial f}{\partial y_1},\ldots,\frac{\partial f}{\partial y_k}\right\rangle_{\mathcal{E}(r;k)}.$$

By the above theorem, if 1 and $a_1,\ldots,a_n \in \mathrm{m}(r;k)$ is a representative of a basis of the vector space, then $f+a_1v_1+\cdots a_nv_n \in \mathrm{m}(r;k+n)$ is a reticular R^+-stable unfolding of f.

In [8], we have given the classification of unimodular function germs under the reticular R-equivalence. This classification includes the classification of function germs whose reticular R-codimensions are lower than 8.

5. Stability of optical caustics with r-corners

In this section we shall investigate the stability of an optical caustic with an r-corner under perturbations of a light source surface with respect to a fixed Hamiltonian function.

Definition 5.1 Let V^0 be the hypersurface germ in (M,q_0) satisfying $\xi_0|_{T_{q_0}V^0}=0$ defined by an immersion $\iota:(\mathbf{H}^r\times\mathbf{R}^k,0)\to(M,q_0)$. We say that V^0 produces *a stable optical caustic with an r-corner at time t_0* if the following condition holds:

For any open neighborhood X of q_0 in M, U of u_0 in M, W of 0 in $\mathbf{R}^{r+k}$, any representative $\tilde{\tau}:X\times U\to\mathbf{R}$ of τ and any representative immersion $\tilde{\iota}:W\to X$ of ι, there exists an open neighborhood $N_{\tilde{\iota}}$ of $\tilde{\iota}$ in the space of immersions from W to X with C^∞-topology such that for every $\tilde{\kappa}\in N_{\tilde{\iota}}$ the regular r-cubic configuration associated the light source surface defined by $\tilde{\kappa}|_{\mathbf{H}^r\times\mathbf{R}^k}$ at $(0,y_0)$ is lagrangian equivalent to one associated with V^0 for some $(0,y_0)\in W$.

We remark that, by Theorem 3.2, the condition defined by changing the part 'the regular r-cubic $\cdots$ for some $(0,y_0)\in W$' in Definition 5.1 to '$(\tilde{\tau}\circ(\tilde{\kappa}\times id_u)-t_0)|_{\mathbf{H}^r\times\mathbf{R}^{k+m}}$ at $(0,y_0,u_0')$ is reticular R^+-equivalent to $\tau\circ(\iota\times id_u)-t_0$ for some $(0,y_0,u_0')\in W\times U$' is equivalent to the original.

Let V be an open set in $\mathbf{R}^{r+k+m}$ with the coordinates $(x_1,\ldots,x_r,$ $y_1,\ldots,y_k,u_1,\ldots,u_m)$. We define the map

$$j_1^l:C^\infty(V,\mathbf{R})\to C^\infty(V,J^l(r+k,1))$$

by setting $j_F^l(x,y,u)$ as the l-jet at 0 of the map $(x',y')\mapsto(F(x+x',y+$

$y', u) - F(x, y, u))$ for $F \in C^\infty(V, \mathbf{R})$.

Now we give the affirmative answer to the probrem (1).

Theorem 5.2 *Let M be an $m(= r + k + 1)$-dimensional differentiable manifold, $H : T^*M\backslash 0 \to \mathbf{R}$ a positive and positively homogeneous Hamilton function, $q_0 \in M$, $\xi_0 \in E_{q_0}$, $t_0 \geq 0$ and τ the ray length function associated with the regular point (t_0, ξ_0) of $\exp_{q_0}$. Let V^0 be the hypersurface germ in (M, q_0) satisfying $\xi_0|_{T_{q_0}V^0} = 0$ defined by an immersion $\iota : (\mathbf{H}^r \times \mathbf{R}^k, 0) \to (M, q_0)$. Then V^0 produces a stable optical caustic with an r-corner at time t_0 if and only if $F := \tau \circ (\iota \times id_u) - t_0$ is a reticular R^+-versal unfolding of $F|_{u=u_0}$.*

By the above remark, this theorem asserts that the stability of F with respect to perturbations of ι is sufficient to one of F as an m-dimensional unfolding. However generally these stabilities are not equivalent. Since the stability as an unfolding means the stability with respect to both of perturbations of the corresponding light source surface and the Hamiltonian function.

Proof. ($\Leftarrow$) Let $\tilde{\iota} : W \to X$ be a representative immersion of ι and $\tilde{\tau} : X \times U \to \mathbf{R}$ be a representative of τ. By shrinking X and U if necessary, we may assume that $\tilde{\tau}|_{X \times u}$ is submersion for every $u \in U$. We denote $\mathrm{Imm}(W, X)$ the set of immersions from W to X and define the continuous map

$$\begin{array}{rccc} \Phi : & \mathrm{Imm}(W, X) & \longrightarrow & C^\infty(W \times U, \mathbf{R}) \\ & \tilde{\kappa} & \longmapsto & \tilde{\tau} \circ (\tilde{\kappa} \times id_u) - t_0 \, . \end{array}$$

Set $\tilde{F} = \Phi(\tilde{\iota})$. Since F is a reticular R^+-stable unfolding of f, there exists a neighborhood $N_{\tilde{F}}$ of $\tilde{F}$ such that, for every function $\tilde{G} \in N_{\tilde{F}}$, the germ $\tilde{G}|_{\mathbf{H}^r \times \mathbf{R}^{k+m}}$ at $(0, y_0, u_0')$ and F are reticular R^+-equivalent for some $(0, y_0, u_0') \in W \times U$. Then $\Phi^{-1}(N_{\tilde{F}})$ is a neighborhood of $\tilde{\iota}$ for which the condition in Definition 5.1 holds.

($\Rightarrow$) We suppose Lemma 5.3. Let $\tilde{\iota'} : W' \to X$ be a representative immersion of ι and $\tilde{\tau} : X \times U \to \mathbf{R}$ be a representative of τ. Choose a relative compact neighborhood W of 0 in $\mathbf{R}^{r+k}$ such that $\overline{W} \subset W'$ and choose a neighborhood $N_{\tilde{\iota}}$ of $\tilde{\iota} := \tilde{\iota'}|_W$ for which the condition in Definition 5.1 holds. We define

$$B_l = \{\tilde{\kappa} \in C^\infty(W', X) \mid j_1^l(\tilde{\tau} \circ (\tilde{\kappa} \times id_u) - t_0)|_{x=0} \text{ is transversal to } O^l_{rR}(j^l f(0))\}$$

for each $l \in \mathbf{N}$, Then B_l is a residual set in $C^\infty(W', X)$ by Lemma 5.3. Since $C^\infty(W', X)$ is a Baire space, $B := \bigcap_{l \in \mathbf{N}} B_l$ is dense.

Set the open set $O = \{\tilde{\kappa} \in C^\infty(W', X) \mid \tilde{\kappa}|_{\overline{W}}$ is an immersion$\}$. Then the map $O \to \mathrm{Imm}(W, X)$ given by $\tilde{\kappa} \mapsto \tilde{\kappa}|_W$ is continuous. Therefore the inverse image $N_{\tilde{\iota}'}$ of $N_{\tilde{\iota}}$ by the above map is open neighborhood of $\tilde{\iota}'$.

Fix $\tilde{\kappa} \in N_{\tilde{\iota}'} \cap B$ sufficiently close to $\tilde{\iota}'$ such that $(\tilde{\tau} \circ (\tilde{\kappa} \times id_u) - t_0)|_{\mathbf{H}^r \times \mathbf{R}^{r+k}}$ at $(0, y_0, u_0')$ and F are reticular R^+-equivalent at $(0, y_0, u_0') \in W \times U$. Define $G \in \mathrm{m}(r; k+m)$ by $G(x, y, u) := \tilde{\tau}(\tilde{\kappa}(x, y+y_0), u+u_0') - t_0$. Then G is reticular R^+-l-transversal unfolding of $g := G|_{u=0}$ for all $l \in \mathbf{N}$. Hence G is a reticular R^+-versal unfolding of g. Therefore F is also a reticular R^+-versal unfolding of f. □

The following completes the proof the Theorem 5.2.

Lemma 5.3 *Let W, X and U be neighborhoods of 0 in $\mathbf{R}^{r+k}$, $\mathbf{R}^m$ and $\mathbf{R}^n$ respectively and we denote their coordinates $(x_1, \ldots, x_r, y_1, \ldots, y_k)$, $(q_1, \ldots, q_m)$ and $(u_1, \ldots, u_n)$ respectively. Let $H : X \times U \to \mathbf{R}$ be a smooth map such that $H|_{X \times u}$ is a submersion for all $u \in U$ and A be a submanifold of $J^l(r+k, 1)$. Then the set*

$$B = \{f \in C^\infty(W, X) \mid j_1^l H \circ (f \times id_u)|_{x=0} \text{ is transversal to } A\}$$

is residual.

Proof. Let $V = W \cap \{x = 0\}$. Then the map

$$\gamma : C^\infty(W, X) \to C^\infty(V \times U, J^l(r+k, 1)) \quad (f \mapsto j_1^l(H \circ (g \times id_u))|_{x=0})$$

is continuous. If $K \subset A$ is a compact subset, then $C = \{F \in C^\infty(V \times U, J^l(r+k, 1)) | F$ is transversal to A on $K\}$ is open. Therefore $B = \gamma^{-1}(C)$ is open.

Choose relatively compact open covering $\{W_i\}_{i \in \mathbf{N}}$ and $\{W_i'\}_{i \in \mathbf{N}}$ of W such that $\overline{W_i} \subset W_i'$ for $i \in \mathbf{N}$. For each $i \in \mathbf{N}$ set

$$B_i = \{f \in C^\infty(W, X) \mid j_1^l H \circ (f \times id_u)|_{x=0} \text{ is transversal to } A \text{ on } \overline{W}_i \cap \{x = 0\}\}.$$

Since $B = \bigcap_{i \in \mathbf{N}} B_i$ and each B_i is open by an analogous proof of the above, it is enough to prove that every B_i is dense in order to complete the proof.

The proof is analogous to that of ordinary transversal lemma. Fix $i \in \mathbf{N}$ and $f \in C^\infty(W, X)$. Let P be the set of all n-tuples of polynomial maps of degree $\leq l$ on x, y. Choose a smooth function $\rho : W \to [0,1]$ such that $\rho = 1$ on $\overline{W_i}$ and $\rho = 0$ on $W - W_i'$. Put $P' = \{\alpha \in P \mid (f + \rho \cdot \alpha)(W) \subset X\}$. Since $P' = \{\alpha \in P \mid (f + \rho \cdot \alpha)(W_i') \subset X\}$ and $\overline{W}_i$ is compact, P' is a neighborhood of 0. We define the following maps for $\alpha \in P'$:

$$\iota_\alpha : V \times U \longrightarrow W' \times U \times P' \ ((y, u) \mapsto (y, u, \alpha))$$

$$\begin{aligned} \mu : V \times U \times P' &\longrightarrow J^l(r + k, 1) \\ &((y, u, \alpha) \mapsto j_1^l(H \circ ((f + \rho \cdot \alpha) \times id_u))(0, y, u))). \end{aligned}$$

Let $\alpha \in P'$. Then $(f + \rho \cdot \alpha) \in B_i$ if and only if $j_1^l(H \circ ((f + \rho \cdot \alpha) \times id_u))|_{x=0}$ is transversal to A on $\overline{W_i} \cap \{x = 0\}$, and this holds if and only if $\mu \circ \iota_\alpha$ is transversal to A on $\overline{W_i} \cap \{x = 0\}$, Since $\rho = 1$ on W_i and $H|_{X \times u}$ is a submersion, μ is submersion and hence this holds if ι_α is transversal to $A' := \mu^{-1}(A)$. Hence ι_α is transversal to A at $(0, y, u, \alpha) \in V \times U \times P'$ if and only if $(0, y, u, \alpha) \notin A'$ or the projection $\pi : A' \to P'$ is regular at $(0, y, u, \alpha)$.

Since the set of critical values of π has measure 0 in P' by the Sard-Brown theorem, there exists α arbitrarily near 0 such that $j_1^l(H \circ ((f + \rho \cdot \alpha) \times id_u))|_{x=0}$ is transversal to A on $\overline{W}_i \cap \{x = 0\}$. This means that there exists $g \in C^\infty(W, X)$ arbitrarily close f such that $j_1^l(H \circ (g \times id_u))|_{x=0}$ is transversal to A on $\overline{W}_i \cap \{x = 0\}$. Hence B_i is dense. □

6. Versality of optical caustics with r-corners

In this section we shall investigate our second problem. Recall that $\tau : (M \times M, (q_0, u_0)) \to (\mathbf{R}, t_0)$ denotes the ray length function. We say that a function germ $f \in \mathrm{m}(r; k)^2$ occur as an *organizer* of a reticular versal caustic at (t_0, ξ_0) if there exists the hypersurface germ V^f in (M, q_0) satisfying $\xi_0|_{T_{q_0}V^0} = 0$ defined by an immersion $\iota_f : (\mathbf{H}^r \times \mathbf{R}^k, 0) \to (M, q_0)$ such that $\tau \circ (\iota_f \times id_u) - t_0$ is a reticular R^+-versal unfolding of f.

Lemma 6.1 *Let a function germ $f \in \mathrm{m}(r; k)^2$ occur as an organizer of a reticular versal caustic at (t_0, ξ_0). If a function germ $g \in \mathrm{m}(r; k)^2$ is reticular R-equivalent to f, then g also does occur as an organizer of a* *reticular versal caustic at* (t_0, ξ_0).

Proof. By the hypothesis, there exists a hypersurface germ V^f and an immersion ι_f to which above condition holds. Since f is reticular R-equivalent to g, there exists $\phi \in \mathcal{B}(r,k)$ such that $g = f \circ \phi$. Consider the coordinate change $(x,y) \mapsto \phi^{-1}(x,y)$ on V^f. Let V^g be the hypersurface germ of (M, q_0) parameterized by ι_g:

$$\begin{array}{ccccc} & \iota_g & V^g & g & \\ & \swarrow & & \searrow & \\ M & & \downarrow\phi & & \mathbf{R} \\ & \nwarrow & & \nearrow & \\ & \iota_f & V^f & f & \end{array} .$$

By the above diagram we have

$$\begin{aligned} G(x,y,u) &:= \tau(\iota_g(x,y),u) - t_0 = \tau(\iota_f(\phi(x,y)),u) - t_0 \\ &= F(\phi(x,y),u). \end{aligned}$$

Since F is reticular R^+-versal unfolding of f, G is reticular R^+-versal unfolding of $G|_{u=0} = f \circ \phi = g$. □

Definition 6.2 [5, 3.2] Let $u \in M$ and $\eta \in E_u$. Then we say that the Hamiltonian function H *has rank* s *at* u *in direction* η if the following condition holds: Let L_η be the line in T_u^*M spanned by η. If we introduce affine coordinates $v_1, \ldots, v_m$ in T_u^*M such that $T_\eta E_u$ is given by $v_m = 1$, the v_m-axis is L_η, and if we represent E locally at η as $v_m = 1 + h(v_1, \ldots, v_{m-1})$, then the Hessian of h at η has rank s.

Theorem 6.3 *Let M be an $m(= r + k + 1)$-dimensional differentiable manifold, $H : T^*M \backslash 0 \to \mathbf{R}$ a positive and positively homogeneous Hamilton function, $q_0 \in M$, $\xi_0 \in E_{q_0}$ and $t_0 \geq 0$. Assume that (t_0, ξ_0) is a regular point of $\exp_{q_0}$, put $u_0 = \exp_{q_0}(t_0, \xi_0)$ and suppose $\eta_0 \in E_{u_0}$ be the image of ξ_0 under the local flow of H at time t_0. Then each of following conditions (1), (2) is sufficient for $f \in \mathrm{m}(r;k)^2$ to occur as an organizer of a reticular versal caustic at (t_0, ξ_0) :*

(1) *The reticular R-codimension $f \leq m$.*

(2) *The reticular R-codimension $f = m + 1$, corank $f \geq 1$ and the rank s of H at u_0 in direction $\eta_0 \geq 1$.*

Proof. Choose coordinates $(u_1, \ldots, u_m)$ of M at u_0 such that, with respect to the corresponding fiber coordinates $(v_1, \ldots, v_m)$ in $T^*_{u_0}M$, H satisfies the conditions in Definition 6.2. By a linear coordinate change of $(u_1, \ldots, u_{m-1})$, we may assume that h has the form $h(v_1, \ldots, v_m) = \sum_{i=1}^{r} \varepsilon_i v_i^2 + \sum_{j=1}^{k} \delta_j v_{r+j}^2 + a$, where $\varepsilon_i, \delta_j = 0$ or ± 1, $a \in \mathrm{m}(r;k)^3$ and in the case (2) $\delta_1 \neq 0$.

Let $f \in \mathrm{m}(r;k)^2$ satisfy the condition (1) or (2). By Splitting Lemma 4.3, there exists a function germ $f_0 \in \mathrm{m}(r;l)^2$ such that f is reticular R-equivalent to $f_0(x_1, \ldots, x_r, y_1, \ldots, y_l) \pm y_{l+1}^2 \pm \cdots \pm y_k^2$ and $f_0|_{x=0} \in \mathrm{m}(0;l)^3$. We may assume that $f = f_0 \pm y_{l+1}^2 \pm \cdots \pm y_k^2$ by Lemma 6.1. Then we have $\mathcal{E}(r;k)/\langle x\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle = \mathcal{E}(r;l)/\langle x\frac{\partial f_0}{\partial x}, \frac{\partial f_0}{\partial y}\rangle$.

First, we prove that $x_1, \ldots, x_r, y_1, \ldots, y_l$ are linearly independent in $\mathcal{E}(r;l)/\langle x\frac{\partial f_0}{\partial x}, \frac{\partial f_0}{\partial y}\rangle$ over $\mathbf{R}$. Let $\alpha_1 x_1 + \cdots + \alpha_r x_r + \beta_1 y_1 + \cdots + \beta_l y_l = 0 \in \mathcal{E}(r;l)/\langle x\frac{\partial f_0}{\partial x}, \frac{\partial f_0}{\partial y}\rangle$ for $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_l \in \mathbf{R}$. Since $\langle \beta, y\rangle = 0$ in $\mathcal{E}(0;l)/\langle \frac{\partial f_0}{\partial y}|_{x=0}\rangle$ and $f_0|_{x=0} \in m^3(0;l)$, we have $\beta = 0$. Suppose that $\alpha_1 \neq 0$. Then there exist $\gamma_0, \ldots, \gamma_l \in \mathrm{m}(1;l)$ such that

$$x_1 + \gamma_0(x_1,y)x_1\frac{\partial f_0}{\partial x_1}(x_1,0,y) + \gamma_1(x_1,y)\frac{\partial f_0}{\partial y_1}(x_1,0,y)$$
$$+\gamma_l(x_1,y)\frac{\partial f_0}{\partial y_l}(x_1,0,y) = 0.$$

Therefore we have $\gamma_i(0) \neq 0$ for some $i \geq 1$, for $x_1\frac{\partial f_0}{\partial x_1}(x_1,0,y) \in \mathrm{m}(1;l)^2$. We may assume that $i = 1$. Then this means that $\frac{\partial f_0}{\partial y_1}|_{x=0} \in \langle \frac{\partial f_0}{\partial y_2}|_{x=0}, \ldots, \frac{\partial f_0}{\partial y_l}|_{x=0}\rangle$ and contradicts that $\dim_{\mathbf{R}} \mathrm{m}(0;l)/\langle \frac{\partial f_0}{\partial y}|_{x=0}\rangle < \infty$.

Consider the case (2). The vector space

$$\mathrm{m}(r;l)/\left(\left\langle x\frac{\partial f_0}{\partial x}, \frac{\partial f_0}{\partial y}\right\rangle + L_{\mathbf{R}}\langle x,y\rangle + \mathrm{m}(r;l)^3\right)$$

must have a positive dimension because if not we have reticular R-codimension $f \leq m$. Therefore we may assume by Lemma 6.1 and Lemma 6.4 below that

$$b_{k+1} := \sum_{i=1}^{r} \varepsilon_i x_i^2 + \sum_{j=1}^{l} \delta_j y_j^2 \neq 0 \quad \text{in}$$
$$\mathrm{m}(r;l)/\left(\left\langle x\frac{\partial f_0}{\partial x}, \frac{\partial f_0}{\partial y}\right\rangle + L_{\mathbf{R}}\langle x,y\rangle\right).$$

Now choose $b_{l+1},\dots,b_t \in \mathrm{m}(r;l)^2$ and $b_{t+1},\dots,b_k \in \mathrm{m}(r;l)^3$ such that $x_1,\dots,x_r,y_1,\dots,y_l$, $b_{l+1},\dots,b_t,b_{k+1}$ is a basis of $\mathrm{m}(r;l)/(\langle x\frac{\partial f_0}{\partial x},\frac{\partial f_0}{\partial y}\rangle + \mathrm{m}(r;l)^3)$ and $x_1,\dots,x_r,y_1,\dots,y_l,b_{l+1},\dots,$ b_{k+1} is a basis of $\mathrm{m}(r;l)/\langle x\frac{\partial f_0}{\partial x},\frac{\partial f_0}{\partial y}\rangle$.

In the case (1), choose $b_{l+1},\dots,b_k$ such that $x_1,\dots,x_r,y_1,\dots,y_l$, $b_{l+1},\dots,b_k$ generate $\mathrm{m}(r;l)/\langle x\frac{\partial f_0}{\partial x},\frac{\partial f_0}{\partial y}\rangle$ over $\mathbf{R}$.

Now define $\phi \in \mathcal{B}(r;k)$ by

$$\begin{aligned}&\phi(x_1,\dots,x_r,y_1,\dots,y_k)\\ &\qquad = (x_1,\dots,x_r,y_1,\dots,y_l,y_{l+1}+b_{l+1},\dots,y_k+b_k).\end{aligned}$$

Since $\exp^-_{u0}$ is invertible, the map $\iota_f : (\mathbf{H}^r\times\mathbf{R}^k,0)\to(M,q_0)$ given by $\iota_f(x,y)=\exp^-_{u0}(f(x,y)+t_0,(\phi(x,y),1+h\circ\phi(x,y)))$ defines a hypersurface germ V_f in (M,q_0). Then we have

$$F(x,y,0) := \tau(\iota_f(x,y),u_0)-t_0 = (f(x,y)+t_0)-t_0 = f(x,y),$$

$$\xi(q_0,u_0)|_{T_{q_0}V^f} = -d_{(x,y)}(\tau\circ\iota_f)((0,0),u_0) = -d_{(x,y)}f(0,0)=0,$$

$$\begin{aligned}&d_uF(x,y,0)\\ &\quad = d_u\tau(\iota_f(x,y),u_0) = \eta(\iota_f(x,y),u_0) = (\phi(x,y),1+h\circ\phi(x,y))\\ &\quad = (x_1,\dots,x_r,y_1,\dots,y_l,y_{l+1}+b_{l+1},\dots,y_k+b_k,b_{k+1}+a+1),\end{aligned}$$

where $a\in\mathrm{m}(r;k)^3$.

In the case (1), we have

$$\begin{aligned}&\mathcal{E}(r;k)/\left\langle x\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right\rangle\\ &\qquad = L_{\mathbf{R}}\langle 1,x_1,\dots,x_r,y_1,\dots,y_l,b_{l+1},\dots,b_k\rangle\\ &\qquad = L_{\mathbf{R}}\langle 1,x_1,\dots,x_r,y_1,\dots,y_l,y_{l+1}+b_{l+1},\dots,y_k+b_k\rangle.\end{aligned}$$

Hence the proof of the case (1) is completed.

In the case (2). since $1,x_1,\dots,x_r,y_1,\dots,y_l,b_{l+1},\dots,b_k,b_{k+1}$ is a basis of $\mathcal{E}(r;k)/\langle x\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\rangle$, there exist $\alpha_0,\alpha_1,\dots,\alpha_r,\beta_1,\dots,\beta_{k+1}\in\mathbf{R}$ such that

$$\begin{aligned}a \equiv\ &\alpha_0+\alpha_1x_1+\cdots+\alpha_rx_r+\beta_1y_1+\cdots+\beta_ly_l+\beta_{l+1}b_{l+1}\\ &+\cdots+\beta_{k+1}b_{k+1} \mod \mathcal{E}(r;k)/\left\langle x\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right\rangle.\end{aligned}$$

Hence

$$0 \equiv \alpha_0 + \alpha_1 x_1 + \cdots + \alpha_r x_r + \beta_1 y_1 + \cdots + \beta_l y_l + \beta_{l+1} b_{l+1} + \cdots + \beta_t b_t + \beta_{k+1} b_{k+1} \mod \mathcal{E}(r;k)\Big/\left(\left\langle x\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right\rangle + \mathrm{m}(r;k)^3\right).$$

Since $x_1, \ldots, x_r, y_1, \ldots, y_l, b_{l+1}, \ldots, b_t, b_{k+1}$ is a basis of $\mathcal{E}(r;k)/(\langle x\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle + \mathrm{m}(r;k)^3)$,

$$\alpha_0 = \alpha_1 = \cdots = \alpha_r = \beta_1 = \cdots = \beta_t = \beta_{k+1} = 0.$$

Hence $a \in L_{\mathbf{R}}\langle b_{t+1}, \ldots, b_k\rangle$ in $\mathcal{E}(r;k)/\langle x\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle$. This means that

$$\mathcal{E}(r;k)\Big/\left\langle x\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right\rangle = L_{\mathbf{R}}\langle 1, x_1, \ldots, x_r, y_1, \ldots, y_l, b_{l+1}, \ldots, b_k, b_{k+1} + a\rangle.$$

Therefore

$$\mathcal{E}(r;k)\Big/\left\langle x\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right\rangle = L_{\mathbf{R}}\langle 1, x_1, \ldots, x_r, y_1, \ldots, y_l, y_{l+1} + b_{l+1}, \ldots, y_k + b_k, b_{k+1} + a + 1\rangle.$$

Hence the proof for the case (2) is completed, supposing Lemma 6.4. □

It remains to show:

Lemma 6.4 *Let* $A = \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_r, \delta_1, \ldots, \delta_l) \in M(r+l, r+l; \mathbf{R})$, $\varepsilon_1, \ldots, \varepsilon_r, \delta_1, \ldots, \delta_l$ *are* 0 *or* ± 1 *and* $\delta_1 \neq 0$. *Then the set* $\mathcal{F}$ *of matrices linearly generated by* $D\phi(0)^t A D\phi(0)$ *for all* $\phi \in \mathcal{B}(r;l)$ *is equal to that of symmetric matrices in* $M(r+l, r+l; \mathbf{R})$.

Proof. We denote $\varepsilon = \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_r)$ and $\delta = \mathrm{diag}(\delta_1, \ldots, \delta_l)$. At first we remark that

$$\{B\delta C \in M(s,t;\mathbf{R}) \mid B \in M(s,l;\mathbf{R}), C \in M(l,t;\mathbf{R})\} = M(s,t;\mathbf{R})$$

for any integer s and t. Let $\phi \in \mathcal{B}(r;l)$ be given. We denote $\phi(x,y) = (x_1 a_1(x,y), \ldots, x_r a_r(x,y), b_1(x,y), \ldots, b_l(x,y))$. Then we have by immediately calculation that

$$D\phi(0)^t AD\phi(0) =$$

$$\begin{pmatrix} \mathrm{diag}(a_1^2(0)\varepsilon_1, \ldots, a_r^2(0)\varepsilon_r) + 2(\frac{\partial b}{\partial x}(0))^t\delta(\frac{\partial b}{\partial x}(0)) & 2(\frac{\partial b}{\partial x}(0))^t\delta(\frac{\partial b}{\partial y}(0)) \\ 2(\frac{\partial b}{\partial y}(0))^t\delta(\frac{\partial b}{\partial x}(0)) & 2(\frac{\partial b}{\partial y}(0))^t\delta(\frac{\partial b}{\partial y}(0)) \end{pmatrix}.$$

By considering the case $\frac{\partial b}{\partial x}(0) = 0$ we have

$$\left\{ \begin{pmatrix} \mathrm{diag}(a_1^2(0)\varepsilon_1, \ldots, a_r^2(0)\varepsilon_r) & 0 \\ 0 & B \end{pmatrix} \in M(r+l, r+l; \mathbf{R}) \mid B \in M(l,l;\mathbf{R}) \right\} \subset \mathcal{F}.$$

This means that

$$\left\{ \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} \in M(r+l, r+l; \mathbf{R}) \mid B \in M(l,l;\mathbf{R}) \right\} \subset \mathcal{F}.$$

Let $\phi, \phi' \in \mathcal{B}(r;l)$ satisfy the conditions that $a(0) = a'(0)$, $\frac{\partial b}{\partial x}(0) = \frac{\partial b'}{\partial x}(0)$, where $\phi(x,y) = (x_1a_1(x,y), \ldots, x_ra_r(x,y), b_1(x,y), \ldots, b_l(x,y))$ and $\phi'(x,y) = (x_1a_1'(x,y), \ldots, x_ra_r'(x,y), b_1'(x,y), \ldots, b_l'(x,y))$. Then

$$D\phi(0)^t AD\phi(0) - D\phi'(0)^t AD\phi'(0)$$
$$= \begin{pmatrix} 0 & 2(\frac{\partial b}{\partial x}(0))^t\delta(\frac{\partial b}{\partial y}(0) - \frac{\partial b'}{\partial y}(0)) \\ 2(\frac{\partial b}{\partial y}(0) - \frac{\partial b'}{\partial y}(0))^t\delta(\frac{\partial b}{\partial x}(0)) & * \end{pmatrix}.$$

Therefore

$$\left\{ \begin{pmatrix} 0 & B \\ B^t & 0 \end{pmatrix} \in M(r+l, r+l; \mathbf{R}) \mid B \in M(r,l;\mathbf{R}) \right\} \subset \mathcal{F}.$$

Similarly we have

$$\left\{ \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \in M(r+l, r+l; \mathbf{R}) \mid B \in M(r,r;\mathbf{R}),\ B^t = B \right\} \subset \mathcal{F}.$$

□

Acknowledgments The author would like to thank Professor S. Izumiya for useful remarks and useful discussions.

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