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Separation and weak separation on Riemann surfaces

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Abstract. We show some necessary and sufficient conditions for weak separation by an algebra A of analytic functions on a Riemann surface R. One of these equivalent conditions is the following. There exists a sequence of relatively compact open sets $\{D_n\}$ in R such that (i) ∂D_n is connected, (ii) $\overline{D}_1 \subset \overline{D}_2 \subset \overline{D}_3 \subset \cdots$, (iii) $R = \cup \overline{D}_n$, and (iv) A separates the points of a neighborhood of ∂D_n.

Key words: weak separation, algebra of analytic functions, Riemann surface.

1. Introduction

Let R be a Riemann surface, and let A be an algebra of analytic functions on R. We always assume that A contains constant functions. We say that points p and q of R are separated by A if there is a function f in A such that $f(p) \neq f(q)$, and when any pair of distinct points are separated by A, we say that the algebra A separates the points of R. For functions f and g in A such that $g \not\equiv 0$, (f/g) is a meromorphic function and so we can consider the value $(f/g)(p)$ at any point p of R. According to Royden [4] we say that points p and q of R are weakly separated by A if there are functions f and g in A as above such that $(f/g)(p) \neq (f/g)(q)$, and when any pair of distinct points are weakly separated by A, we say that the algebra A weakly separates the points of R.

On the other hand, in Gamelin-Hayashi [2] it was defined that A weakly separates the points of R if there is a discrete subset Λ of R such that A separates the points of $R\backslash\Lambda$ in case A is the algebra of bounded analytic functions $H^{\infty}(R)$. These two definitions for weak separation coincides each other.

In this paper we study some necessary and sufficient conditions for weak separation, and show that separation on a rather narrow set means weak separation on R. We also include a proof of equivalence of two definitions for weak separation. It will be convenient since the proof is not given in [2]. For the moment we use the terminology "weak separation" in the sense of

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Royden.

2. Preparation

We want to use the Royden's resolution $\tilde{R}$ of R with respect to A and the canonical map $\varphi : R \to \tilde{R}$, and also some lemmas which are used to construct $\tilde{R}$ in Royden [4]. Following lemmas and proposition are implicitly included in Royden [3]. See also Bishop [1]. We include the proof for the sake of convenience.

Lemma 1 *If points p, q of R are weakly separated by A, then there are neighborhoods U of p and V of q such that A separates any pair of points (p', q') in $U \times V$ except (p, q).*

Proof. Let f and g be functions in A such that $(f/g)(p) \neq (f/g)(q)$. If $f(p) \neq f(q)$ or $g(p) \neq g(q)$ then the conclusion easily follows, so we can assume that $f(p) = f(q)$ and $g(p) = g(q)$. Then $(f/g)(p) \neq (f/g)(q)$ can occur only when $f(p) = f(q) = g(p) = g(q) = 0$. As g is not identically 0, there exist neighborhoods U of p and V of q such that $g \neq 0$ in $(U \backslash \{p\}) \cup (V \backslash \{q\})$ and $(f/g)(U) \cap (f/g)(V) = \emptyset$. Then any pair of points (p', q') in $(U \times V) \backslash \{(p, q)\}$ are separated by functions f or g in A. □

For a point p of R, let

$$M(p) = \{f/g : f,\ g \in A,\ g \not\equiv 0,\ (f/g)(p) = 0\}$$

and let $\nu(p)$ be the minimal order of meromorphic functions in $M(p)$ at p.

Lemma 2 *For a point p of R, let h be a function in $M(p)$ with order $\nu(p)$ at p. Then for any function f in A, there exists a neighborhood U of p such that f is represented as*

$$f = \sum_{n=0}^{\infty} c_n h^n$$

in U.

Proof. By some local coordinate (U, z) with $z(p) = 0$, h can be represented as $h = z^{\nu(p)}$ and

$$f = \sum_{m=0}^{\infty} a_m z^m$$

in U. We want to show that the set $\{m : a_m \neq 0,\ m$ is not a multiple of $\nu(p)\}$ is empty. If not, let s be the smallest number of this set, and let t be the integer with $t\nu(p) < s < (t+1)\nu(p)$. Then

$$\frac{f - \sum_{k=0}^{t} a_{k\nu(p)} h^k}{h^t} = \frac{f - \sum_{k=0}^{t} a_{k\nu(p)} z^{k\nu(p)}}{z^{t\nu(p)}} = a_s z^{s-t\nu(p)} + \cdots$$

is an element of $M(p)$ and the order of this function at p is less than $\nu(p)$. This is a contradiction. □

Lemma 3 *Suppose that A weakly separates the points of R. Then for any point p of R, there exists a neighborhood U of p such that A separates the points of U.*

Proof. Let U and h be as in the proof of Lemma 2. Lemma 2 shows that a pair of points in U are not weakly separated by A if these points are not separated by h. Since $h = z^{\nu(p)}$, it must be that $\nu(p) = 1$. Then h itself separates the points of U.

Let $h = f/g$ with $f, g \in A$. By replacing U with a smaller neighborhood of p if necessary, we can assume that $f \neq 0$ and $g \neq 0$ in $U \backslash \{p\}$. Then any pair of points in U are separated by functions f or g in A. □

The Royden's resolution $\tilde{R}$ of R with respect to A and the cannonical map $\varphi : R \to \tilde{R}$ is defined in [4] in terms of the homomorphisms of algebras. Here $\tilde{R}$ is a Riemann surface and φ is an analytic map satisfying that $\varphi(p) = \varphi(q)$ for $p, q \in R$ if and only if there exist non-constant analytic maps ρ and σ from a neighborhood of p and q respectively into the complex plane satisfying that $\rho(p) = 0$, $\sigma(q) = 0$, and every f in A takes a same value on $\rho^{-1}(z) \cup \sigma^{-1}(z)$ for any complex number z in the images of ρ or σ.

We use only this property of $\tilde{R}$ and φ in the proof of following proposition, and so we may use this equivalence relation $p \sim q$ to define a Riemann surface $R/\sim$, which is enough for the purpose of this paper, although $\varphi(R) = R/\sim$ is a subsurface of the Royden's resolution $\tilde{R}$ in general.

Proposition 1 *For p and q of R, $\varphi(p) = \varphi(q)$ if and only if p and q are not weakly separated by A. Especially, the map φ is injective on R if and only if A weakly separates the points of R.*

Proof. If $\varphi(p) = \varphi(q)$, then A does not separate $\rho^{-1}(z)$ and $\sigma^{-1}(z)$ for any z and so by Lemma 1, A does not weakly separate p and q.

For the reverse implication, we assume p and q are not weakly separated by A. Let h_p, h_q be functions in $M(p)$, $M(q)$ respectively as in the statement of Lemma 2. Since p and q are not weakly separated, $h_q(p) = h_p(p) = 0$, $h_p(q) = h_q(q) = 0$, and $(h_p/h_q)(p) = (h_p/h_q)(q)$. First two equations imply $h_q \in M(p)$, $h_p \in M(q)$ and so $(h_p/h_q)(p) \neq 0$, $(h_p/h_q)(q) \neq \infty$. Hence h_p and h_q have the same order at q, and so we can take the same function $h = h_p$ in Lemma 2 for both p and q.

For any $f \in A$, there are neighborhoods U of p and V of q such that $f = \sum_{n=0}^{\infty} a_n h^n$ in U and $f = \sum_{n=0}^{\infty} b_n h^n$ in V. If $a_n = b_n$ for $n = 0, 1, \ldots, k-1$, then the function

$$\frac{f - \sum_{n=0}^{k-1} a_n h^n}{h^k}$$

is a member of quotient field of A and takes values a_k at p and b_k at q. So $a_k = b_k$ and this shows that $a_n = b_n$ for all n. Therefore, if we take $\rho = \sigma = h$, f takes a same value on $\rho^{-1}(z) \cup \sigma^{-1}(z) = h^{-1}(z)$ for any complex number z in $h(U \cup V)$. □

3. Main Theorem

For two sets U and E in R, We say that A is separating on U with respect to E if every point in U is separated by A from any other point in $U \cup E$.

Theorem 1 *Let A be an algebra of analytic functions on a Riemann surface R. Then the following four conditions are equivalent.*

(a) *A weakly separates the points of R.*

(b) *There exists a discrete subset Λ of R such that A separates the points of $R \backslash \Lambda$.*

(c) *There exists a sequence of compact sets $\{K_n\}$ in R such that* (i) *$K_1 \subset K_2 \subset K_3 \subset \cdots$,* (ii) *$R = \cup K_n$, and* (iii) *$A$ is separating on a neighborhood of ∂K_n with respect to K_n.*

(d) *There exists a sequence of relatively compact open sets $\{D_n\}$ in R such that* (i) *∂D_n is connected,* (ii) *$\overline{D}_1 \subset \overline{D}_2 \subset \overline{D}_3 \subset \cdots$,* (iii) *$R = \cup \overline{D}_n$, and* (iv) *$A$ separates the points of a neighborhood of ∂D_n.*

Proof. (a) $\Rightarrow$ (b): Let

$$\Gamma = \{(p, q) \in R \times R : p \neq q,\ p \text{ and } q \text{ are not separated by } A\}.$$

By Lemma 1 and Lemma 3, Γ is a discrete subset of $R \times R$. Let $\{R_n\}$ be an exhaustion of R by relatively compact subregions R_n of R. For $p \in R$, let $\chi(p) = \min\{n : p \in R_n\}$ and we set

$$\Lambda = \{p \in R : \text{there exists a } q \in R \text{ such that } (p,q) \in \Gamma \text{ and } \chi(q) \leq \chi(p)\}.$$

First, we show that A separates the points of $R \backslash \Lambda$. If not, there exists a pair of points p, $q \in R\backslash\Lambda$ which are not separated by A, so $(p,q) \in \Gamma$. Then either p or q is a member of Λ according to $\chi(q) \leq \chi(p)$ or $\chi(p) \leq \chi(q)$. This is a contradiction.

Next, we show that Λ is a discrete subset of R. If not, there exists a sequence $\{p_m\}$ of points in Λ such that $\{p_m\}$ converges to a point p in R. Then all points of $\{p_m\}$ are contained in an R_n. By the definition of Λ, for each p_m there exists a point $q_m \in R$ such that $(p_m, q_m) \in \Gamma$ and q_m is also contained in R_n. Since R_n is relatively compact, there is a subsequence of $\{(p_m, q_m)\}$ which converges to a point in $R \times R$. This contradicts the fact that Γ is a discrete subset of $R \times R$.

(b) $\Rightarrow$ (d): We can take an exhaustion $\{R_n\}$ of R such that ∂R_n consists of finite number of smooth Jordan closed curves and $\partial R_n \cap \Lambda = \emptyset$ for all n. We can also join every component of ∂R_n by finite number of disjoint smooth Jordan arcs in R_n without passing Λ. Let L_n be the union of these Jordan arcs. Then $D_n = R_n \backslash L_n$ satisfies the conditions of (d).

(c) $\Rightarrow$ (a): We use the Royden's resolution $\tilde{R}$ of R with respect to A and the canonical map $\varphi : R \to \tilde{R}$. It suffices to show that φ is injective on each K_n.

First we show that there exists a neighborhood V of $\varphi(\partial K_n)$ such that for $w \in V$, the number of points in $\varphi^{-1}(w) \cap K_n$ is 1 or 0. Let U be a neighborhood of ∂K_n such that A is separating on U with respect to K_n. Since φ is an open mapping, $\varphi(U)$ is a neighborhood of $\varphi(\partial K_n)$. Let $w \in \varphi(U)$ and $p \in U$ be such as $\varphi(p) = w$. If there exists another point $q \in K_n$, $\varphi(q) = w$, then p and q are not separated by A which contradicts the assumption. Hence $V = \varphi(U)$ suffices our request.

Now we show that φ is injective on $\operatorname{int} K_n$ by reduction to absurdity. So we assume that there exist points $a, b \in \operatorname{int} K_n$ such that $a \neq b$ and $\varphi(a) = \varphi(b)$. If $\varphi(a) \in \varphi(S)$ where $S = \{p \in K_n : \frac{d\varphi}{d\zeta}(p) = 0\}$ (the set of singular points of the map φ), we can take $c \notin \varphi(S)$ near $\varphi(a)$ and $\tilde{a} \in \varphi^{-1}(c) \cap \operatorname{int} K_n$ near a and $\tilde{b} \in \varphi^{-1}(c) \cap \operatorname{int} K_n$ near b so that $\tilde{a} \neq \tilde{b}$. Hence we can assume that $\varphi(a) \notin \varphi(S)$.

We can join $\varphi(a)$ with a point $x \in V$ by a Jordan arc γ in $\tilde{R}\backslash(\varphi(S) \cup \varphi(\partial K_n))$. In fact we can join $\varphi(a)$ with any point $y \in V\backslash\varphi(S)$ by an Jordan arc $\tilde{\gamma}$ in $\tilde{R}\backslash\varphi(S)$ with the equation $u : [0,1] \to \tilde{R}$, $u(0) = \varphi(a)$, $u(1) = y$. If $\tilde{\gamma} \cap \varphi(\partial K_n) = \emptyset$, we can take $x = y$ and $\gamma = \tilde{\gamma}$. If $\tilde{\gamma} \cap \varphi(\partial K_n) \neq \emptyset$, let $t_0 = \min\{t : u(t) \in \varphi(\partial K_n)\}$ and take t_1 such that $u(t_1) \in V$ and $t_1 < t_0$. Then we can take $x = u(t_1)$ and the subarc γ of $\tilde{\gamma}$ for $0 \leq t \leq t_1$.

By usual lifting argument, we see that there exist arcs γ_a and γ_b in R with initial points a and b respectively, such that $\varphi(\gamma_a) = \varphi(\gamma_b) = \gamma$ and $\gamma_a \cap \gamma_b = \emptyset$. Since γ_a and γ_b do not meet ∂K_n, these are contained in $\operatorname{int} K_n$. Accordingly $\varphi^{-1}(x) \cap K_n$ contains at least two points $\varphi^{-1}(x) \cap \gamma_a$ and $\varphi^{-1}(x) \cap \gamma_b$. This contradicts $x \in V$ and so we see that φ is injective on $\operatorname{int} K_n$. This with the assumptin of (c) shows that φ is injective on K_n.

(d) $\Rightarrow$ (c): We again use the Royden's resolution $\tilde{R}$ of R and $\varphi : R \to \tilde{R}$. Let U be a neighborhood of ∂D_n such that A separates the points of U. We can take an arcwise connected compact set B with $\partial D_n \subset \operatorname{int} B \subset B \subset U$. For example, we can cover ∂D_n by finite number of coordinate disks V_m with $\overline{V}_m \subset U$ and $V_m \cap \partial D_n \neq \emptyset$. Then $\cup V_m$ is an open connected set and we can take $B = \overline{\cup V_m}$ as an arcwise connected compact set.

We want to show that A is separating on B with respect to D_n by reduction to absurdity. If not, there exists a point $p \in D_n\backslash B$ with $\varphi(p) \in \varphi(B)$. Let E be a component of $\varphi^{-1}(\varphi(B)) \cap (D_n \cup B)$ containing p. As φ is injective on U, $\varphi(U\backslash B)$ does not meet $\varphi(B)$ and so $\varphi^{-1}(\varphi(B)) \cap (D_n \cup B)$ is contained in the union of mutually disjoint compact sets $D_n\backslash U$ and B. This shows that $E \subset D_n\backslash U \subset D_n\backslash B$.

Now we can use lifting argument to show that $\varphi(E) = \varphi(B)$. In fact, for any point w in $\varphi(B)$, we can join $\varphi(p)$ and w by an arc γ in $\varphi(B)$, and we can take a maximal arc γ_p in R with initial point p such that $\varphi(\gamma_p) \subset \gamma$. If $\varphi(\gamma_p)$ is a proper subset of γ, then the arc γ_p continues to the outside of the set D_n, and $\gamma_p \cap \partial D_n \neq \emptyset$. Hence, the set E intersects with the set B, a contradiction. Thus, $\varphi(\gamma_p) = \gamma$. Then $w \in \varphi(\gamma_p) \subset \varphi(E)$ and so $\varphi(E) = \varphi(B)$.

From $E \subset D_n$ and $\partial D_n \subset B$, it follows that $\varphi(\partial D_n) \subset \varphi(B) = \varphi(E) \subset \varphi(D_n)$. For any function f in A, we can take an analytic function $\tilde{f}$ on $\tilde{R}$ such that $f = \tilde{f} \circ \varphi$. Then $f(\partial D_n) = \tilde{f}(\varphi(\partial D_n)) \subset \tilde{f}(\varphi(D_n)) = f(D_n)$ and by the maximal modulus principle, f must be a constant function. This contradicts the assumption of (d), and we conclude that A is separating on B with respect to D_n.

Let $K_n = \overline{D}_n$. Since $\partial K_n \subset \partial D_n \subset \operatorname{int} B \subset B$ and $K_n \cup \operatorname{int} B \subset D_n \cup B$, conditions of (c) are satisfied if we take $\operatorname{int} B$ as a neighborhood of ∂K_n. □

The condition (i) "∂D_n is connected" of (d) in Theorem 1 can not be removed, and also we can not remove "a neighborhood of" in the condition (iv) of (d).

To show this we use a Riemann surface R which is known as Myrberg's example ([3]), and we take A as the algebra of bounded analytic functions $H^\infty(R)$. Let a_n, b_n be two sequences of real numbers such that $0 < a_{n+1} < b_{n+1} < a_n < b_n$ $(n = 1, 2, \ldots)$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$. We define a Riemann surface R as a two sheeted unbounded covering surface of punctured disk $\Delta_0 = \{0 < |z| < 1\}$ which has branch points over $\{a_n\}$ and $\{b_n\}$. Let $\pi : R \to \Delta_0$ be a projection, and let $C_r = \{|z| = r\}$. We also assume that $\pi^{-1}(C_r)$ is connected for $a_n \leq r \leq b_n$ $(n = 1, 2, \ldots)$ and $\pi^{-1}(C_r)$ has two components for $b_{n+1} < r < a_n$ $(n = 1, 2, \ldots)$ and for $b_1 < r < 1$. It is known that every bounded analytic function on R takes a same value on $\pi^{-1}(z)$ for $z \in \Delta_0$, and so $H^\infty(R)$ can not weakly separates the points of R.

We can take connected open sets $\{D_n\}$ in R such that ∂D_n has four components and each component is a component of $\pi^{-1}(C_r)$ for $r = c_n, d_n, s_n, t_n$ respectively, where $b_{n+1} < c_n < d_n < a_n$ and $b_1 < s_n < t_n < s_{n+1} < t_{n+1} < 1$ $(n = 1, 2, \ldots)$, $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = 1$. Note that we must take components of $\pi^{-1}(C_r)$ on "different sheets" of R for $r = c_n$ and $r = d_n$, and also for $r = s_n$ and $r = t_n$. The conditions (ii) and (iii) of (d) are satisfied by the construction, and (iv) is satisfied since $H^\infty(R)$ contains the function $z \circ \pi$ where z is the coordinate function on Δ_0. Now all conditions of (d) are satisfied except (i).

For another example which shows necessity of "a neighborhood of" in the condition (iv) of (d), we modify the Riemann surface R such as R has branch points also over $\{s_n\}$ and $\{t_n\}$. Again every bounded analytic function on R takes a same value on $\pi^{-1}(z)$ for $z \in \Delta_0$. Let $\Gamma_{n,1}$ be a subset of $\pi^{-1}(C_{a_n})$ which form a closed Jordan curve, and let $\Gamma_{n,2}$ be another closed Jordan curve on R such that $\pi(\Gamma_{n,2})$ is the circle whose diameter is the segment $[-b_n, a_n]$ and such that $R \backslash (\Gamma_{n,1} \cup \Gamma_{n,2})$ has no relatively compact components. We take $\Gamma_{n,3}$ and $\Gamma_{n,4}$ in the same manner, such as $\Gamma_{n,3} \subset \pi^{-1}(C_{s_n})$ and $\pi(\Gamma_{n,4})$ is the circle whose diameter is the segment $[-t_n, s_n]$. Now we can take connected open sets $\{\tilde{D}_n\}$ in R such that $\partial\tilde{D}_n$ has two components $\Gamma_{n,1} \cup \Gamma_{n,2}$ and $\Gamma_{n,3} \cup \Gamma_{n,4}$. We can join these two components by a Jordan arc L_n in $\tilde{D}_n$ where $\pi(L_n)$ is a segment $[a_n, s_n]$. Then $D_n = \tilde{D}_n \backslash L_n$ satisfies all conditions of (d) if we remove "a neighborhood of" in the condition (iv) of (d).

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