Hokkaido Mathematical Journal Vol. 27 (1998) p. 605–623

Irrational foliations of $S^3 \times S^3$

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(Received June 4, 1997)

Abstract. The Godbillon-Vey class is a characteristic cohomology class of dimension 3 for foliations of codimension 1 whose transition functions are transversely Lipschitz and with derivatives of bounded variations. We show that for a foliation $\mathcal{F}$ of $S^3 \times S^3$ of codimension 1, the ratio a/b of the Godbillon-Vey class $GV(\mathcal{F}) = (a, b) \in \mathbf{R} \oplus \mathbf{R} \cong H^3(S^3 \times S^3; \mathbf{R})$ takes any real value. It has been known that this ratio is invariant under the deformation of smooth foliations.

Key words: codimension 1 foliations, classifying spaces, Godbillon-Vey class, rationality.

Introduction

Let $\mathcal{F}$ be a codimension-one foliation of of $S^3 \times S^3$. For a codimension-one foliation, the Godbillon-Vey class is defined as a 3-dimensional cohomology class ([6]). Hence in this case, $GV(\mathcal{F}) \in H^3(S^3 \times S^3; \mathbf{R}) \cong \mathbf{R} \oplus \mathbf{R}$. We call $\mathcal{F}$ rational if $GV(\mathcal{F}) = (a, b) \in H^3(S^3 \times S^3; \mathbf{R})$ satisfies $a/b \in \mathbf{Q} \cup \{\infty\}$, and call $\mathcal{F}$ irrational if $a/b \in \mathbf{R} - \mathbf{Q}$. Gel'fand-Feigin-Fuks ([2]) noticed that this ratio a/b is invariant under a deformation of codimension-one foliations. Hence rationality or irrationality of foliations of $S^3 \times S^3$ is invariant under deformation.

This definition of rationality and irrationality imitates the one for the linear foliations of the 2-dimensional torus T^2. (See [12], [13] for the interesting progress in piecewise linear foliations on T^2.) A rational linear foliation of T^2 is defined by a submersion to the circle S^1. In a similar way, we can construct examples of rational foliations of $S^3 \times S^3$ by defining a Haefliger structure on $S^3 \times S^3$ as a pull-back by an appropriate map to S^3 and using the theorem of existence of foliations ([16]). An irrational linear foliation of T^2 is easy to construct. But it has not been known whether there exist irrational foliations of $S^3 \times S^3$. The question of the existence of irrational foliations of $S^3 \times S^3$ was raised in Gel'fand-Feigin-Fuks [2] and

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1991 Mathematics Subject Classification : Primary 57R32; Secondary 57R30, 57R50, 58H10.

The author was supported in part by Grant-in-Aid for Scientific research 07454013 and 09440028, Ministry of Education, Science, Sports and Culture, Japan.

discussed in Morita [14]. Since rationality or irrationality of foliations of $S^3 \times S^3$ is invariant under deformation, the existence of rational foliations gives no insurance for the existence of irrational foliations.

The domain of definition of the Godbillon-Vey class has been enlarged by several authors (see [5], [3], [10], [17]). Then the same question of the existence of irrational foliations is raised for each class of foliations.

In the case of transversely piecewise linear foliations, we showed in [20] that all transversely piecewise linear foliations are rational with respect to the discrete Godbillon-Vey class defined by Ghys and Sergiescu ([5], [3]). The discontinuous invariants defined by Morita ([14]) and the description by Greenberg ([7]) of the classifying space for the transversely piecewise linear foliations were essential to prove the rationality of transversely piecewise linear foliations.

In this paper we show the following theorem.

Theorem *For any $(a,b) \in H^3(S^3 \times S^3; \mathbf{R})$, there exists a foliation $\mathcal{F}$ of class $C^{L,\mathcal{V}_1}$ such that $GV(\mathcal{F}) = k(a,b)$ for some positive integer k.*

Foliations of class $C^{L,\mathcal{V}_1}$ ([21]) are transversely Lipschitz foliations such that the derivatives of transition functions are with bounded variations. These foliations were called of class P after [9]. For these foliations, the Godbillon-Vey class GV is still defined (see also [17]). This Godbillon-Vey class is the sum $GV_{\text{reg}} + GV_{\text{atom}} + GV_{\text{sing}}$ of the usual Godbillon-Vey class GV_{reg} ([6]), the discrete Godbillon-Vey class $GV_{\text{atom}} = \overline{GV}$ ([5], [3]) and the singular Godbillon-Vey class GV_{sing} ([17]). Our main theorem of course says that there exist irrational foliations of class $C^{L,\mathcal{V}_1}$ of $S^3 \times S^3$. For our examples, $GV(\mathcal{F}) = GV_{\text{atom}}(\mathcal{F})$. The property of the Godbillon-Vey class under the deformations of foliations of class $C^{L,\mathcal{V}_1}$ is not yet clear.

As we mentioned before, the existence of rational foliations is well known. In order to show the existence of irrational foliations of $S^3 \times S^3$, we need to use a result of Morita ([14]).

In fact, Morita translated the question of rationality into that of graded commutativity of $*$-product defined on the homology of the group of diffeomorphisms of $\mathbf{R}$ with compact support ([14]). Thus we look at the homology of the group $\mathbf{G}_c^{L,\mathcal{V}_1}(\mathbf{R})$ of homeomorphisms of class $C^{L,\mathcal{V}_1}$ of $\mathbf{R}$ with compact support, and we in fact show the graded commutativity for the $*$-product on 2-dimensional homology classes coming from the homology of the group $PL_c(\mathbf{R})$ of piecewise linear homeomorphisms of $\mathbf{R}$ with compact support.

This paper is organized as follows.

In §1, we review the results of Morita ([14]) on the relationship between the Whitehead products on the homotopy groups of the classifying space for the codimension one foliations and the $*$-products on the homology groups of the group of diffeomorphisms. Then we reduce the proof of our main theorem to the commutativity of the $*$-product.

In §2, we review the fact that the second homology of the group $PL_c(\mathbf{R})$ of piecewise linear homeomorphisms of $\mathbf{R}$ with compact support is isomorphic to $\mathbf{R} \otimes_{\mathbf{Q}} \mathbf{R}$ ([7]) and its image in the second homology of the group $\mathbf{G}_c^{L,\mathcal{V}_1}(\mathbf{R})$ isomorphic to $\mathbf{R}$ ([18]). This fact was used in [18] to show that the foliated cobordism class as foliations of class $C^{L,\mathcal{V}_1}$ of transversely oriented transversely piecewise linear foliations of closed oriented 3-manifolds is characterized by its (discrete) Godbillon-Vey class.

§3 is the heart of this paper. Let $B\mathbf{G}_c^{L,\mathcal{V}_1}(\mathbf{R})^\delta$ and $BPL_c(\mathbf{R})^\delta$ denote the classifying spaces for the groups $\mathbf{G}_c^{L,\mathcal{V}_1}(\mathbf{R})$ and $PL_c(\mathbf{R})$ with the discrete topology, respectively. We show that the $*$-product in $H_*(B\mathbf{G}_c^{L,\mathcal{V}_1}(\mathbf{R})^\delta; \mathbf{Z})$ of elements of $H_2(BPL_c(\mathbf{R})^\delta; \mathbf{Z})$ is commutative. This implies the existence of the irrational foliations of class $C^{L,\mathcal{V}_1}$ of $S^3 \times S^3$. We use an explicit construction similar to that used in [18].

This work was essentially done during my stay at Pontificia Universidade Católica do Rio de Janeiro in January 1992. I thank it for warm hospitality.

I tried to construct smooth irrational foliations of $S^3 \times S^3$ for several years. But for this interesting question, we have made little progress.

1. Whitehead product and $*$-product

In order to construct an irrational foliation of $S^3 \times S^3$, we can try the following thing. Choose two foliations F_1 and F_2 of S^3 such that $GV(F_1) = a$ and $GV(F_2) = b$, respectively. In $S^3 \times S^3$, we put F_1 and F_2 on $S^3 \times \{*\}$ and on $\{*\} \times S^3$, respectively, and extend it as a Haefliger structure in a regular neighborhood N of $S^3 \times \{*\} \cup \{*\} \times S^3$. Now we try to extend the Haefliger structure on $\partial N \cong S^5$ to the rest which is diffeomorphic to a 6 dimensional disk. This is precisely the problem of calculating the Whitehead product

$$\pi_3(B\overline{\Gamma}_1) \times \pi_3(B\overline{\Gamma}_1) \longrightarrow \pi_5(B\overline{\Gamma}_1)$$

for the elements of $\pi_3(B\overline{\Gamma}_1)$ represented by F_1 and F_2. Here $B\overline{\Gamma}_1$ denotes the classifying space for the transversely oriented codimension-one Haefliger structures.

Let $\mathrm{Diff}_c(\mathbf{R})$ denotes the group of diffeomorphisms of the real line with compact support and let $B\,\mathrm{Diff}_c(\mathbf{R})^\delta$ denotes the classifying space for the group $\mathrm{Diff}_c(\mathbf{R})$ with the discrete topology. Under the isomorphism $H_*(B\,\mathrm{Diff}_c(\mathbf{R})^\delta; \mathbf{Z}) \cong H_*(\Omega B\overline{\Gamma}_1; \mathbf{Z})$ due to Mather ([11], see also [15], [5], [7]), the Whitehead product corresponds to the $*$-product defined in [14] as follows.

Let $\mu : \mathrm{Diff}_c(\mathbf{R}) \times \mathrm{Diff}_c(\mathbf{R}) \longrightarrow \mathrm{Diff}_c(\mathbf{R})$ be the composition of two isomorphisms $\mathrm{Diff}_c(\mathbf{R}) \cong \mathrm{Diff}_c((-\infty, 0))$ and $\mathrm{Diff}_c(\mathbf{R}) \cong \mathrm{Diff}_c((0,\infty))$, and the inclusion

$$\mathrm{Diff}_c((-\infty, 0)) \times \mathrm{Diff}_c((0,\infty)) \longrightarrow \mathrm{Diff}_c(\mathbf{R}).$$

Then μ induces a product $*$ on the homology of $B\,\mathrm{Diff}_c(\mathbf{R})^\delta$.

Morita showed the following proposition ([14]).

Proposition 1.1 *Let F_1 and F_2 be foliations of S^3 and u_1 and u_2, the corresponding elements of $H_2(B\,\mathrm{Diff}_c(\mathbf{R})^\delta; \mathbf{Z})$ by Mather's theorem. If $u_1 * u_2 = u_2 * u_1$, then the Whitehead product of the two elements of $\pi_3(B\overline{\Gamma}_1)$ represented by F_1 and F_2 has finite order.*

We will show that for transversely piecewise linear foliations F_1 and F_2 of S^3, the corresponding elements u_1 and u_2 in $H_2(B\mathbf{G}_c^{L,\mathcal{V}_1}(\mathbf{R})^\delta; \mathbf{Z})$ satisfies $u_1 * u_2 = u_2 * u_1$. We know that the Godbillon-Vey class of PL foliations of S^3 takes any real value. Hence by Proposition 1.1, for any $(a, b) \in H^3(S^3 \times S^3; \mathbf{R})$, there is a Haefliger structure $\mathcal{H}$ of class $C^{L,\mathcal{V}_1}$ on $S^3 \times S^3$ such that $GV(\mathcal{H}) = k(a, b)$ for some positive integer k. Using the theorem of existence of foliations ([16]), we obtain a foliation $\mathcal{F}$ of $S^3 \times S^3$ such that $GV(\mathcal{F}) = k(a, b)$. This proves our main theorem.

2. Second homology of the group of piecewise linear homeomorphisms

Let $PL_c(\mathbf{R})$ be the group of piecewise linear homeomorphisms of $\mathbf{R}$ with compact support. We know that the second homology group of $BPL_c(\mathbf{R})^\delta$ is isomorphic to $\mathbf{R} \otimes_{\mathbf{Q}} \mathbf{R}$ ([7]). We also know the generators. Let f_a be a piecewise linear homeomorphism of $\mathbf{R}$ with support in $[-1, 0]$ such that

$\log f_a'(-0) = a$ and let g_b be a piecewise linear homeomorphisms of $\mathbf{R}$ with support in $[0,1]$ such that $\log g_b'(+0) = b$. Then $(f_a, g_b) - (g_b, f_a)$ is a 2-cycle of $BPL_c(\mathbf{R})^\delta$ representing the element $a \otimes_{\mathbf{Q}} b \in \mathbf{R} \otimes_{\mathbf{Q}} \mathbf{R}$. We note that this 2-cycle corresponds to the the piecewise linear Reeb foliation of S^3 whose compact toral leaf has the germs at 0 of f_a and g_b above as holonomies, and the Godbillon-Vey invariant of this foliation is ab.

Let $\mathbf{G}_c^{L,\mathcal{V}_1}(\mathbf{R})$ denote the group of the Lipschitz homeomorphisms f of $\mathbf{R}$ with compact support such that $\log f'(x-0)$ exists and is a function of bounded variation. Let $|||\log f'|||_1$ denote the total variation of $\log f'$.

If we look at piecewise linear homeomorphisms in the group $\mathbf{G}_c^{L,\mathcal{V}_1}(\mathbf{R})$, we know the following ([18]). The image of $H_2(BPL_c(\mathbf{R})^\delta; \mathbf{Z})$ in $H_2(B\mathbf{G}_c^{L,\mathcal{V}_1}(\mathbf{R})^\delta; \mathbf{Z})$ is isomorphic to $\mathbf{R}$ and this isomorphism is given by the Godbillon-Vey invariant. Hence as a foliation of class $C^{L,\mathcal{V}_1}$, any transversely piecewise linear foliation of a 3-manifold is foliated cobordant to a single piecewise linear Reeb foliation of S^3.

This was shown by using a kind of infinite juxtaposition construction and the following proposition ([19]) which we also need in this paper. A piecewise linear homeomorphism of $\mathbf{R}$ with compact support is said to be elementary if it has at most 3 nondifferentiable points.

Proposition 2.1 ([19]) *Let (a_i, b_i) $(i = 1, 2, \cdots)$ be disjoint open intervals whose union is bounded. Let f_i be a piecewise linear homeomorphism of $\mathbf{R}$ with support in $[(7a_i + b_i)/8, (a_i + 7b_i)/8]$ which is a composition of at most k elementary piecewise linear homeomorphisms. Suppose that $\sum |||\log f_i'|||_1{}^{1/2} < \infty$. Then $f = \prod f_i$ is written as a product (composition) of $3k$ commutators of piecewise linear homeomorphisms of $\mathbf{R}$ as follows.*

$$f = \prod_{j=1}^{3k} [g_{2j-1}, g_{2j}],$$

where $g_i \in \mathbf{G}_c^{L,\mathcal{V}_1}(\mathbf{R})$, the supports of g_i $(i = 1, \ldots, 6k)$ are contained in the closure $\overline{\bigcup [a_i, b_i]}$ of $\bigcup [a_i, b_i]$.

Remark. In the above proposition, the condition on the support of f_i can be replaced, for example, by $\operatorname{Supp} f_i \subset [(15a_i + b_i)/16, (a_i + 15b_i)/16]$.

Let $c = \sum_i (f_1^{(i)}, \ldots, f_n^{(i)})$ be an n-chain of $B\mathbf{G}_c^{L,\mathcal{V}_1}(\mathbf{R})^\delta$. The support $\operatorname{Supp} c$ of the chain c is defined to be the union $\bigcup_{i,j} \operatorname{Supp} f_j^{(i)}$ of the supports of $f_j^{(i)}$.

For an n-chain c and an n'-chain c' such that the elements of $\mathbf{G}_c^{L,\mathcal{V}_1}(\mathbf{R})$ appearing in c commute with the ones appearing in c', we have the Cartesian product $c \times c'$ such that

$$\partial(c \times c') = (\partial c) \times c' + (-1)^n c \times (\partial c').$$

Hence if we have an n-cycle c and an n'-cycle c' such that $\mathrm{Int}\,\mathrm{Supp}\,c \cup \mathrm{Int}\,\mathrm{Supp}\,c' = \emptyset$, then we obtain the Cartesian product $c \times c'$ which is an $(n+n')$-cycle.

Corollary 2.2 *Let (a_i, b_i), f_i and f be as in Proposition 2.1. Let c be an n-cycle of $B\mathbf{G}_c^{L,\mathcal{V}_1}(\mathbf{R})$ such that $\mathrm{Int}\,\mathrm{Supp}\,c \cap \bigcup_i (a_i, b_i) = \emptyset$. Then the $(n+1)$-cycle $(f) \times c$ is homologous to zero.*

In the rest of this section, we prepare notations and give several simple consequences.

When f and g are commuting homeomorphisms of $\mathbf{R}$ of class $C^{L,\mathcal{V}_1}$, we write the homology class of the 2-cycle $(f) \times (g) = (f, g) - (g, f)$ of $B\mathbf{G}_c^{L,\mathcal{V}_1}(\mathbf{R})^\delta$ by $\{f, g\}$. This is represented by the homomorphism $\pi_1(T^2) \cong \mathbf{Z}^2 \longrightarrow \mathbf{G}_c^{L,\mathcal{V}_1}(\mathbf{R})$, which sends the generators to f and g. It is easy to see that $\{f, g\} = -\{g, f\}$ and if f_i $(i = 1, \dots, k)$ commutes with g_j $(j = 1, \dots, \ell)$, then $\{\prod_i f_i, \prod_j g_j\} = \sum_{i,j} \{f_i, g_j\}$.

Let h be a piecewise linear homeomorphism with support in $[-2, 2]$ such that $h(x) = (x+2)/2$ for $x \in [-1, 2]$. Put $U = (-2/3, 2/3)$. Then $h^j(U)$ are disjoint.

For a real number u such that $|u| \leq 1$, let f_u be an elementary PL homeomorphism of $\mathbf{R}$ with support in $[-1/2^4, 0]$ such that $\log f_u'(x) = u$ for $x \in [-1/2^6, 0)$ and $||| \log f_u' |||_1 \leq 4|u|$. In the same way, for a real number v such that $|v| \leq 1$, let g_v be an elementary PL homeomorphism of $\mathbf{R}$ with support in $[0, 1/2^4]$ such that $\log g_v'(x) = v$ for $x \in (0, 1/2^6]$ and $||| \log g_v' |||_1 \leq 4|v|$.

Let L denote the linear map defined by $L(x) = 2^{-1}x$. For a real number w, let T^w denote the translation defined by $T^w(x) = x + w$. For a real number w such that $|w| \leq 2^{-2}$ and a sequence of real numbers $\{c_j\}_{j=0,1,2,\cdots}$ such that $|c_j| \leq 2^{-2j}$, let $F_{(w;c_0,c_1,c_2,\cdots)}$ and $G_{(w;c_0,c_1,c_2,\cdots)}$ be the homeomorphisms defined by

$$F_{(w;c_0,c_1,c_2,\cdots)} = \prod_{j=0}^{\infty} h^j T^w L^j f_{c_j} L^{-j} T^{-w} h^{-j} \quad \text{and}$$
$$G_{(w;c_0,c_1,c_2,\cdots)} = \prod_{j=0}^{\infty} h^j T^w L^j g_{c_j} L^{-j} T^{-w} h^{-j},$$

respectively. Since the support of $T^w L^j f_{c_j} L^{-j} T^{-w}$ is contained in $[-1/2^{4+j} + w, w] \subset U$,

$$||| \log(h^j T^w L^j f_{c_j} L^{-j} T^{-w} h^{-j})' |||_1 = ||| \log(f_{c_j})' |||_1 \leq 2^{-2j+2}$$

and $\sum 2^{-2j+2} < \infty$, $F_{(w;c_0,c_1,c_2,\cdots)}$ is an element of $\mathbf{G}_c^{L,\mathcal{V}_1}(\mathbf{R})$. In a similar way, $G_{(w;c_0,c_1,c_2,\cdots)}$ is also an element of $\mathbf{G}_c^{L,\mathcal{V}_1}(\mathbf{R})$.

We show the following proposition which is similar to that in [18].

Proposition 2.3 $\{T^w f_1 T^{-w}, T^w g_{\hat{c}} T^{-w}\} = \{F_{(w;0,\frac{1}{2^2},\frac{1}{2^4},\cdots)}, G_{(w;0,c_1,c_2,\cdots)}\}$, *where* $\hat{c} = \sum_{j=1}^{\infty} \frac{c_j}{2^{2j}}$.

Proof. Put

$$s_j = 2^{2j} \sum_{i=j+1}^{\infty} \frac{c_i}{2^{2i}}.$$

Then

$$|s_j| \leq 2^{2j} \sum_{i=j+1}^{\infty} 2^{-4i} \leq \frac{1}{2^{2j}}$$

and s_j satisfies

$$s_0 = \hat{c} \quad \text{and} \quad 2^2 s_j - s_{j+1} = 2^{2j+2} \frac{c_{j+1}}{2^{2j+2}} = c_{j+1}.$$

We compute the second homology class $\{F_{(w;1,\frac{1}{2^2},\frac{1}{2^4},\cdots)}, G_{(w;s_0,s_1,s_2,\cdots)}\}$ in two ways.

First, we have the following lemma.

Lemma 2.4

$$\{F_{(w;1,\frac{1}{2^2},\frac{1}{2^4},\cdots)}, G_{(w;s_0,s_1,s_2,\cdots)}\}$$
$$= 2^2 \{F_{(w;0,\frac{1}{2^2},\frac{1}{2^4},\frac{1}{2^6},\cdots)}, G_{(w;0,s_0,s_1,s_2,\cdots)}\}.$$

Proof. Since $F_{(w;1,\frac{1}{2^2},\frac{1}{2^4},\cdots)}$ and $(F_{(w;\frac{1}{2^2},\frac{1}{2^4},\frac{1}{2^6},\cdots)})^4$ coincide near the points in $\{h^j(w); j = 0, 1, 2, \cdots\}$, the support of

$$F_{(w;1,\frac{1}{2^2},\frac{1}{2^4},\cdots)}(F_{(w;\frac{1}{2^2},\frac{1}{2^4},\frac{1}{2^6},\cdots)})^{-4}$$

is contained in $\bigcup_j h^j([-1/2^{4+j}+w, -1/2^{6+j}+w])$. Since it is a product of 5 elementary PL homeomorphisms on each $h^j([-1/2^{4+j}+w, -1/2^{6+j}+w])$. Thus by Proposition 2.1, it is written as a product of commutators with support in $h^j([-1/2^{3+j}+w, w])$, and as in Corollary 2.2, we have

$$\{F_{(w;1,\frac{1}{2^2},\frac{1}{2^4},\cdots)}(F_{(w;\frac{1}{2^2},\frac{1}{2^4},\frac{1}{2^6},\cdots)})^{-4}, G_{(w;s_0,s_1,s_2,\cdots)}\} = 0.$$

That is,

$$\{F_{(w;1,\frac{1}{2^2},\frac{1}{2^4},\cdots)}, G_{(w;s_0,s_1,s_2,\cdots)}\} = 2^2\{F_{(w;\frac{1}{2^2},\frac{1}{2^4},\frac{1}{2^6}\cdots)}, G_{(w;s_0,s_1,s_2,\cdots)}\}.$$

Since $F_{(w;\frac{1}{2^2},\frac{1}{2^4},\cdots)}$ and $h^{-1}F_{(w;0,\frac{1}{2^2},\frac{1}{2^4},\frac{1}{2^6},\cdots)}h$ coincides near the points in $\{h^j(w); j = 0, 1, 2, \cdots\}$. Hence the support of

$$F_{(w;\frac{1}{2^2},\frac{1}{2^4},\cdots)}(h^{-1}F_{(w;0,\frac{1}{2^2},\frac{1}{2^4},\frac{1}{2^6},\cdots)}h)^{-1}$$

is contained in $\bigcup_{j=0}^{\infty} h^j([-1/2^{4+j}+w, -1/2^{7+j}+w])$ and it is a product of 2 elementary PL homeomorphisms on each $h^j([-1/2^{4+j}+w, -1/2^{7+j}+w])$. Again by Proposition 2.1, it is written as a product of commutators with support in $h^j([-1/2^{3+j}+w, w])$, and as in Corollary 2.2, we have

$$\{F_{(w;\frac{1}{2^2},\frac{1}{2^4},\cdots)}(h^{-1}F_{(w;0,\frac{1}{2^2},\frac{1}{2^4},\frac{1}{2^6},\cdots)}h)^{-1}, G_{(w;s_0,s_1,s_2,\cdots)}\} = 0.$$

Thus

$$\begin{aligned}2^2\{F_{(w;\frac{1}{2^2},\frac{1}{2^4},\frac{1}{2^6}\cdots)}, G_{(w;s_0,s_1,s_2,\cdots)}\}\\ = 2^2\{h^{-1}F_{(w;0,\frac{1}{2^2},\frac{1}{2^4},\frac{1}{2^6}\cdots)}h, G_{(w;s_0,s_1,s_2,\cdots)}\}.\end{aligned}$$

By a similar reason for

$$G_{(w;s_0,s_1,s_2,\cdots)}(h^{-1}G_{(w;0,s_0,s_1,s_2,\cdots)}h)^{-1},$$

the right-hand-side is equal to the following.

$$\begin{aligned}2^2\{h^{-1}F_{(w;0,\frac{1}{2^2},\frac{1}{2^4},\frac{1}{2^6}\cdots)}h, h^{-1}G_{(w;0,s_0,s_1,s_2,\cdots)}h\}\\ = 2^2\{F_{(w;0,\frac{1}{2^2},\frac{1}{2^4},\frac{1}{2^6}\cdots)}, G_{(w;0,s_0,s_1,s_2,\cdots)}\}.\end{aligned}$$

Here the equality holds because the conjugation acts as the identity on the homology. □

Secondly, we have the following lemma.

Lemma 2.5

$$\begin{aligned}&\{F_{(w;1,\frac{1}{2^2},\frac{1}{2^4},\cdots)}, G_{(w;s_0,s_1,s_2,\cdots)}\}\\&\quad = \{T^w f_1 T^{-w}, T^w g_{s_0} T^{-w}\} + \{F_{(w;0,\frac{1}{2^2},\frac{1}{2^4},\cdots)}, G_{(w;0,s_1,s_2,\cdots)}\}.\end{aligned}$$

Proof. Since

$$\begin{aligned}F_{(w;1,\frac{1}{2^2},\frac{1}{2^4},\cdots)} &= T^w f_1 T^{-w} F_{(w;0,\frac{1}{2^2},\frac{1}{2^4},\cdots)} \quad \text{and}\\ G_{(w;s_0,s_1,s_2,\cdots)} &= T^w g_{s_0} T^{-w} G_{(w;0,s_1,s_2,\cdots)},\end{aligned}$$

we have

$$\begin{aligned}&\{F_{(w;1,\frac{1}{2^2},\frac{1}{2^4},\cdots)}, G_{(w;s_0,s_1,s_2,\cdots)}\}\\&\quad = \{T^w f_1 T^{-w}, T^w g_{s_0} T^{-w}\} + \{T^w f_1 T^{-w}, G_{(w;0,s_1,s_2,\cdots)}\}\\&\qquad + \{F_{(w;0,\frac{1}{2^2},\frac{1}{2^4},\cdots)}, T^w g_{s_0} T^{-w}\} + \{F_{(w;0,\frac{1}{2^2},\frac{1}{2^4},\cdots)}, G_{(w;0,s_1,s_2,\cdots)}\}.\end{aligned}$$

Here by the perfectness of $PL_c(\mathbf{R})$ the second and the third summands are zero. □

By Lemmas 2.4 and 2.5,

$$\begin{aligned}&\{T^w f_1 T^{-w}, T^w g_{\hat{c}} T^{-w}\}\\&\quad = \{T^w f_1 T^{-w}, T^w g_{s_0} T^{-w}\}\\&\quad = 2^2\{F_{(w;0,\frac{1}{2^2},\frac{1}{2^4},\frac{1}{2^6}\cdots)}, G_{(w;0,s_0,s_1,s_2,\cdots)}\}\\&\qquad - \{F_{(w;0,\frac{1}{2^2},\frac{1}{2^4},\cdots)}, G_{(w;0,s_1,s_2,\cdots)}\}\\&\quad = \{F_{(w;0,\frac{1}{2^2},\frac{1}{2^4},\frac{1}{2^6}\cdots)}, G_{(w;0,2^2s_0-s_1,2^2s_1-s_2,2^2s_2-s_3,\cdots)}\}\\&\quad = \{F_{(w;0,\frac{1}{2^2},\frac{1}{2^4},\frac{1}{2^6}\cdots)}, G_{(w;0,c_1,c_2,c_3,\cdots)}\}.\end{aligned}$$

Here the third equality holds because the support of

$$(G_{(w;0,s_0,s_1,s_2,\cdots)})^4 (G_{(w;0,s_1,s_2,\cdots)})^{-1} (G_{(w;0,2^2s_0-s_1,2^2s_1-s_2,2^2s_2-s_3,\cdots)})^{-1}$$

is contained in $\bigcup_{j=1}^{\infty} h^j([-1/2^{4+j}+w, -1/2^{6+j}+w])$ and it is a product of 6 elementary PL homeomorphisms on each $h^j([-1/2^{4+j}+w, -1/2^{6+j}+w])$.

Thus we have proved Proposition 2.3. $\square$

3. Commutativity of the $*$-product

In this section, we show that the $*$-product in $H_*(B\mathbf{G}_c^{L,\mathcal{V}_1}(\mathbf{R})^\delta;\mathbf{Z})$ of elements of $H_2(BPL_c(\mathbf{R})^\delta;\mathbf{Z})$ is commutative.

When f_1, f_2, f_3 and f_4 are commuting elements of $\mathbf{G}_c^{L,\mathcal{V}_1}(\mathbf{R})$, we write the homology class of the 4-cycle

$$(f_1)\times(f_2)\times(f_3)\times(f_4)=\sum_\sigma \operatorname{sign}(\sigma)(f_{\sigma(1)},f_{\sigma(2)},f_{\sigma(3)},f_{\sigma(4)})$$

of $B\mathbf{G}_c^{L,\mathcal{V}_1}(\mathbf{R})^\delta$ by $\{f_1,f_2,f_3,f_4\}$. This is represented by the homomorphism $\pi_1(T^4)\cong\mathbf{Z}^4\longrightarrow\mathbf{G}_c^{L,\mathcal{V}_1}(\mathbf{R})$, which sends the generators to f_1, f_2, f_3 and f_4.

We know that the image of any element of $H_2(BPL_c(\mathbf{R})^\delta;\mathbf{Z})$ in $H_2(B\mathbf{G}_c^{L,\mathcal{V}_1}(\mathbf{R})^\delta;\mathbf{Z})$ is written as $\{f_1,g_a\}$ where a is the Godbillon-Vey invariant. The commutativity of the $*$-product on the image of $H_2(BPL_c(\mathbf{R})^\delta;\mathbf{Z})$ in $H_*(B\mathbf{G}_c^{L,\mathcal{V}_1}(\mathbf{R})^\delta;\mathbf{Z})$ is precisely the following proposition.

Proposition 3.1 *For any real numbers a and b,*

$$\begin{aligned}&\{T^{-1/2^2}f_1T^{1/2^2},T^{-1/2^2}g_aT^{1/2^2},T^{1/2^2}f_1T^{-1/2^2}T^{1/2^2}g_bT^{-1/2^2}\}\\&\quad=\{T^{-1/2^2}f_1T^{1/2^2},T^{-1/2^2}g_bT^{1/2^2},T^{1/2^2}f_1T^{-1/2^2},T^{1/2^2}g_aT^{-1/2^2}\}.\end{aligned}$$

Here is a sequence of remarks on this proposition. First, the proposition is easily proved if a/b is a rational number. Secondly, it is sufficient to prove the proposition when $0<a<1$ and $0<b<1$. Thirdly, since we know that $\{f_1,g_b\}=\{f_{b^{1/2}},g_{b^{1/2}}\}$, it is sufficient to prove the proposition when $0<a<1$ and $b=1$. This is because we can multiply the unit by $b^{1/2}$ and the argument for $b=1$ is translated to the general case. Finally, it is enough to show the proposition for a which is written as

$$a=\sum_{i=1}^\infty\frac{a_i}{2^{8i}}\quad(a_i\in\{0,1\}).$$

This is because any real number a can be written as a sum of 2^8-1 real numbers of the type above, considering the 2^8-adic expansion of a.

We begin the proof of the proposition by computing the 4-dimensional homology class

$$\{F_{(-\frac{1}{2^2};1,\frac{1}{2^2},\frac{1}{2^4},\dots)}, G_{(-\frac{1}{2^2};p_0,p_1,p_2,\dots)}, F_{(\frac{1}{2^2};1,\frac{1}{2^2},\frac{1}{2^4},\dots)}, G_{(\frac{1}{2^2};q_0,q_1,q_2,\dots)}\}$$

in two ways, where $|p_i| \leq 2^{-2i}$ and $|q_j| \leq 2^{-2j}$.

First as in the proof of Lemma 2.4, we have the following equality.

$$\begin{aligned}
&\{F_{(-\frac{1}{2^2};1,\frac{1}{2^2},\frac{1}{2^4},\dots)}, G_{(-\frac{1}{2^2};p_0,p_1,p_2,\dots)}, F_{(\frac{1}{2^2};1,\frac{1}{2^2},\frac{1}{2^4},\dots)}, G_{(\frac{1}{2^2};q_0,q_1,q_2,\dots)}\} \\
&= 2^8\{h^{-1}F_{(-\frac{1}{2^2};0,\frac{1}{2^2},\frac{1}{2^4},\frac{1}{2^6},\dots)}h, h^{-1}G_{(-\frac{1}{2^2};0,\frac{p_0}{2^2},\frac{p_1}{2^2},\frac{p_2}{2^2},\dots)}h, \\
&\qquad h^{-1}F_{(\frac{1}{2^2};0,\frac{1}{2^2},\frac{1}{2^4},\frac{1}{2^6},\dots)}h, h^{-1}G_{(\frac{1}{2^2};0,\frac{q_0}{2^2},\frac{q_1}{2^2},\frac{q_2}{2^2},\dots)}h\} \\
&= 2^8\{F_{(-\frac{1}{2^2};0,\frac{1}{2^2},\frac{1}{2^4},\frac{1}{2^6},\dots)}, G_{(-\frac{1}{2^2};0,\frac{p_0}{2^2},\frac{p_1}{2^2},\frac{p_2}{2^2},\dots)}, \\
&\qquad F_{(\frac{1}{2^2};0,\frac{1}{2^2},\frac{1}{2^4},\frac{1}{2^6},\dots)}, G_{(\frac{1}{2^2};0,\frac{q_0}{2^2},\frac{q_1}{2^2},\frac{q_2}{2^2},\dots)}\}.
\end{aligned}$$

Secondly, by decomposing the homeomorphisms into the parts supported on U and $(2/3, 2]$ and using the perfectness of $PL_c(\mathbf{R})$ as in the proof of Lemma 2.5, we have the following equality.

$$\begin{aligned}
&\{F_{(-\frac{1}{2^2};1,\frac{1}{2^2},\frac{1}{2^4},\dots)}, G_{(-\frac{1}{2^2};p_0,p_1,p_2,\dots)}, F_{(\frac{1}{2^2};1,\frac{1}{2^2},\frac{1}{2^4},\dots)}, G_{(\frac{1}{2^2};q_0,q_1,q_2,\dots)}\} \\
&= \{T^{-1/2^2}f_1T^{1/2^2}F_{(-\frac{1}{2^2};0,\frac{1}{2^2},\frac{1}{2^4},\dots)}, T^{-1/2^2}g_{p_0}T^{1/2^2}G_{(-\frac{1}{2^2};0,p_1,p_2,\dots)}, \\
&\qquad T^{1/2^2}f_1T^{-1/2^2}F_{(\frac{1}{2^2};0,\frac{1}{2^2},\frac{1}{2^4},\dots)}, T^{1/2^2}g_{q_0}T^{-1/2^2}G_{(\frac{1}{2^2};0,q_1,q_2,\dots)}\} \\
&= \{T^{-1/2^2}f_1T^{1/2^2}, T^{-1/2^2}g_{p_0}T^{1/2^2}, T^{1/2^2}f_1T^{-1/2^2}, T^{1/2^2}g_{q_0}T^{-1/2^2}\} \\
&\quad + \{T^{-1/2^2}f_1T^{1/2^2}, T^{-1/2^2}g_{p_0}T^{1/2^2}, F_{(\frac{1}{2^2};0,\frac{1}{2^2},\frac{1}{2^4},\dots)}, G_{(\frac{1}{2^2};0,q_1,q_2,\dots)}\} \\
&\quad + \{F_{(-\frac{1}{2^2};0,\frac{1}{2^2},\frac{1}{2^4},\dots)}, G_{(-\frac{1}{2^2};0,p_1,p_2,\dots)}, \\
&\qquad T^{1/2^2}f_1T^{-1/2^2}, T^{1/2^2}g_{q_0}T^{-1/2^2}\} \\
&\quad + \{F_{(-\frac{1}{2^2};0,\frac{1}{2^2},\frac{1}{2^4},\dots)}, G_{(-\frac{1}{2^2};0,p_1,p_2,\dots)}, F_{(\frac{1}{2^2};0,\frac{1}{2^2},\frac{1}{2^4},\dots)}, G_{(\frac{1}{2^2};0,q_1,q_2,\dots)}\}.
\end{aligned}$$

Here by using Proposition 2.3 and the fact that conjugation acts as the identity, the second summand and the third summand are computed as follows.

$$\{T^{-1/2^2}f_1T^{1/2^2}, T^{-1/2^2}g_{p_0}T^{1/2^2}, F_{(\frac{1}{2^2};0,\frac{1}{2^2},\frac{1}{2^4},\dots)}, G_{(\frac{1}{2^2};0,q_1,q_2,\dots)}\}$$

$$
\begin{aligned}
&= \{T^{-3}f_1T^3, T^{-3}g_{p_0}T^3, F_{(\frac{1}{2^2};0,\frac{1}{2^2},\frac{1}{2^4},\cdots)}, G_{(\frac{1}{2^2};0,q_1,q_2,\cdots)}\}\\
&= \{T^{-3}f_1T^3, T^{-3}g_{p_0}T^3, T^{1/2^2}f_1T^{-1/2^2}, T^{1/2^2}g_{\hat{q}}T^{-1/2^2}\}\\
&= \{T^{-1/2^2}f_1T^{1/2^2}, T^{-1/2^2}g_{p_0}T^{1/2^2}, T^{1/2^2}f_1T^{-1/2^2}, T^{1/2^2}g_{\hat{q}}T^{-1/2^2}\},
\end{aligned}
$$

where $\hat{q} = \sum_{j=1}^{\infty} \frac{q_j}{2^{2j}}$.

$$
\begin{aligned}
&\{F_{(-\frac{1}{2^2};0,\frac{1}{2^2},\frac{1}{2^4},\cdots)}, G_{(-\frac{1}{2^2};0,p_1,p_2,\cdots)}, T^{1/2^2}f_1T^{-1/2^2}, T^{1/2^2}g_{q_0}T^{-1/2^2}\}\\
&\quad= \{F_{(-\frac{1}{2^2};0,\frac{1}{2^2},\frac{1}{2^4},\cdots)}, G_{(-\frac{1}{2^2};0,p_1,p_2,\cdots)}, T^{-3}f_1T^3, T^{-3}g_{q_0}T^3\}\\
&\quad= \{T^{-1/2^2}f_1T^{1/2^2}, T^{-1/2^2}g_{\hat{p}}T^{1/2^2}, T^{-3}f_1T^3, T^{-3}g_{q_0}T^3\}\\
&\quad= \{T^{1/2^2}f_1T^{-1/2^2}, T^{1/2^2}g_{\hat{p}}T^{-1/2^2}, T^{-1/2^2}f_1T^{1/2^2}, T^{-1/2^2}g_{q_0}T^{1/2^2}\}\\
&\quad= \{T^{-1/2^2}f_1T^{1/2^2}, T^{-1/2^2}g_{q_0}T^{1/2^2}, T^{1/2^2}f_1T^{-1/2^2}, T^{1/2^2}g_{\hat{p}}T^{-1/2^2}\},
\end{aligned}
$$

where $\hat{p} = \sum_{j=1}^{\infty} \frac{p_j}{2^{2j}}$.

Thus we obtain the following lemma.

Lemma 3.2

$$
\begin{aligned}
&2^8\{F_{(-\frac{1}{2^2};0,\frac{1}{2^2},\frac{1}{2^4},\frac{1}{2^6},\cdots)}, G_{(-\frac{1}{2^2};0,\frac{p_0}{2^2},\frac{p_1}{2^2},\frac{p_2}{2^2},\cdots)},\\
&\qquad\qquad F_{(\frac{1}{2^2};0,\frac{1}{2^2},\frac{1}{2^4},\frac{1}{2^6},\cdots)}, G_{(\frac{1}{2^2};0,\frac{q_0}{2^2},\frac{q_1}{2^2},\frac{q_2}{2^2},\cdots)}\}\\
&\quad - \{F_{(-\frac{1}{2^2};0,\frac{1}{2^2},\frac{1}{2^4},\cdots)}, G_{(-\frac{1}{2^2};0,p_1,p_2,\cdots)},\\
&\qquad\qquad F_{(\frac{1}{2^2};0,\frac{1}{2^2},\frac{1}{2^4},\cdots)}, G_{(\frac{1}{2^2};0,q_1,q_2,\cdots)}\}\\
&= \{T^{-1/2^2}f_1T^{1/2^2}, T^{-1/2^2}g_{p_0}T^{1/2^2}, T^{1/2^2}f_1T^{-1/2^2}, T^{1/2^2}g_{q_0}T^{-1/2^2}\}\\
&\quad + \{T^{-1/2^2}f_1T^{1/2^2}, T^{-1/2^2}g_{p_0}T^{1/2^2}, T^{1/2^2}f_1T^{-1/2^2}, T^{1/2^2}g_{\hat{q}}T^{-1/2^2}\}\\
&\quad + \{T^{-1/2^2}f_1T^{1/2^2}, T^{-1/2^2}g_{q_0}T^{1/2^2}, T^{1/2^2}f_1T^{-1/2^2}, T^{1/2^2}g_{\hat{p}}T^{-1/2^2}\}.
\end{aligned}
$$

We will use this lemma to prove Proposition 3.1. As we remarked before, it is enough to show

$$
\begin{aligned}
&\{T^{-1/2^2}f_1T^{1/2^2}, T^{-1/2^2}g_1T^{1/2^2}, T^{1/2^2}f_1T^{-1/2^2}, T^{1/2^2}g_{c_0}T^{-1/2^2}\}\\
&\quad= \{T^{-1/2^2}f_1T^{1/2^2}, T^{-1/2^2}g_{c_0}T^{1/2^2}, T^{1/2^2}f_1T^{-1/2^2}, T^{1/2^2}g_1T^{-1/2^2}\}
\end{aligned}
$$

for a real number c_0 such that

$$c_0 = \sum_{i=1}^{\infty} \frac{a_i}{2^{8i}} \quad (a_i \in \{0, 1\}).$$

Let c_j be a sequence of real numbers such that $|c_j| \leq 1/2^{2j}$.

Put $p_j = 1/2^{2j}$ and $q_j = c_j$ in Lemma 3.2. Then by the argument as in the proof of Lemma 2.4, we have the following equality.

$$\begin{aligned}
\{&F_{(-\frac{1}{2^2};0,\frac{1}{2^2},\frac{1}{2^4},\frac{1}{2^6},\cdots)}, G_{(-\frac{1}{2^2};0,\frac{1}{2^2},\frac{1}{2^4},\frac{1}{2^6},\cdots)}, \\
&\quad F_{(\frac{1}{2^2};0,\frac{1}{2^2},\frac{1}{2^4},\frac{1}{2^6},\cdots)}, G_{(\frac{1}{2^2};0,2^6c_0-c_1,2^6c_1-c_2,2^6c_3-c_3,\cdots)}\} \\
&= \{T^{-1/2^2} f_1 T^{1/2^2}, T^{-1/2^2} g_1 T^{1/2^2}, T^{1/2^2} f_1 T^{-1/2^2}, T^{1/2^2} g_{c_0} T^{-1/2^2}\} \\
&\quad + \{T^{-1/2^2} f_1 T^{1/2^2}, T^{-1/2^2} g_1 T^{1/2^2}, T^{1/2^2} f_1 T^{-1/2^2}, T^{1/2^2} g_{\hat{c}} T^{-1/2^2}\} \\
&\quad + \{T^{-1/2^2} f_1 T^{1/2^2}, T^{-1/2^2} g_{c_0} T^{1/2^2}, \\
&\qquad T^{1/2^2} f_1 T^{-1/2^2}, T^{1/2^2} g_{1/15} T^{-1/2^2}\}.
\end{aligned}$$

On the other hand, put $p_j = c_j$ and $q_j = 1/2^{2j}$, we have the following equality.

$$\begin{aligned}
\{&F_{(-\frac{1}{2^2};0,\frac{1}{2^2},\frac{1}{2^4},\frac{1}{2^6},\cdots)}, G_{(-\frac{1}{2^2};0,2^6c_0-c_1,2^6c_1-c_2,2^6c_3-c_3,\cdots)}, \\
&\quad F_{(\frac{1}{2^2};0,\frac{1}{2^2},\frac{1}{2^4},\frac{1}{2^6},\cdots)}, G_{(\frac{1}{2^2};0,\frac{1}{2^2},\frac{1}{2^4},\frac{1}{2^6},\cdots)}\} \\
&= \{T^{-1/2^2} f_1 T^{1/2^2}, T^{-1/2^2} g_{c_0} T^{1/2^2}, T^{1/2^2} f_1 T^{-1/2^2}, T^{1/2^2} g_1 T^{-1/2^2}\} \\
&\quad + \{T^{-1/2^2} f_1 T^{1/2^2}, T^{-1/2^2} g_{c_0} T^{1/2^2}, \\
&\qquad T^{1/2^2} f_1 T^{-1/2^2}, T^{1/2^2} g_{1/15} T^{-1/2^2}\} \\
&\quad + \{T^{-1/2^2} f_1 T^{1/2^2}, T^{-1/2^2} g_1 T^{1/2^2}, T^{1/2^2} f_1 T^{-1/2^2}, T^{1/2^2} g_{\hat{c}} T^{-1/2^2}\}.
\end{aligned}$$

Now put

$$c_j = 2^{6j} \sum_{i=j+1}^{\infty} \frac{a_i}{2^{8i}},$$

then we have

$$2^6 c_{j-1} - c_j = 2^{6j} \frac{a_j}{2^{8j}} = \frac{a_j}{2^{2j}}.$$

By the above equalities, we have the following equality.

$$\begin{aligned}
&\{T^{-1/2^2} f_1 T^{1/2^2}, T^{-1/2^2} g_1 T^{1/2^2}, T^{1/2^2} f_1 T^{-1/2^2}, T^{1/2^2} g_{c_0} T^{-1/2^2}\}\\
&\quad - \{T^{-1/2^2} f_1 T^{1/2^2}, T^{-1/2^2} g_{c_0} T^{1/2^2}, T^{1/2^2} f_1 T^{-1/2^2}, T^{1/2^2} g_1 T^{-1/2^2}\}\\
&\quad = \{F_{(-\frac{1}{2^2};0,\frac{1}{2^2},\frac{1}{2^4},\frac{1}{2^6},\dots)}, G_{(-\frac{1}{2^2};0,\frac{1}{2^2},\frac{1}{2^4},\frac{1}{2^6},\dots)},\\
&\qquad\qquad F_{(\frac{1}{2^2};0,\frac{1}{2^2},\frac{1}{2^4},\frac{1}{2^6},\dots)}, G_{(\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)}\}\\
&\qquad - \{F_{(-\frac{1}{2^2};0,\frac{1}{2^2},\frac{1}{2^4},\frac{1}{2^6},\dots)}, G_{(-\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)},\\
&\qquad\qquad F_{(\frac{1}{2^2};0,\frac{1}{2^2},\frac{1}{2^4},\frac{1}{2^6},\dots)}, G_{(\frac{1}{2^2};0,\frac{1}{2^2},\frac{1}{2^4},\frac{1}{2^6},\dots)}\}.
\end{aligned}$$

Since

$$G_{(\pm\frac{1}{2^2};0,\frac{1}{2^2},\frac{1}{2^4},\frac{1}{2^6},\dots)} = G_{(\pm\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\frac{1-a_3}{2^6},\dots)} G_{(\pm\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)},$$

the right-hand-side is equal to

$$\begin{aligned}
&\{F_{(-\frac{1}{2^2};0,\frac{1}{2^2},\frac{1}{2^4},\frac{1}{2^6},\dots)}, G_{(-\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\frac{1-a_3}{2^6},\dots)},\\
&\qquad F_{(\frac{1}{2^2};0,\frac{1}{2^2},\frac{1}{2^4},\frac{1}{2^6},\dots)}, G_{(\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)}\}\\
&\quad - \{F_{(-\frac{1}{2^2};0,\frac{1}{2^2},\frac{1}{2^4},\frac{1}{2^6},\dots)}, G_{(-\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)},\\
&\qquad F_{(\frac{1}{2^2};0,\frac{1}{2^2},\frac{1}{2^4},\frac{1}{2^6},\dots)}, G_{(\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\frac{1-a_3}{2^6},\dots)}\}.
\end{aligned}$$

By Corollary 2.2, the right hand side is equal to the following.

$$\begin{aligned}
&\{F_{(-\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\frac{1-a_3}{2^6},\dots)}, G_{(-\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\frac{1-a_3}{2^6},\dots)},\\
&\qquad F_{(\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)}, G_{(\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)}\}\\
&\quad - \{F_{(-\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)}, G_{(-\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)},\\
&\qquad F_{(\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\frac{1-a_3}{2^6},\dots)}, G_{(\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\frac{1-a_3}{2^6},\dots)}\}\\
&= \{F_{(-\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\frac{1-a_3}{2^6},\dots)}, G_{(-\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\frac{1-a_3}{2^6},\dots)},\\
&\qquad F_{(\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)}, G_{(\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)}\}\\
&\quad - \{F_{(\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\frac{1-a_3}{2^6},\dots)}, G_{(\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\frac{1-a_3}{2^6},\dots)},\\
&\qquad F_{(-\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)}, G_{(-\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)},\}.
\end{aligned}$$

Thus we have shown the following lemma.

Lemma 3.3

$$\begin{aligned}
&\{T^{-1/2^2} f_1 T^{1/2^2}, T^{-1/2^2} g_1 T^{1/2^2}, T^{1/2^2} f_1 T^{-1/2^2}, T^{1/2^2} g_{c_0} T^{-1/2^2}\} \\
&\quad - \{T^{-1/2^2} f_1 T^{1/2^2}, T^{-1/2^2} g_{c_0} T^{1/2^2}, T^{1/2^2} f_1 T^{-1/2^2}, T^{1/2^2} g_1 T^{-1/2^2}\} \\
&= \{F_{(-\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\frac{1-a_3}{2^6},\dots)}, G_{(-\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\frac{1-a_3}{2^6},\dots)}, \\
&\qquad F_{(\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)}, G_{(\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)}\} \\
&\quad - \{F_{(\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\frac{1-a_3}{2^6},\dots)}, G_{(\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\frac{1-a_3}{2^6},\dots)}, \\
&\qquad F_{(-\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)}, G_{(-\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)}\}.
\end{aligned}$$

We are going to show the following lemma which together with the previous lemma implies Proposition 3.1, hence our main theorem.

Lemma 3.4

$$\begin{aligned}
&\{F_{(-\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\frac{1-a_3}{2^6},\dots)}, G_{(-\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\frac{1-a_3}{2^6},\dots)}, \\
&\qquad F_{(\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)}, G_{(\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)}\} \\
&= \{F_{(\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\frac{1-a_3}{2^6},\dots)}, G_{(\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\frac{1-a_3}{2^6},\dots)}, \\
&\qquad F_{(-\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)}, G_{(-\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)}\}.
\end{aligned}$$

Proof. The restrictions of the 4-cycles

$$\begin{aligned}
&(F_{(-\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\frac{1-a_3}{2^6},\dots)}) \times (G_{(-\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\frac{1-a_3}{2^6},\dots)}) \\
&\qquad \times (F_{(\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)}) \times (G_{(\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)}) \quad \text{and} \\
&(F_{(\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\frac{1-a_3}{2^6},\dots)}) \times (G_{(\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\frac{1-a_3}{2^6},\dots)}) \\
&\qquad \times (F_{(-\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)}) \times (G_{(-\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)})
\end{aligned}$$

to $h^j(U)$ are degenerate chains which differ by the conjugation by a translation by $\pm 1/2^{j+1}$, the sign depending on whether $a_j = 0$ or 1.

For the sequence $\{a_j\}_{j=0,1,2,\dots}$ such that $a_j \in \{0,1\}$, let $\widetilde{F}_{(\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)}$

and $\widetilde{G}_{(\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)}$ be the homeomorphisms of $\mathbf{R}$ defined by

$$\widetilde{F}_{(\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)} = \prod_{j=1}^{\infty} h^j \Big(\prod_{i=-2^j+1}^{2^j-1} T^{-i/2^{j+2}} L^j f_{a_j/2^{2j}} L^{-j} T^{i/2^{j+2}} \Big) h^{-j}$$

and

$$\widetilde{G}_{(\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)} = \prod_{j=1}^{\infty} h^j \Big(\prod_{i=-2^j+1}^{2^j-1} T^{-i/2^{j+2}} L^j g_{a_j/2^{2j}} L^{-j} T^{i/2^{j+2}} \Big) h^{-j},$$

respectively. Then

$$\begin{aligned} ||| \log(\widetilde{F}_{(\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)})' |||_1 &\leq \sum_j \sum_i ||| \log(f_{a_j/2^{2j}})' |||_1 \\ &\leq \sum_j 2 \cdot 2^j \cdot 4 \cdot 2^{-2j} < \infty \end{aligned}$$

and $\widetilde{F}_{(\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)}$ is an element of $\mathbf{G}_c^{L,\nu_1}(\mathbf{R})$. In a similar way, so is $\widetilde{G}_{(\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)}$.

For a real number w such that $|w| \leq 1/2^3$, let t_w denote a homeomorphism of $\mathbf{R}$ satisfying the following conditions:

the support of t_w is contained in $U = (-2/3, 2/3)$,
$t_w(x) = x + w$ for $x \in [-3/8, 3/8]$, and
$||| \log(t_w)' |||_1 \leq 4|w|$.

Now let $H_{(\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)}$ be a homeomorphism defined by

$$H_{(\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)} = \prod_{j=1}^{\infty} h^j t_{a_j/2^{j+2}} h^{-j}.$$

Then we have

$$\begin{aligned} &H_{(\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)} \big(F_{(-\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)} \widetilde{F}_{(\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)}\big) \big(H_{(\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)}\big)^{-1} \\ &\quad = \widetilde{F}_{(\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)} F_{(\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)} \quad \text{and} \\ &H_{(\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)} \big(G_{(-\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)} \widetilde{G}_{(\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)}\big) \big(H_{(\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)}\big)^{-1} \\ &\quad = \widetilde{G}_{(\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)} G_{(\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)}. \end{aligned}$$

We also have

$$H_{(\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)} F_{(\pm\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\frac{1-a_3}{2^6},\dots)} (H_{(\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)})^{-1} = F_{(\pm\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\frac{1-a_3}{2^6},\dots)} \quad \text{and}$$

$$H_{(\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)} G_{(\pm\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\frac{1-a_3}{2^6},\dots)} (H_{(\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)})^{-1} = G_{(\pm\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\frac{1-a_3}{2^6},\dots)}.$$

Hence we have the following equality.

$$\begin{aligned}
&\{F_{(-\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\frac{1-a_3}{2^6},\dots)}, G_{(-\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\frac{1-a_3}{2^6},\dots)},\\
&\quad F_{(-\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)} \widetilde{F}_{(\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)}, G_{(-\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)} \widetilde{G}_{(\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)}\}\\
&= \{F_{(-\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\frac{1-a_3}{2^6},\dots)}, G_{(-\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\frac{1-a_3}{2^6},\dots)},\\
&\quad \widetilde{F}_{(\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)} F_{(\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)}, \widetilde{G}_{(\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)} G_{(\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\frac{a_3}{2^6},\dots)}\}
\end{aligned}$$

By the bilinearity, the both sides are decomposed into the sum of four 4-dimensional homology classes.

$$\begin{aligned}
&\{F_{(-\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\dots)}, G_{(-\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\dots)},\\
&\qquad F_{(-\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\dots)}, G_{(-\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\dots)}\}\\
&+\{F_{(-\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\dots)}, G_{(-\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\dots)}, F_{(-\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\dots)}, \widetilde{G}_{(\frac{a_1}{2^2},\frac{a_2}{2^4},\dots)}\}\\
&+\{F_{(-\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\dots)}, G_{(-\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\dots)}, \widetilde{F}_{(\frac{a_1}{2^2},\frac{a_2}{2^4},\dots)}, G_{(-\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\dots)}\}\\
&+\{F_{(-\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\dots)}, G_{(-\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\dots)}, \widetilde{F}_{(\frac{a_1}{2^2},\frac{a_2}{2^4},\dots)}, \widetilde{G}_{(\frac{a_1}{2^2},\frac{a_2}{2^4},\dots)}\}\\
&= \{F_{(-\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\dots)}, G_{(-\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\dots)},\\
&\qquad F_{(\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\dots)}, G_{(\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\dots)}\}\\
&+\{F_{(-\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\dots)}, G_{(-\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\dots)}, F_{(\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\dots)}, \widetilde{G}_{(\frac{a_1}{2^2},\frac{a_2}{2^4},\dots)}\}\\
&+\{F_{(-\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\dots)}, G_{(-\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\dots)}, \widetilde{F}_{(\frac{a_1}{2^2},\frac{a_2}{2^4},\dots)}, G_{(\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\dots)}\}\\
&+\{F_{(-\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\dots)}, G_{(-\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\dots)}, \widetilde{F}_{(\frac{a_1}{2^2},\frac{a_2}{2^4},\dots)}, \widetilde{G}_{(\frac{a_1}{2^2},\frac{a_2}{2^4},\dots)}\}
\end{aligned}$$

Since the second terms and the third terms of the both sides are zero by Corollary 2.2, and the fourth terms coincide, we obtain the following equality.

$$\begin{aligned}\{&F_{(-\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\dots)}, G_{(-\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\dots)},\\ &\qquad F_{(-\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\dots)}, G_{(-\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\dots)}\}\\ =\{&F_{(-\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\dots)}, G_{(-\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\dots)},\\ &\qquad F_{(\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\dots)}, G_{(\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\dots)}\}.\end{aligned}$$

In a similar way, we have the following equality.

$$\begin{aligned}\{&F_{(-\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\dots)}, G_{(-\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\dots)},\\ &\qquad F_{(-\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\dots)}, G_{(-\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\dots)}\}\\ =\{&F_{(\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\dots)}, G_{(\frac{1}{2^2};0,\frac{1-a_1}{2^2},\frac{1-a_2}{2^4},\dots)},\\ &\qquad F_{(-\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\dots)}, G_{(-\frac{1}{2^2};0,\frac{a_1}{2^2},\frac{a_2}{2^4},\dots)}\}.\end{aligned}$$

These equalities show Lemma 3.4. □

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