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On the basis of twisted de Rham cohomology

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Abstract. The study of logarithmic form is essential to compute the cohomology group. First we will show the condition to represent (homogeneous) logarithmic $(n-1)$-forms by the logarithmic forms of type $\frac{dP}{P}$. By using this result, we can choose a basis for the twisted rational de Rham cohomology.

Key words: logarithmic forms.

Introduction

Let $D_j \subset \mathbb{C}^n$, $1 \leq j \leq m$ be a divisor defined by a polynomial $P_j(u) \in \mathbb{C}[u_1, \ldots, u_n]$, $1 \leq j \leq m$, and set D as the union of these. Define a covariant derivative

$$\nabla_\omega = d + \sum_{j=1}^{m} \frac{dP_j}{P_j} \wedge$$

on $M := \mathbb{C}^n - D$. The kernel of ∇_ω is the set of ∇_ω-horizontal sections. It defines a rank one local system $\mathcal{L}_\omega$. A general theory [KN] provided a nice interpretation of several integral representation of special function by means of duality between de Rham cohomology of ∇_ω and certain twisted cycles. In case that $P_j(u)$, $1 \leq j \leq m$, are all linear and in general position, [A1], [K] give beautiful representation of a basis for the top-dimensional de Rham cohomology group by logarithmic forms. The purpose of this article is to extend this study to our setting. The goal of this paper is Theorem 7.3.1 which gives a method to find an explicit basis of the top-dimensional twisted de Rham cohomology group. This is a natural generalization of a theorem of [K]. In this paper, we employ the condition Assumption 1.1.1 on $\bar{D} := \{\bar{P}_1 \cdots \bar{P}_m = 0\}$ and Assumption 1.2.1 on D as in [KN]. It is shown in [K] that there exists a gap, which is essential to our study, between the space of Saito's logarithmic forms $\Omega^p(\log D)$ and the space of ordinary logarithmic forms $\Omega^p\langle\mathcal{D}\rangle$. Let $\Omega^p(*D)$ be the space of rational p-forms with poles along D. The Grothendiek-Deligne comparison theorem asserts that there exists

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a canonical isomorphism

$$H^p(M, \mathcal{L}_\omega) \simeq H^p(\Omega^\cdot(*D), \nabla_\omega)$$

for $p = 0, 1, \ldots$. [K] showed that the natural inclusion $(\Omega^\cdot(\log D), \nabla_\omega) \longrightarrow (\Omega^\cdot(*D), \nabla_\omega)$ induces an isomorphism of the cohomology of these complexes. To know the structure of logarithmic complex, we shall study the gap between Saito's logarithmic complex and ordinary logarithmic complex by using the degree filtration on these complexes, which leads to determination of the structure of $H^n(\Omega^\cdot(\log D), \nabla_\omega)$.

From §4 and on, we consider the case that $P_j(u)$, $1 \leq j \leq m-1$ are linear and $P_m(u)$ is a non-constant polynomial of degree $q+1 > 0$ which satisfy Assumptions 1.1.1 and 1.2.1. Our aim is to find a basis for $H^n(\Omega^\cdot(\log D), \nabla_\omega)$. The process to get a basis is analogous to [AKOT §9]. Let $\mathcal{G} = \{H_1, \ldots, H_{m-1}\}$ be an affine n-arrangement in general position and $H_j = \{P_j = 0\}$, $1 \leq j \leq m-1$. Let $\bar{\mathcal{G}}$ denote the arrangement obtained by parallel translation of each hyperplane of $\mathcal{G}$ to the origin. Let L^+ denote the set of intersections of elements of $\mathcal{G}$ of positive dimension. Given $X \in L^+$, let $\bar{X}$ denote its parallel translate to the origin. Let $n \geq 2$. We say that a polynomial $f \in S$ of positive degree is $\bar{\mathcal{G}}$-transverse if the restriction $\bar{f}|_{\bar{X}}$ of $\bar{f}$ to $\bar{X}$ is not constantly equal to 0 and has no critical point outside the origin for any $X \in L^+$.

Our basis for $H^n(\Omega^\cdot(\log D), \nabla_\omega)$ is a certain subset $\widetilde{B}'$ of $\Omega^n(\log D)$. For $X = H_{j_1} \cap \cdots \cap H_{j_p} \in L^+(\mathcal{G})$, we define $Q_X = P_{j_1} \cdots P_{j_p}$. Let $\Delta(P_m)$ denote the Jacobi ideal of P_m; $\Delta(P_m) := \left(\frac{\partial P_m}{\partial u_1}, \ldots, \frac{\partial P_m}{\partial u_n}\right)$. The quotient $S/\Delta(P_m)$is called the associated Milnor algebra. Let $\bar{X} \in L(\bar{\mathcal{G}})$ be the parallel translate of X containing the origin, $I_{\bar{X}}$ the ideal in S consisting of the polynomials vanishing on $\bar{X}$, and $S_{\bar{X}} := S/I_{\bar{X}}$ the coordinate ring of $\bar{X}$. The Milnor algebra of the restriction $\bar{P}_m|_{\bar{X}}$ of $\bar{P}_m$ to $\bar{X}$ is equal to $S_{\bar{X}}/\Delta(\bar{P}_m|_{\bar{X}})$. There is a natural surjective map $\phi_X : S \to S_{\bar{X}}/\Delta(\bar{P}_m|_{\bar{X}})$. Let $MB_X \subset S$ be a set of homogeneous polynomials on which ϕ_X is injective and so that $\phi_X(MB_X)$ is a basis for $S_{\bar{X}}/\Delta(\bar{P}_m|_{\bar{X}})$. We call MB_X a Milnor basis at X. Let τ be the volume form $du_1 \wedge \cdots \wedge du_n$. Define

$$\widetilde{\mathcal{P}}_X{}' = \left\{ \frac{b\tau}{Q_X P_m} \,\middle|\, b \in MB_X \right\}$$

for $X \in L^+(\mathcal{G})$ and $\widetilde{\mathcal{P}}' = \bigcup_{X \in L^+} \widetilde{\mathcal{P}}_X$. Define

$$\widetilde{\mathcal{NP}}' = \left\{ \frac{\tau}{P_{j_1} \cdots P_{j_{n-1}} P_k \cdots P_{m-1} P_m} \;\middle|\; n \leq k \leq m-1,\ 1 \leq j_1 < \cdots < j_{n-1} \leq k-1 \right\}.$$

Let $\widetilde{\mathcal{B}}' = \widetilde{\mathcal{P}}' \cup \widetilde{\mathcal{NP}}'$. Then we obtain Main theorem:

Main Theorem (Theorem 7.3.1) *Let $\mathcal{G}$ be an affine n-arrangement in general position with $m-1$ hyperplanes defined by $H_j = \{P_j(u) = 0\}$, $1 \leq j \leq m-1$. Let $P_m(u)$ be a $\bar{\mathcal{G}}$-transverse polynomial of degree $q+1 > 0$. Suppose that Assumption* 1.2.1 *holds for P_j, $1 \leq j \leq m$. If $\sum_{j=1}^m l_j \alpha_j \neq l$, $l-1, \ldots$, then*

(1) *The set $\widetilde{\mathcal{B}}'$ gives a basis for $H^n(\Omega^{\cdot}(\log D), \nabla_\omega)$*

(2) $\dim H^n(\Omega^{\cdot}(\log D), \nabla_\omega) = \sum_{i=0}^{n} \binom{m-1}{n-i} q^i.$

Consider the case $q = 0$, that is, $P_j(u)$, $1 \leq j \leq m$ are all linear and the arrangement $\{P_j = 0\}_{1 \leq j \leq m}$ is in general position. we can easily check that $\widetilde{\mathcal{P}}' = \emptyset$. By using the trick of partial fractional decomposition (see [K, pp. 74–75]), we can rewrite $\widetilde{\mathcal{NP}}'$ as

$$\left\{ \frac{dP_{j_1}}{P_{j_1}} \wedge \cdots \wedge \frac{dP_{j_n}}{P_{j_n}} \;\middle|\; 1 \leq j_1 < \cdots < j_n \leq m-1 \right\}$$

which is the same result as shown in [A1], [K].

In §§1–2, we will study the representation theorem for logarithmic forms. In §3, we have a refined result about its representation the one than shown in [KN], and we will show the representation theorem for the Euler form φ_E. From §4 and on, our situation is that $\mathcal{G} = \{H_1, \ldots, H_{m-1}\}$ is in general position and P_m is a $\bar{\mathcal{G}}$-transverse polynomial of degree $q+1$. The argument to get a basis for $H^n(\Omega^{\cdot}(\log D), \nabla_\omega)$ is analogous to the method of [AKOT].

We set

$\mathbb{C}^n$: complex n-dimensional affine space with coordinates $u = (u_1, \ldots, u_n)$

S: the coordinate ring of $\mathbb{C}^n$,

$P_j(u)$, $1 \leq j \leq m$: non-constant polynomials in u,

D_j: the divisor defined by $P_j(u)$,

$\mathcal{D} := \{D_1, \ldots, D_m\}$,

D: the divisor defined by $P := P_1 \cdots P_m$,

$U(u) := \prod_{j=1}^{m} P_j(u)^{\alpha_j}$,

$\omega := dU/U = \sum_{j=1}^{m} \alpha_j \frac{dP_j}{P_j}$,

d: the exterior differentiation on $\mathbb{C}^n$,

$\nabla_\omega := d + \omega\wedge$: the covariant differentiation with respect to ω,

$\Omega^p(\mathbb{C}^n)$: space of polynomial p-forms,

$\Omega^p(*D)$: space of rational p-forms with poles along D,

$\Omega^p(\log D)$: space of logarithmic p-forms with poles along D,

$\Omega^p\langle\mathcal{D}\rangle$: space of logarithmic p-forms with respect to $\mathcal{D}$.

1. The Kita-Noumi representation theorem for logarithmic forms

In this section, we shall study a representation theorem for logarithmic forms in the sense of Saito [S2]. Define

$$\Omega^p(\log D) := \{\varphi \in \Omega^p(*D) \mid P\varphi \in \Omega^p(\mathbb{C}^n) \quad \text{and} \quad dP \wedge \varphi \in \Omega^{p+1}(\mathbb{C}^n)\},$$

which is called the space of *logarithmic p-forms with poles along D*. Define

$$\Omega^p\langle\mathcal{D}\rangle := \bigwedge^p \langle dP_1/P_1, \ldots, dP_m/P_m, du_1, \ldots, du_n\rangle,$$

which is called the space of *logarithmic p-forms with respect to $\mathcal{D}$*. Clearly, $\Omega^p\langle\mathcal{D}\rangle \subset \Omega^p(\log D)$, but the converse is not true in general. The gap is essential in our application and influential for the top cohomology $H^n(\Omega^\cdot(\log D), \nabla_\omega)$.

1.1. Let $\bar{P}_j(u)$, $1 \le j \le m$, be non-constant *homogeneous* polynomials in S. By abuse of notations we denote by $(d\bar{P}_{j_1} \wedge \cdots \wedge d\bar{P}_{j_r}, \bar{P}_{j_1}, \ldots, \bar{P}_{j_r})$ the ideal of S generated by $\bar{P}_{j_1}, \ldots, \bar{P}_{j_r}$ and the minors $\frac{\partial(\bar{P}_{j_1},\ldots,\bar{P}_{j_r})}{\partial(u_{i_1},\ldots,u_{i_r})}$, $1 \le i_1 < \cdots < i_r \le n$ of the Jacobian matrix $\left(\frac{\partial \bar{P}_{j_k}}{\partial u_i}\right)$, $1 \le k \le r$, $1 \le i \le n$. Throughout this paper, we make the following assumption:

Assumption 1.1.1 (1) For $1 \le r \le \min\{m, n-1\}$, the algebraic set defined by the ideal $(d\bar{P}_{j_1} \wedge \cdots \wedge d\bar{P}_{j_r}, \bar{P}_{j_1}, \dots, \bar{P}_{j_r})$ is either empty or the origin.

(2) $\bar{P}_{j_1}, \dots, \bar{P}_{j_s}$ form a regular sequence in S for $1 \leq s \leq \min\{m, n\}$.

Under the assumption above, the following lemma is known (see [K], [KN] for proof):

Lemma 1.1.2 (Representation theorem for logarithmic forms) *Let $\bar{P}_j(u)$, $1 \leq j \leq m$, be polynomials satisfying Assumption* 1.1.1. *Let $0 \leq p \leq n-2$ and $\varphi \in \Omega^p(\log \bar{D})$. Then φ can be written in the form*

$$\varphi = \varphi_0 + \sum_{j=1}^{m} \frac{d\bar{P}_j}{\bar{P}_j} \wedge \varphi_j + \cdots$$

$$\cdots + \sum_{1 \leq j_1 < \cdots < j_p \leq m} \frac{d\bar{P}_{j_1}}{\bar{P}_{j_1}} \wedge \cdots \wedge \frac{d\bar{P}_{j_p}}{\bar{P}_{j_p}} \cdot \varphi_{j_1 \cdots j_p} \tag{1.1.1}$$

where $\varphi_{j_1 \cdots j_\nu} \in \Omega^{p-\nu}(\mathbb{C}^n)$.

Remark. Lemma 1.1.2 implies that $\Omega^p\langle \bar{\mathcal{D}} \rangle = \Omega^p(\log \bar{D})$ for $0 \leq p \leq n-2$.

1.2. Let $P_j(u)$, $1 \leq j \leq m$, be non-constant polynomials in S. We make the following assumption:

Assumption 1.2.1 (1) For $1 \leq r \leq \min\{m, n\}$, the ideal $(dP_{j_1} \wedge \cdots \wedge dP_{j_r}, P_{j_1}, \dots, P_{j_r})$ coincides with S.

(2) $P_{j_1}, \dots, P_{j_s}$ form a regular sequence in S for $1 \leq s \leq \min\{m, n+1\}$ (In case $s = n+1$, we say that $P_{j_1}, \dots, P_{j_{n+1}}$ form a regular sequence when $P_{j_1}, \dots, P_{j_n}$ form a regular sequence and $P_{j_{n+1}}$ is a unit of $S/(P_{j_1}, \dots, P_{j_n})$).

Then we have

Lemma 1.2.2 *We suppose Assumption* 1.2.1. *Let $r + p \leq n$ and let ψ be a polynomial p-form such that*

$$dP_{j_1} \wedge \cdots \wedge dP_{j_r} \wedge \psi \equiv 0 \pmod{P_{j_1}, \dots, P_{j_r}},$$

then

$$\psi \equiv 0 \pmod{dP_{j_1}, \dots, dP_{j_r}, P_{j_1}, \dots, P_{j_r}}.$$

Proof. We set

$I := (dP_{j_1} \wedge \cdots \wedge dP_{j_r}, P_{j_1}, \dots, P_{j_r}),$

$A := S/(P_{j_1}, \ldots, P_{j_r})$,
$\mathfrak{a} := I/(P_{j_1}, \ldots, P_{j_r})$; an ideal of A,
$\pi : S \to A$; the canonical homomorphism.

Then we have $I = \pi^{-1}(\mathfrak{a})$ and hence $S/I \simeq A/\mathfrak{a}$. By Assumption 1.2.1 (1), $I = S$ and hence $\mathfrak{a} = A$. Therefore we may apply [S1, THEOREM (i)] to this case and we obtain our assertion. □

The following lemma is shown in [KN] without proof. We shall give a proof which is useful to understand the proof of Theorem 3.3.1.

Lemma 1.2.3 (Representation theorem for logarithmic forms) *Let $P_j(u)$, $1 \le j \le m$, be polynomials satisfying Assumption 1.2.1. Let $0 \le p \le n$ and ψ be a polynomial p-form such that*

$$dP_j \wedge \psi \equiv 0 \pmod{P_j} \quad \text{for} \quad 1 \le j \le m. \tag{1.2.1}$$

Then ψ can be written in the form

$$\begin{aligned} \psi \;=\; P_1 \cdots P_m \Big\{ \psi_0 + \sum_{j=1}^{m} \frac{dP_j}{P_j} \wedge \psi_j + \cdots \\ \cdots + \sum_{1 \le j_1 < \cdots < j_p \le m} \frac{dP_{j_1}}{P_{j_1}} \wedge \cdots \wedge \frac{dP_{j_p}}{P_{j_p}} \cdot \psi_{j_1 \cdots j_p} \Big\} \end{aligned} \tag{1.2.2}$$

where $\psi_{j_1 \cdots j_\nu} \in \Omega^{p-\nu}(\mathbb{C}^n)$.

Proof. Since this lemma is trivially true for $p = n$ by [S1, THEOREM (i)], we may assume $0 \le p \le n-1$. We shall prove this lemma, just the same way of the proof of Lemma 1.1.3, by induction on the number m of polynomials P_j, $1 \le j \le m$. In case $m = 1$, by (1.2.1), $0 \le p \le n-1$ and Lemma 1.2.2, we have

$$\psi \equiv 0 \pmod{dP_1, P_1}.$$

Hence ψ can be written as

$$\psi = P_1 \psi_0 + dP_1 \wedge \psi_1 = P_1 \left\{ \psi_0 + \frac{dP_1}{P_1} \wedge \psi_1 \right\}$$

where $\psi_0 \in \Omega^p(\mathbb{C}^n)$ and $\psi_1 \in \Omega^{p-1}(\mathbb{C}^n)$, which means that Lemma 1.2.3 holds for $m = 1$. We assume Lemma 1.2.3 is true for m; let $\psi \in \Omega^p(\mathbb{C}^n)$ such that $dP_j \wedge \psi \equiv 0 \pmod{P_j}$ for $1 \le j \le m+1$. By induction, ψ can be written in the form (1.2.2). Let N be be the largest integer for which there exists some $\psi_{j_1\cdots j_N} \neq 0$ in (1.2.2). By induction on N, we shall show that ψ can be written in the form

$$\psi = P_1 \cdots P_m P_{m+1} \left\{ \gamma_0 + \sum_{j=1}^{m+1} \frac{dP_j}{P_j} \wedge \gamma_j + \cdots \right.$$
$$\left. \cdots + \sum_{1 \leq j_1 < \cdots < j_{n-1} \leq m+1} \frac{dP_{j_1}}{P_{j_1}} \wedge \cdots \wedge \frac{dP_{j_p}}{P_{j_p}} \cdot \gamma_{j_1 \cdots j_p} \right\}. \quad (1.2.3)$$

In case $N = 0$, we have $\psi = P_1 \cdots P_m \psi_0$. Since $dP_{m+1} \wedge \psi \equiv 0 \pmod{P_{m+1}}$, we have

$$P_1 \cdots P_m dP_{m+1} \wedge \psi_0 \equiv 0 \pmod{P_{m+1}}.$$

Since $0 \leq p \leq n-1$ and Assumption 1.2.1 (2) implies that $\{P_j, P_{m+1}\}$ form a regular sequence for $1 \leq j \leq m$ and hence $\psi_0 \equiv 0 \pmod{dP_{m+1}, P_{m+1}}$ by Lemma 1.2.2. Thus ψ_0 can be written in the form

$$\psi_0 = P_{m+1}\alpha + dP_{m+1} \wedge \beta, \quad \alpha \in \Omega^p(\mathbb{C}^n), \quad \beta \in \Omega^{p-1}(\mathbb{C}^n),$$

and hence

$$\psi = P_1 \cdots P_{m+1} \left\{ \alpha + \frac{dP_{m+1}}{P_{m+1}} \wedge \beta \right\},$$

which means that Lemma 1.2.3 holds for $N = 0$. Suppose that the statement is true for $N-1$ and we shall show that it holds for N. We consider first the index $(1, \ldots, N)$. Since we have $\psi_{j_1 \cdots j_\nu} = 0$ for $\nu \geq N+1$, from (1.2.2) we have

$$\psi - P_{N+1} \cdots P_m dP_1 \wedge \cdots \wedge dP_N \wedge \psi_{1\cdots N} \equiv 0 \pmod{P_1, \ldots, P_N}.$$

Since $N \leq p \leq n-1$, Assumption 1.2.1 (2) implies that $\{P_1, \ldots, P_N, P_j, P_{m+1}\}$ form a regular sequence for $N+1 \leq j \leq m$. Using $dP_{m+1} \wedge \psi \equiv 0 \pmod{P_{m+1}}$, we have

$$dP_1 \wedge \cdots \wedge dP_N \wedge dP_{m+1} \wedge \psi_{1\cdots N} \equiv 0 \pmod{P_1, \ldots, P_N, P_{m+1}}.$$

Since $\psi_{1\cdots N} \in \Omega^{p-N}(\mathbb{C}^n)$ and $N \leq p \leq n-1$, by Lemma 1.2.2 we obtain

$$\psi_{1\cdots N} \equiv 0 \pmod{dP_1, \ldots, dP_N, dP_{m+1}, P_1, \ldots, P_N, P_{m+1}}$$

and hence we can write $\psi_{1\cdots N}$ as

$$\begin{aligned}\psi_{1\cdots N} &= \sum_{j=1}^{N} P_j \alpha_{1\cdots N;j} + P_{m+1}\alpha_{1\cdots N;m+1} \\ &\quad + \sum_{j=1}^{N} dP_j \wedge \beta_{1\cdots N;j} + dP_{m+1} \wedge \beta_{1\cdots N;m+1}\end{aligned}$$

where $\alpha_{1\cdots N;j} \in \Omega^{p-N}(\mathbb{C}^n)$ and $\beta_{1\cdots N;j} \in \Omega^{p-N-1}(\mathbb{C}^n)$.

By the same reasoning we have

$$\begin{aligned}\psi_{j_1\cdots j_N} &= \sum_{k=1}^{N} P_{j_k} \alpha_{j_1\cdots j_N;j_k} + P_{m+1}\alpha_{j_1\cdots j_N;m+1} \\ &\quad + \sum_{k=1}^{N} dP_{j_k} \wedge \beta_{j_1\cdots j_N;j_k} + dP_{m+1} \wedge \beta_{j_1\cdots j_N;m+1} \qquad (1.2.4)\end{aligned}$$

where $\alpha_{j_1\cdots j_N;j_k} \in \Omega^{p-N}(\mathbb{C}^n)$ and $\beta_{j_1\cdots j_N;j_k} \in \Omega^{p-N-1}(\mathbb{C}^n)$. Substituting (1.2.4) into (1.2.2), we get

$$\begin{aligned}\psi &= P_1\cdots P_m\Big\{\sum_{\nu=0}^{N-1}\sum_{1\le j_1<\cdots<j_\nu\le m}\frac{dP_{j_1}}{P_{j_1}}\wedge\cdots\wedge\frac{dP_{j_\nu}}{P_{j_\nu}}\wedge\psi_{j_1\cdots j_\nu}\Big\} \\ &\quad + P_1\cdots P_m\Big\{\sum_{1\le j_1<\cdots<j_N\le m}\frac{dP_{j_1}}{P_{j_1}}\wedge\cdots\wedge\frac{dP_{j_N}}{P_{j_N}} \\ &\qquad\qquad \wedge\Big(\sum_{k=1}^{N} P_{j_k}\alpha_{j_1\cdots j_N;j_k} + P_{m+1}\alpha_{j_1\cdots j_N;m+1}\Big)\Big\} \\ &\quad + P_1\cdots P_m\Big\{\sum_{1\le j_1<\cdots<j_N\le m}\frac{dP_{j_1}}{P_{j_1}}\wedge\cdots\wedge\frac{dP_{j_N}}{P_{j_N}} \\ &\qquad\qquad \wedge\Big(\sum_{k=1}^{N} dP_{j_k}\beta_{j_1\cdots j_N;j_k} + dP_{m+1}\wedge\beta_{j_1\cdots j_N;m+1}\Big)\Big\}. \end{aligned} \qquad (1.2.5)$$

Notice that the last term in the right hand side in (1.2.5) is equal to

$$P_1\cdots P_m\Big\{\sum_{1\le j_1<\cdots<j_N\le m}\frac{dP_{j_1}}{P_{j_1}}\wedge\cdots\wedge\frac{dP_{j_N}}{P_{j_N}}\wedge dP_{m+1}\wedge\beta_{j_1\cdots j_N;m+1}\Big\}.$$

We set

$$\eta = \psi - P_1 \cdots P_{m+1}\Big\{ \sum_{1\le j_1<\cdots<j_N\le m} \frac{dP_{j_1}}{P_{j_1}} \wedge \cdots \wedge \frac{dP_{j_N}}{P_{j_N}} \wedge \alpha_{j_1\cdots j_N;m+1} + \sum_{1\le j_1<\cdots<j_N\le m} \frac{dP_{j_1}}{P_{j_1}} \wedge \cdots \wedge \frac{dP_{j_N}}{P_{j_N}} \wedge \frac{dP_{m+1}}{P_{m+1}} \wedge \beta_{j_1\cdots j_N;m+1}\Big\}. \tag{1.2.6}$$

By using (1.2.5), we can write η in the form

$$\eta = P_1 \cdots P_m\Big\{ \sum_{\nu=0}^{N-1} \sum_{1\le j_1<\cdots<j_\nu\le m} \frac{dP_{j_1}}{P_{j_1}} \wedge \cdots \wedge \frac{dP_{j_\nu}}{P_{j_\nu}} \wedge \eta_{j_1\cdots j_\nu}\Big\}, \tag{1.2.7}$$

$\eta_{j_1\cdots j_\nu} \in \Omega^{p-\nu}(\mathbb{C}^n)$. By (1.2.6) and $dP_j \wedge \psi \equiv 0 \pmod{P_j}$, $1 \le j \le m+1$, we can easily see that $dP_j \wedge \eta \equiv 0 \pmod{P_j}$ for $1 \le j \le m+1$. Hence in view of (1.2.7), η satisfies the assumption of induction and hence it is written in the form

$$\eta = P_1 \cdots P_m P_{m+1}\Big\{ \tilde\eta_0 + \sum_{j=1}^{m+1} \frac{dP_j}{P_j} \wedge \tilde\eta_j + \cdots + \sum_{1\le j_1<\cdots<j_p\le m+1} \frac{dP_{j_1}}{P_{j_1}} \wedge \cdots \wedge \frac{dP_{j_p}}{P_{j_p}} \wedge \tilde\eta_{j_1\cdots j_p}\Big\}. \tag{1.2.8}$$

Substituting (1.2.8) into (1.2.6), we see that ψ can be written in the form (1.2.3). This completes induction. □

Remark. By Lemma 1.2.3, we have $\Omega^p\langle\mathcal{D}\rangle = \Omega^p(\log D)$ for $0 \le p \le n$.

2. Filtration of $\Omega^p(\log D)$ and $\Omega^p\langle\mathcal{D}\rangle$

Let $P_j(u)$, $1 \le j \le m$, be non-zero polynomials in S of which the homogeneous part of maximal degree is denoted by $\bar P_j(u)$ and we set $l_j := \deg \bar P_j$ for $1 \le j \le m$.

2.1. We define an increasing filtration on the complex $\Omega^{\cdot}(\log D)$. Let $\varphi \in \Omega^p(\log D)$, then by definition, φ is written as $\varphi = \alpha/P$, $\alpha \in \Omega^p(\mathbb{C}^n)$. If each coefficient of α is a polynomial of degree at most $\mu - p$, then we say that the degree of α is $\le \mu$ and write $\deg \alpha \le \mu$. We may formally consider the degree of $1/P$ as $-l := \sum_{j=1}^m l_j$ and say that the degree $\varphi \le \mu$ if $\deg \alpha \le$

$\mu + l$. Let $F_\mu\Omega^p(\log D)$ denote the subspace of $\Omega^p(\log D)$ which consists of the logarithmic p-forms φ of degree $\leq \mu$. Then the family $F_\mu\Omega^p(\log D)$, $\mu \geq -l+p$ defines an increasing filtration F of $\Omega^p(\log D)$.

Next, we define an increasing filtration on the complex $\Omega^{\cdot}\langle\mathcal{D}\rangle$. By definition, a p-form φ in $\Omega^p\langle\mathcal{D}\rangle$ can be written in the form

$$\begin{aligned}\varphi &= \varphi_0 + \sum_{j=1}^{m} \frac{dP_j}{P_j} \wedge \varphi_j + \cdots \\ &\quad \cdots + \sum_{1\leq j_1<\cdots<j_p\leq m} \frac{dP_{j_1}}{P_{j_1}} \wedge \cdots \wedge \frac{dP_{j_p}}{P_{j_p}} \cdot \varphi_{j_1\cdots j_p} \end{aligned} \tag{2.1.1}$$

where $\varphi_{j_1\cdots j_\nu} \in \Omega^{p-\nu}(\mathbb{C}^n)$. Considering $\frac{dP_j}{P_j}$ formally as a p-form of degree 0. Let $G_\mu\Omega^p\langle\mathcal{D}\rangle$ denote the subspace of $\Omega^p\langle\mathcal{D}\rangle$ which consists of the logarithmic p-forms that can be written in the form (2.1.1) with $\varphi_{j_1\cdots j_\nu}$ of degree at most μ for $1 \leq j_1 < \cdots < j_\nu \leq m$ and $0 \leq \nu \leq p$. Then the family $G_\mu\Omega^p\langle\mathcal{D}\rangle$, $\mu \in \mathbb{Z}_{\geq 0}$ defines an increasing filtration G of $\Omega^p\langle\mathcal{D}\rangle$.

Define

$$\begin{aligned}\mathrm{Gr}_\mu^F\Omega^{\cdot}(\log D) &:= F_\mu\Omega^{\cdot}(\log D)/F_{\mu-1}\Omega^{\cdot}(\log D)\\ \mathrm{Gr}_\mu^G\Omega^{\cdot}\langle\mathcal{D}\rangle &:= G_\mu\Omega^{\cdot}\langle\mathcal{D}\rangle/G_{\mu-1}\Omega^{\cdot}\langle\mathcal{D}\rangle\end{aligned}$$

The former (resp. latter) is an associated graded complex to the filtered complex $(F.\Omega^{\cdot}(\log D), \nabla_\omega)$ (resp. $(G.\Omega^{\cdot}\langle\mathcal{D}\rangle, \nabla_\omega)$) equipped with the differential $\mathrm{Gr}_\mu^F(\nabla_\omega)$ (resp. $\mathrm{Gr}_\mu^G(\nabla_\omega)$) induced by ∇_ω.

Let $\bar{P}_j(u)$, $1 \leq j \leq m$, be homogeneous polynomials in S and $\bar{D}_j$ be the divisors defined by $\bar{P}_j$. And we set $\bar{\mathcal{D}} := \{\bar{D}_1, \ldots, \bar{D}_m\}$. Let $\bar{D}$ be the divisor defined by $\bar{P} := \bar{P}_1 \cdots \bar{P}_m$. Then $\Omega^p(\log \bar{D})$ and $\Omega^p\langle\bar{\mathcal{D}}\rangle$ are decomposed into the direct sum as

$$\begin{aligned}\Omega^p(\log \bar{D}) &= \bigoplus_{\mu\in\mathbb{Z}} \Omega^p(\log \bar{D})_\mu,\\ \Omega^p\langle\bar{\mathcal{D}}\rangle &= \bigoplus_{\mu\in\mathbb{Z}} \Omega^p\langle\bar{\mathcal{D}}\rangle_\mu.\end{aligned}$$

The space $\Omega^p(\log \bar{D})_\mu$ (resp. $\Omega^p\langle\bar{\mathcal{D}}\rangle_\mu$) is called the homogeneous part of $\Omega^p(\log \bar{D})$ (resp. $\Omega^p\langle\bar{\mathcal{D}}\rangle_\mu$) of degree μ.

2.2. Now we will see two canonical linear mappings σ_μ^p and τ_μ^p which are studied in [KN, pp. 141–142]. First we define σ^p_μ:

$$\sigma^p_\mu : \mathrm{Gr}^F_\mu \Omega^p(\log D) \longrightarrow \Omega^p(\log \bar{D})_\mu.$$

The mapping σ^p_μ is defined as follows:let $\varphi \in F_\mu\Omega^p(\log D)$; then $\varphi = \alpha/P$, $\alpha \in \Omega^p(\mathbb{C}^n)$ and $\deg\alpha \leq \mu + l$. We denote by $\bar{\alpha}$ the homogeneous part of α of degree $\mu + l$ and set $\bar{\varphi} := \bar{\alpha}/\bar{P}$; then we see easily $\bar{\varphi} \in \Omega^p(\log \bar{D})_\mu$. We define the the mapping σ^p_μ as sending φ mod $F_{\mu-1}\Omega^p(\log D)$ to $\bar{\varphi}$. Then by definition of $\bar{\varphi}$, it is clear that σ^p_μ is well-defined and *injective.*

Next, we turn to the filtration $G_\cdot$. Let φ be in $G_\mu\Omega^p\langle\mathcal{D}\rangle$. Then φ can be expressed in the form (2.1.1) where each $\varphi_{j_1\cdots j_\mu}$ is a polynomial $(p-\nu)$-form of degree at most μ. Setting $\varphi = \psi/P$ where ψ is in $\Omega^p(\mathbb{C}^n)$, we have

$$\psi = P_1\cdots P_m\Big\{\psi_0 + \sum_{j=1}^m \frac{dP_j}{P_j}\wedge\psi_j + \cdots \\ \cdots + \sum_{1\leq j_1<\cdots<j_p\leq m} \frac{dP_{j_1}}{P_{j_1}}\wedge\cdots\wedge\frac{dP_{j_p}}{P_{j_p}}\cdot\psi_{j_1\cdots j_p}\Big\}. \qquad (2.2.1)$$

Let $\bar{\psi}_{j_1\cdots j_\nu}$ be the homogeneous part of degree μ of $\psi_{j_1\cdots j_\nu}$. Taking the homogeneous part of degree $\mu + l$ of the both sides of (2.2.1), we have

$$\bar{\varphi} = \bar{\psi}/\bar{P} = \bar{\varphi}_0 + \sum_{j=1}^m \frac{d\bar{P}_j}{\bar{P}_j}\wedge\bar{\varphi}_j + \cdots \\ \cdots + \sum_{1\leq j_1<\cdots<j_p\leq m} \frac{d\bar{P}_{j_1}}{\bar{P}_{j_1}}\wedge\cdots\wedge\frac{d\bar{P}_{j_p}}{\bar{P}_{j_p}}\cdot\bar{\varphi}_{j_1\cdots j_p}. \qquad (2.2.2)$$

From the definition of $\Omega^p\langle\mathcal{D}\rangle_\mu$, it follows that $\bar{\varphi}$ is in $\Omega^p\langle\bar{\mathcal{D}}\rangle_\mu$. Thus we can define a natural *surjective* mapping:

$$\tau^p_\mu : \mathrm{Gr}^G_\mu\Omega^p\langle\mathcal{D}\rangle \longrightarrow \Omega^p\langle\bar{\mathcal{D}}\rangle_\mu.$$

If $\bar{\varphi}$ is in $\Omega^p\langle\bar{\mathcal{D}}\rangle_\mu$, then $\bar{\varphi}$ can be expressed in the form (2.2.2) where each $\bar{\varphi}$ is homogeneous of degree μ for $1 \leq j_1 < \cdots < j_\nu \leq m$ and $0 \leq \nu \leq p$. Setting

$$\varphi = \bar{\varphi}_0 + \sum_{j=1}^m \frac{dP_j}{P_j}\wedge\bar{\varphi}_j + \cdots \\ \cdots + \sum_{1\leq j_1<\cdots<j_p\leq m} \frac{dP_{j_1}}{P_{j_1}}\wedge\cdots\wedge\frac{dP_{j_p}}{P_{j_p}}\cdot\bar{\varphi}_{j_1\cdots j_p},$$

we obtain $\tau_\mu^p(\varphi \bmod G_{\mu-1}\Omega^p\langle\mathcal{D}\rangle) = \bar{\varphi}$. The following lemma is essential.

Lemma 2.2.1 *Under the Assumption* (1.1.1) *on the homogeneous parts* $\bar{P}_j$ *of* P_j, $1 \le j \le m$, *we have*

(1) $\sigma_\mu^p : \mathrm{Gr}_\mu^F \Omega^p(\log D) \longrightarrow \Omega^p(\log \bar{D})_\mu$ *is an isomorphism for* $p \neq n-1$ *and* $\mu \in \mathbb{Z}$.

(2) $\tau_\mu^p : \mathrm{Gr}_\mu^G \Omega^p\langle\mathcal{D}\rangle \longrightarrow \Omega^p\langle\bar{\mathcal{D}}\rangle_\mu$ *is an isomorphism for* $0 \le p \le n-1$ *and* $\mu \in \mathbb{Z}$.

Proof. Assertions (1) and (2) are proved in [KN, LEMMA 9]. □

2.3. We set

$$N^p(\log D)_\mu := \Omega^p(\log \bar{D})_\mu / \mathrm{Im}\, \sigma_\mu^p.$$

Since σ_μ^p is injective,we have a short exact sequence of complexes:

$$0 \longrightarrow \mathrm{Gr}_\mu^F \Omega^{\cdot}(\log D) \longrightarrow \Omega^{\cdot}(\log \bar{D})_\mu \longrightarrow N^{\cdot}(\log D)_\mu \longrightarrow 0. \qquad (2.3.1)$$

[K, THEOREM 3.4.1] showed that if $\sum_{j=1}^m l_j\alpha_j \neq l,\ l-1, \dots$, we have $H^p(\Omega^{\cdot}(\log \bar{D})_\mu, \nabla_{\bar{\omega}}) = 0$ and hence by passing to long exact sequence of cohomology of (2.3.1),we obtain

$$H^{p-1}(N^{\cdot}(\log D)_\mu) \simeq H^p(\mathrm{Gr}_\mu^F \Omega^{\cdot}(\log D), \mathrm{Gr}_\mu^F(\nabla_\omega)).$$

Under the Assumption 1.1.1, if $\alpha_j \notin \mathbb{Z}_{>0}$ for $1 \le j \le m$ and $\sum_{j=1}^m l_j\alpha_j \notin \mathbb{Z}$, the following statements are known (see [K], [KN] for proofs):

$$H^p(\Omega^{\cdot}(*D), \nabla_\omega) = 0 \quad \text{for} \quad p \neq n,$$
$$H^n(\Omega^{\cdot}(*D), \nabla_\omega) \simeq H^n(\Omega^{\cdot}(\log D), \nabla_\omega),$$
$$H^n(\mathrm{Gr}_\mu^F(\Omega^{\cdot}(\log D), \mathrm{Gr}_\mu^F(\nabla_\omega)) \simeq \mathrm{Gr}_\mu^F(H^n(\Omega^{\cdot}(\log D), \nabla_\omega)).$$

Lemma 2.2.1 (1) yields that

$$N^p(\log D)_\mu = 0 \quad \text{for} \quad p \neq n-1.$$

Hence in order to compute the twisted rational de Rham cohomology, our main interest is in the isomorphism as follows:

$$N^{n-1}(\log D)_\mu \simeq H^n(\mathrm{Gr}_\mu^F \Omega^{\cdot}(\log D), \mathrm{Gr}_\mu^F(\nabla_\omega)).$$

3. More investigation about a representation theorem for logarithmic forms

In this section, we will explain a representation theorem of logarithmic forms in terms of degree filtration. From this section, we shall assume that Assumption 1.1.1 holds for the homogeneous parts $\bar{P}_j$ of P_j, $1 \leq j \leq m$ and Assumption 1.2.1 holds for P_j, $1 \leq j \leq m$. We set $\deg P_j = q_j + 1 > 0$, $1 \leq j \leq m$. We use the following notations:

$\Omega^p(\mathbb{C}^n)_\nu$: space of homogeneous polynomial p-forms of degree ν

$\Omega^p(\mathbb{C}^n)_{\leq\nu}$: space of polynomial p-forms of degree $\leq \nu$.

3.1. As a generalization of Lemma 1.2.3, we will show the following proposition about a representation theorem of logarithmic forms in the relation about its degree.

Proposition 3.1.1 *Let $0 \leq p \leq n-1$ and $\varphi \in F_\mu\Omega^p(\log D)$. Suppose $\mu \geq 0$, then φ can be written in the form*

$$\varphi = \varphi_0 + \sum_{j=1}^{m} \frac{dP_j}{P_j} \wedge \varphi_j + \cdots$$
$$\cdots + \sum_{1\leq j_1<\cdots<j_p\leq m} \frac{dP_{j_1}}{P_{j_1}} \wedge \cdots \wedge \frac{dP_{j_p}}{P_{j_p}} \cdot \varphi_{j_1\cdots j_p} \tag{3.1.1}$$

where $\varphi_{j_1\cdots j_\nu} \in \Omega^{p-\nu}(\mathbb{C}^n)_{\leq\mu}$. Moreover, if $\deg\varphi \leq -1$, we have $\varphi = 0$.

Proof. By Lemma 1.2.3, the assertion of this proposition is that we can replace each $\varphi_{j_1\cdots j_\nu}$ by its degree $\leq \mu$ for arbitrary expression (3.1.1). Suppose that $\varphi \in G_\lambda\Omega^p\langle\mathcal{D}\rangle$ and $\lambda > \mu$. Taking the homogeneous part of degree λ of the both sides of (3.1.1), we have

$$0 = \bar{\varphi}_0 + \sum_{j=1}^{m} \frac{d\bar{P}_j}{\bar{P}_j} \wedge \bar{\varphi}_j + \cdots$$
$$\cdots + \sum_{1\leq j_1<\cdots<j_p\leq m} \frac{d\bar{P}_{j_1}}{\bar{P}_{j_1}} \wedge \cdots \wedge \frac{d\bar{P}_{j_p}}{\bar{P}_{j_p}} \cdot \bar{\varphi}_{j_1\cdots j_p}.$$

This means that $\tau^p_\lambda(\varphi \bmod G_{\lambda-1}\Omega^p\langle\mathcal{D}\rangle) = 0$. Therefore by Lemma 2.2.1 (2), we have $\varphi = 0$ in $\mathrm{Gr}^G_\lambda\Omega^p\langle\mathcal{D}\rangle$. And hence we have $\varphi \in G_{\lambda-1}\Omega^p\langle\mathcal{D}\rangle$. Continuing these computations $(\lambda - \mu)$ times, we obtain that $\varphi \in G_\mu\Omega^p\langle\mathcal{D}\rangle$ which is our desired result. □

Clearly, we can see that $G_\mu\Omega^p\langle\mathcal{D}\rangle \subset F_\mu\Omega^p(\log D)$ for all p. Proposition 3.1.1 implies that $F_\mu\Omega^p(\log D) \subset G_\mu\Omega^p\langle\mathcal{D}\rangle$ for $0 \le p \le n-1$. Hence we have

$$F_\mu\Omega^p(\log D) = G_\mu\Omega^p\langle\mathcal{D}\rangle$$

for $0 \le p \le n-1$. Therefore we obtain the following:

Theorem 3.1.2 *Under Assumption* 1.1.1 *on the homogeneous parts* $\bar{P}_j$ *of* $\bar{P}_j$, $1 \le j \le m$, *and Assumption* 1.2.1 *on* P_j, $1 \le j \le m$, *we have*

$$\mathrm{Gr}_\mu^F\Omega^p(\log D) = \mathrm{Gr}_\mu^G\Omega^p\langle\mathcal{D}\rangle$$

for $0 \le p \le n-1$.

3.2. From now, we will see the gap between $\Omega^{n-1}(\log \bar{D})$ and $\Omega^{n-1}\langle\bar{\mathcal{D}}\rangle$ which is essential to give a basis for $H^n(\Omega^\cdot(\log D), \nabla_\omega)$. Let $v = \sum_{i=1}^n u_i\frac{\partial}{\partial u_i}$ be the Euler vector field on $\mathbb{C}^n$ and i_v the interior product with respect to v. We can easily check that i_v acts on the complex $\Omega^\cdot(\log \bar{D})$. Remember that i_v is a skew-derivation of degree -1. By definition, i_v is an S-linear operator such that

$$i_v(du_{i_1}\wedge\cdots\wedge du_{i_p}) = \sum_{j=1}^p(-1)^{j-1}u_{i_j}du_{i_1}\wedge\cdots\wedge\widehat{du_{i_j}}\wedge\cdots\wedge du_{i_p}.$$

Here $\widehat{du_{i_j}}$ denotes deletion of du_{i_j}. Let $\varphi_1 = d\bar{P}_1/\bar{P}_1$. Consider the mapping

$$\partial : \Omega^p(\log \bar{D}) \longrightarrow \Omega^{p+1}(\log \bar{D})$$

by $\partial(\varphi) = \varphi_1 \wedge \varphi$ for $\varphi \in \Omega^p(\log \bar{D})$.

Lemma 3.2.1 *The sequence*

$$0 \longrightarrow \Omega^0(\log \bar{D}) \xrightarrow{\partial} \Omega^1(\log \bar{D}) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Omega^n(\log \bar{D}) \longrightarrow 0 \quad (3.2.1)$$

is exact.

Proof. Suppose that $\varphi \in \Omega^p(\log \bar{D})$ satisfies $\partial\varphi = \varphi_1\wedge\varphi = 0$. Since i_v is a skew-derivation,

$$i_v(\varphi_1\wedge\varphi) = i_v(\varphi_1)\wedge\varphi - \varphi_1\wedge i_v(\varphi).$$

Simple computation shows that $i_v(\varphi_1) = q_1 + 1 \neq 0$, therefore we have

$$\varphi = \frac{1}{q_1 + 1}\partial(i_v(\varphi)).$$

Remember that i_v acts on the complex $\Omega^{\cdot}(\log \bar{D})$, hence $i_v(\varphi) \in \Omega^{p-1}(\log \bar{D})$. □

The Euler form, $\varphi_E \in \Omega^{n-1}(\log \bar{D})$, is defined by

$$\varphi_E := \frac{\sum_{i=1}^{n}(-1)^{i-1}u_i du_1 \wedge \cdots \wedge \widehat{du_i} \wedge \cdots \wedge du_n}{\bar{P}_1 \cdots \bar{P}_m}.$$

Proposition 3.2.2 *$\Omega^{n-1}(\log \bar{D})$ is spanned over $\Omega^{n-1-\nu}(\mathbb{C}^n)$ by the forms*

$$\frac{d\bar{P}_{j_1}}{\bar{P}_{j_1}} \wedge \cdots \wedge \frac{d\bar{P}_{j_\nu}}{\bar{P}_{j_\nu}}$$

for $1 \leq j_1 < \cdots < j_\nu \leq m$, $0 \leq \nu \leq n-1$, together with the Euler form φ_E over S.

Proof. Recall the exact sequence (3.2.1)

$$0 \longrightarrow \Omega^0(\log \bar{D}) \xrightarrow{\partial} \Omega^1(\log \bar{D}) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Omega^n(\log \bar{D}) \longrightarrow 0.$$

We can easily check that

$$\partial(\varphi_E) = (q_1 + 1)\frac{du_1 \wedge \cdots \wedge du_n}{\bar{P}_1 \cdots \bar{P}_m}$$

and notice that

$$\Omega^n(\log \bar{D}) = S\frac{du_1 \wedge \cdots \wedge du_n}{\bar{P}_1 \cdots \bar{P}_m}.$$

which is a free module of rank one. Hence we obtain a short exact sequence

$$0 \longrightarrow \partial\Omega^{n-2}(\log \bar{D}) \longrightarrow \Omega^{n-1}(\log \bar{D}) \xrightarrow{\partial} \Omega^n(\log \bar{D}) \longrightarrow 0$$

which splits. Therefore we have

$$\Omega^{n-1}(\log \bar{D}) = \partial\Omega^{n-2}(\log \bar{D}) \oplus S\varphi_E.$$

This result, together with Lemma 1.1.2, yields our assertion. □

Remark. Proposition 3.2.2 implies that $\Omega^{n-1}(\log \bar{D}) = \Omega^{n-1}\langle \bar{\mathcal{D}} \rangle + S\varphi_E$.

In order to have Proposition 3.2.5 which shows a representation theorem for the Euler form, we prepare a tool. Let $1 \leq j_1, \ldots, j_N \leq n$ be integers. Assume that $j_p \neq j_q$ for $p \neq q$. Let $k_1, \ldots, k_N$ be a permutation of j_ν's, $1 \leq \nu \leq N$ as $k_1 < \cdots < k_N$. Define

$$\begin{aligned} *(du_{j_1} \wedge \cdots \wedge du_{j_N}) := (-1)^{j_1 + \cdots + j_N - \frac{N(N+1)}{2}} \operatorname{sgn} \begin{pmatrix} j_1 \cdots j_N \\ k_1 \cdots k_N \end{pmatrix} \\ \times du_1 \wedge \cdots \wedge \widehat{du_{k_1}} \wedge \cdots \wedge \widehat{du_{k_N}} \wedge \cdots \wedge du_n. \end{aligned}$$

Notice that $du_{j_1} \wedge \cdots \wedge du_{j_N} \wedge *(du_{j_1} \wedge \cdots \wedge du_{j_N}) = du_1 \wedge \cdots \wedge du_n$. If $j_p = j_q$ for some $p \neq q$, we consider $*(du_{j_1} \wedge \cdots \wedge du_{j_N})$ as 0. For example,

$$\begin{aligned} &*du_j = (-1)^{j-1} du_1 \wedge \cdots \wedge \widehat{du_j} \wedge \cdots \wedge du_n, \\ &*1 = du_1 \wedge \cdots \wedge du_n, \\ &*(du_i \wedge du_i) = 0 \quad \text{etc.} \end{aligned}$$

Lemma 3.2.3 *Let $du_{j_1} \wedge \cdots \wedge du_{j_N} \neq 0$, then*

$$\begin{aligned} &du_{j_1} \wedge \cdots \wedge \widehat{du_{j_k}} \wedge \cdots \wedge du_{j_N} \wedge *(du_{j_1} \wedge \cdots \wedge du_{j_N}) \\ &\quad = (-1)^{k-1} * du_{j_k}. \end{aligned}$$

Proof. We shall prove this lemma by induction. In case $N = 1$, this lemma is trivially true. We assume Lemma 3.2.3 is true for $N - 1$. Then

$$\begin{aligned} &du_{j_1} \wedge \cdots \wedge \widehat{du_{j_k}} \wedge \cdots \wedge du_{j_N} \wedge *(du_{j_1} \wedge \cdots \wedge du_{j_N}) \\ &\quad = (du_{j_1} \wedge \cdots \wedge \widehat{du_{j_k}} \wedge \cdots \wedge du_{j_{N-1}}) \wedge du_{j_N} \\ &\qquad \wedge (-1)^{j_1 + \cdots + j_N - \frac{N(N+1)}{2}} du_1 \wedge \cdots \wedge \widehat{du_{j_1}} \wedge \cdots \wedge \widehat{du_{j_N}} \wedge \cdots \wedge du_n \\ &\quad = (-1)^{j_1 + \cdots + j_N - \frac{N(N+1)}{2}} du_{j_1} \wedge \cdots \wedge \widehat{du_{j_k}} \wedge \cdots \wedge du_{j_{N-1}} \\ &\qquad \wedge \{(-1)^{j_N - N} du_1 \wedge \cdots \wedge \widehat{du_{j_1}} \wedge \cdots \wedge \widehat{du_{j_{N-1}}} \wedge \cdots \wedge du_n\} \\ &\quad = du_{j_1} \wedge \cdots \wedge \widehat{du_{j_k}} \wedge \cdots \wedge du_{j_{N-1}} \wedge *(du_{j_1} \wedge \cdots \wedge du_{j_{N-1}}). \end{aligned}$$

Therefore by the induction hypothesis, Lemma 3.2.3 is true for N, this complete induction. □

Lemma 3.2.4 *Let $P_1, \ldots, P_N$ be polynomials in S. Then*

$$\sum_{k=1}^{N} (-1)^{k-1} \left| \frac{\partial(P_1, \ldots, \widehat{P_l}, \ldots, P_N)}{\partial(u_{j_1}, \ldots, \widehat{u_{j_k}}, \ldots, u_{j_N})} \right| *du_{j_k}$$

is equal to

$$dP_1 \wedge \cdots \wedge \widehat{dP_l} \wedge \cdots \wedge dP_N \wedge *(du_{j_1} \wedge \cdots \wedge du_{j_N}).$$

Proof. Simple computation shows that the latter is

$$\sum_{k=1}^{N} \left| \frac{\partial(P_1, \ldots, \widehat{P_l}, \ldots, P_N)}{\partial(u_{j_1}, \ldots, \widehat{u_{j_k}}, \ldots, u_{j_N})} \right| du_{j_1} \wedge \cdots \wedge \widehat{du_{j_k}} \wedge \cdots \wedge du_{j_N} \wedge *(du_{j_1} \wedge \cdots \wedge du_{j_N}).$$

By Lemma 3.2.3, this agrees with the former. □

By using Lemma 3.2.4, we will show a representation theorem for the Euler form as follows:

Proposition 3.2.5 (Representation theorem for the Euler form) *Let $\bar{P}_j$, $1 \le j \le N$, be homogeneous polynomials each degree of $q_j + 1$, $1 \le j \le N$ and $\varphi_E = \frac{\sum_{i=1}^n u_i * du_i}{\bar{P}_1 \cdots \bar{P}_N}$ the Euler form. Let $g_{j_1 \cdots j_N}$ be $\left|\frac{\partial(\bar{P}_1, \ldots, \bar{P}_N)}{\partial(u_{j_1}, \ldots, u_{j_N})}\right|$. Then $g_{j_1 \cdots j_N} \varphi_E$ can be written in the form*

$$\sum_{l=1}^{N} (-1)^{l-1}(q_l+1) \frac{d\bar{P}_1}{\bar{P}_1} \wedge \cdots \wedge \widehat{\frac{d\bar{P}_l}{\bar{P}_l}} \wedge \cdots \wedge \frac{d\bar{P}_N}{\bar{P}_N} \wedge *(du_{j_1} \wedge \cdots \wedge du_{j_N})$$
$$+ \frac{d\bar{P}_1}{\bar{P}_1} \wedge \cdots \wedge \frac{d\bar{P}_N}{\bar{P}_N} \wedge \sum_{i \ne j_1, \ldots, j_N} u_i * (du_i \wedge du_{j_1} \wedge \cdots \wedge du_{j_N}).$$

Proof. Note that $\sum_{i=1}^n u_i \frac{\partial \bar{P}_j}{\partial u_i} = (q_j + 1)\bar{P}_j$, for $1 \le j \le N$. Therefore

$$g_{j_1 \cdots j_N} \sum_{i=1}^{n} u_i * du_i$$
$$= \sum_{k=1}^{N} \begin{vmatrix} \frac{\partial \bar{P}_1}{\partial u_{j_1}} & \cdots & u_{j_k} \frac{\partial \bar{P}_1}{\partial u_{j_k}} & \cdots & \frac{\partial \bar{P}_1}{\partial u_{j_N}} \\ \vdots & & \vdots & & \vdots \\ \frac{\partial \bar{P}_N}{\partial u_{j_1}} & \cdots & u_{j_k} \frac{\partial \bar{P}_N}{\partial u_{j_k}} & \cdots & \frac{\partial \bar{P}_N}{\partial u_{j_N}} \end{vmatrix} * du_{j_k}$$
$$+ g_{j_1 \cdots j_N} \sum_{i \ne j_1, \ldots, j_N} u_i * du_i$$

$$= \sum_{k=1}^{N} \begin{vmatrix} \dfrac{\partial \bar{P}_1}{\partial u_{j_1}} & \cdots & (q_1+1)\bar{P}_1 - \sum\limits_{i \neq j_k} u_i \dfrac{\partial \bar{P}_1}{\partial u_i} & \cdots & \dfrac{\partial \bar{P}_1}{\partial u_{j_N}} \\ \vdots & & \vdots & & \vdots \\ \dfrac{\partial \bar{P}_N}{\partial u_{j_1}} & \cdots & (q_l+1)\bar{P}_N - \sum\limits_{i \neq j_k} u_i \dfrac{\partial \bar{P}_N}{\partial u_i} & \cdots & \dfrac{\partial \bar{P}_N}{\partial u_{j_N}} \end{vmatrix} * du_{j_k}$$

$$+ g_{j_1 \cdots j_N} \sum_{i \neq j_1, \ldots, j_N} u_i * du_i$$

$$= \sum_{k=1}^{N} \begin{vmatrix} \dfrac{\partial \bar{P}_1}{\partial u_{j_1}} & \cdots & (q_1+1)\bar{P}_1 & \cdots & \dfrac{\partial \bar{P}_1}{\partial u_{j_N}} \\ \vdots & & \vdots & & \vdots \\ \dfrac{\partial \bar{P}_N}{\partial u_{j_1}} & \cdots & (q_l+1)\bar{P}_N & \cdots & \dfrac{\partial \bar{P}_N}{\partial u_{j_N}} \end{vmatrix} * du_{j_k}$$

$$+ \sum_{k=1}^{N} \sum_{i \neq j_k} (-1)^k u_i \begin{vmatrix} \dfrac{\partial \bar{P}_1}{\partial u_i} & \dfrac{\partial \bar{P}_1}{\partial u_{j_1}} & \cdots & \widehat{\dfrac{\partial \bar{P}_1}{\partial u_{j_k}}} & \cdots & \dfrac{\partial \bar{P}_1}{\partial u_{j_N}} \\ \vdots & \vdots & & \vdots & & \vdots \\ \dfrac{\partial \bar{P}_N}{\partial u_i} & \dfrac{\partial \bar{P}_N}{\partial u_{j_1}} & \cdots & \widehat{\dfrac{\partial \bar{P}_N}{\partial u_{j_k}}} & \cdots & \dfrac{\partial \bar{P}_N}{\partial u_{j_N}} \end{vmatrix} * du_{j_k}$$

$$+ g_{j_1 \ldots j_N} \sum_{i \neq j_1, \ldots, j_N} u_i * du_i$$

$$= \sum_{k=1}^{N} \sum_{l=1}^{N} (-1)^{k+l} (q_l+1) \bar{P}_l \left| \frac{\partial(\bar{P}_1, \ldots, \widehat{\bar{P}_l}, \ldots, \bar{P}_N)}{\partial(u_{j_1}, \ldots, \widehat{u_{j_k}}, \ldots, u_{j_N})} \right| * du_{j_k}$$

$$+ \sum_{i \neq j_1, \ldots, j_N} u_i \left\{ \left| \frac{\partial(\bar{P}_1, \ldots, \bar{P}_N)}{\partial(u_{j_1}, \ldots, u_{j_N})} \right| * du_i \right.$$

$$\left. + \sum_{k=1}^{N} (-1)^k \left| \frac{\partial(\bar{P}_1, \ldots, \bar{P}_N)}{\partial(u_i, u_{j_1}, \ldots, \widehat{u_{j_k}}, \ldots, u_{j_N})} \right| * du_{j_k} \right\}$$

$$= \sum_{l=1}^{N} (-1)^{l-1} (q_l+1) \bar{P}_l \sum_{k=1}^{N} (-1)^{k-1} \left| \frac{\partial(\bar{P}_1, \ldots, \widehat{\bar{P}_l}, \ldots, \bar{P}_N)}{\partial(u_{j_1}, \ldots, \widehat{u_{j_k}}, \ldots, u_{j_N})} \right| * du_{j_k}$$

$$+ \sum_{i \neq j_1, \ldots, j_N} u_i d\bar{P}_1 \wedge \cdots \wedge d\bar{P}_N \wedge *(du_i \wedge du_{j_1} \wedge \cdots \wedge du_{j_N})$$

$$= \sum_{l=1}^{N}(-1)^{l-1}(q_l+1)\bar{P}_l d\bar{P}_1 \wedge \cdots \wedge \widehat{d\bar{P}_l} \wedge \cdots$$
$$\wedge\, d\bar{P}_N \wedge *(du_{j_1} \wedge \cdots \wedge du_{j_N})$$
$$+ d\bar{P}_1 \wedge \cdots \wedge d\bar{P}_N \wedge \sum_{i\neq j_1,\ldots,j_N} u_i * (du_i \wedge du_{j_1} \wedge \cdots \wedge du_{j_N}).$$

Thus we obtain the desired representation. □

Remark. Proposition 3.2.5 implies that $g_{j_1\cdots j_N}\varphi_E$ is in $\Omega^{n-1}\langle\bar{\mathcal{D}}\rangle$.

3.3. Investigating in more detail the proof of [K, LEMMA 6.6.1], we obtain the following:

Theorem 3.3.1 *Let $\bar{\varphi} \in \Omega^{n-1}(\log \bar{D})_\mu$ and $q_L = \max\{q_1,\ldots,q_m\}$. If $\mu \geq (n-1)q_L$, then $\bar{\varphi}$ can be written in the form*

$$\bar{\varphi} = \bar{\varphi}_0 + \sum_{j=1}^{m} \frac{d\bar{P}_j}{\bar{P}_j} \wedge \bar{\varphi}_j + \cdots$$
$$\cdots + \sum_{1\leq j_1<\cdots<j_{n-1}\leq m} \frac{d\bar{P}_{j_1}}{\bar{P}_{j_1}} \wedge \cdots \wedge \frac{d\bar{P}_{j_{n-1}}}{\bar{P}_{j_{n-1}}} \cdot \bar{\varphi}_{j_1\cdots j_{n-1}} \tag{3.3.1}$$

where $\bar{\varphi}_{j_1\cdots j_\nu} \in \Omega^{n-1-\nu}(\mathbb{C}^n)_\mu$.

Proof. We shall prove Proposition 3.3.1 by induction on the number m of polynomials $\bar{P}_j$, $1 \leq j \leq m$. In case $m = 1$, by Proposition 3.1.1, it suffices to show that every mapping σ_μ^{n-1} is an isomorphism for $\mu \geq (n-1)q_1$. Let $\varphi_E = \sum_{i=1}^n u_i * du_i/\bar{P}_1$. By Proposition 3.1.1 and Lemma 3.2.2, we have

$$\mathrm{Gr}_\mu^F \Omega^{n-1}(\log D) = \Omega^{n-1}(\mathbb{C}^n)_\mu + \left[\frac{dP_1}{P_1}\right] \wedge \Omega^{n-2}(\mathbb{C}^n)_\mu,$$
$$\Omega^{n-1}(\log \bar{D})_\mu = \Omega^{n-1}(\mathbb{C}^n)_\mu + \frac{d\bar{P}_1}{\bar{P}_1} \wedge \Omega^{n-2}(\mathbb{C}^n)_\mu + \varphi_E S_{\mu+q_1+1-n}.$$

By Proposition 3.2.5, we have

$$\frac{\partial \bar{P}_1}{\partial u_j}\varphi_E = (q_1+1) * du_j + \frac{d\bar{P}_1}{\bar{P}_1} \wedge \sum_{i\neq j} u_i * (du_i \wedge du_j),$$
$$\text{for} \quad 1 \leq j \leq n.$$

This means that

$$\left(\frac{\partial \bar{P}_1}{\partial u_1}, \dots, \frac{\partial \bar{P}_1}{\partial u_n}\right) \subset \operatorname{Ann} N^{n-1}(\log D).$$

Conversely, let $h \in \operatorname{Ann} N^{n-1}(\log D)$, then $\xi := h \sum_{i=1}^{n} u_i {*} du_i \equiv 0 \pmod{\bar{P}_1, d\bar{P}_1}$ and $d\bar{P}_1 \wedge \xi \equiv 0 \pmod{\bar{P}_1 d\bar{P}_1}$. We can easily see that

$$d\bar{P}_1 \wedge \xi = (q_1 + 1)\bar{P}_1 h \, du_1 \wedge \cdots \wedge du_n.$$

This means that $h \, du_1 \wedge \cdots \wedge du_n \equiv 0 \pmod{d\bar{P}_1}$, hence $h \in \left(\frac{\partial \bar{P}_1}{\partial u_1}, \dots, \frac{\partial \bar{P}_1}{\partial u_n}\right)$. Therefore we have

$$\left(\frac{\partial \bar{P}_1}{\partial u_1}, \dots, \frac{\partial \bar{P}_1}{\partial u_n}\right) = \operatorname{Ann} N^{n-1}(\log D).$$

By Assumption 1.1.1 (1), the n partial derivatives of $\bar{P}_1$ each homogeneous of degree q_1 form a regular sequence. Hence we obtain

$$\begin{aligned} &\operatorname{Poin}(N^{n-1}(\log D), t) \\ &\quad = t^{n-q_1-1} \operatorname{Poin}\left(\mathbb{C}[u] / \left(\frac{\partial \bar{P}_1}{\partial u_1}, \dots, \frac{\partial \bar{P}_1}{\partial u_n}\right), t\right) \\ &\quad = t^{n-q_1-1}(1 + t + t^2 + \cdots + t^{q_1-1})^n. \end{aligned} \tag{3.3.2}$$

And hence

$$N^{n-1}(\log D)_\mu = 0 \quad \text{for} \quad \mu \geq (n-1)q_1.$$

This means that every mapping σ_μ^{n-1} is an isomorphism for $\mu \geq (n-1)q_1$, which shows that Proposition 3.3.1 holds for $m = 1$. We assume Proposition 3.3.1 is true for m and show that it is also true for $m+1$. Let $\bar{\varphi} \in \Omega^{n-1}(\log \bar{D})_\mu$ with $\mu \geq (n-1)q_L$ where $q_L := \max\{q_1, \dots, q_m, q_{m+1}\}$; then we can write $\bar{\varphi}$ as $\bar{\varphi} = \bar{\psi} / \bar{P}_1 \cdots \bar{P}_m \bar{P}_{m+1}$ where $\bar{\psi}$is a homogeneous polynomial $(n-1)$-form of degree $\mu + m + 1 + \sum_{i=1}^{m+1} q_i$. We have, by definition,

$$d\bar{P}_j \wedge \bar{\psi} \equiv 0 \pmod{\bar{P}_j} \quad \text{for } 1 \leq j \leq m+1.$$

Then by induction assumption, $\bar{\psi}$ can be written in the form

$$\bar{\psi} = \bar{P}_1 \cdots \bar{P}_m \left\{ \bar{\psi}_0 + \sum_{j=1}^{m} \frac{d\bar{P}_j}{\bar{P}_j} \wedge \bar{\psi}_j + \cdots \right.$$
$$\left. \cdots + \sum_{1 \le j_1 < \cdots < j_{n-1} \le m} \frac{d\bar{P}_{j_1}}{\bar{P}_{j_1}} \wedge \cdots \wedge \frac{d\bar{P}_{j_{n-1}}}{\bar{P}_{j_{n-1}}} \cdot \bar{\psi}_{j_1 \cdots j_{n-1}} \right\} \quad (3.3.3)$$

where $\bar{\psi}_{j_1 \cdots j_\nu} \in \Omega^{n-1-\nu}(\mathbb{C}^n)_{\mu+1+q_{m+1}}$. We will show the statement by induction on the largest number N for which $\bar{\psi}_{j_1 \cdots j_N} \neq 0$. In case $N = 0$, we have $\bar{\psi} = \bar{P}_1 \cdots \bar{P}_m \bar{\psi}_0$. Since $d\bar{P}_{m+1} \wedge \bar{\psi} \equiv 0 \pmod{\bar{P}_{m+1}}$, we have

$$\bar{P}_1 \cdots \bar{P}_m d\bar{P}_{m+1} \wedge \bar{\psi}_0 \equiv 0 \pmod{\bar{P}_{m+1}}.$$

Since $0 \le p \le n$ and Assumption 1.1.1 (2) implies that $\{\bar{P}_j, \bar{P}_{m+1}\}$ form a regular sequence for $1 \le j \le m$ we have

$$d\bar{P}_{m+1} \wedge \bar{\psi}_0 \equiv 0 \pmod{\bar{P}_{m+1}}.$$

This means that $\bar{\psi}_0/\bar{P}_{m+1} \in \Omega^{n-1}(\log \bar{D})_\mu$. Apply this to the case $m = 1$ we have

$$\frac{\bar{\psi}_0}{\bar{P}_{m+1}} = \bar{\alpha} + \frac{d\bar{P}_{m+1}}{\bar{P}_{m+1}} \wedge \bar{\beta}, \quad \bar{\alpha} \in \Omega^p(\mathbb{C}^n)_\mu, \quad \bar{\beta} \in \Omega^{p-1}(\mathbb{C}^n)_\mu,$$

and hence

$$\psi = \bar{P}_1 \cdots \bar{P}_{m+1} \left\{ \bar{\alpha} + \frac{d\bar{P}_{m+1}}{\bar{P}_{m+1}} \wedge \bar{\beta} \right\},$$

which means that Proposition 3.3.1 holds for $N = 0$. If $N \le n-2$, then the proof of [K, PROPOSITION 2.2.3] works in this case and hence the statement is true for $N \le n-2$. In case $N = n-1$, notice that $\bar{\psi}_{j_1 \cdots j_{n-1}} \in \Omega^0(\mathbb{C}^n)_{\mu+1+q_{m+1}}$. Since $\bar{P}_{j_1}, \ldots, \bar{P}_{j_{n-1}}, \bar{P}_{m+1}$ form a regular sequence, it follows that

$$(\bar{P}_{j_1}, \ldots, \bar{P}_{j_{n-1}}, \bar{P}_{m+1})_\nu = (u_1, \ldots, u_n)_\nu$$

for $\nu \ge q_{j_1} + \cdots + q_{j_{n-1}} + q_{m+1} + 1$. If $\mu \ge (n-1)q_L$, we can easily check that $\mu + 1 + q_{m+1} \ge q_{j_1} + \cdots + q_{j_{n-1}} + q_{m+1} + 1$. Therefore we can write $\psi_{j_1 \cdots j_{n-1}}$ as

$$\bar{\psi}_{j_1 \cdots j_{n-1}} = \sum_{k=1}^{n-1} \bar{P}_{j_k} \bar{\alpha}_{j_1 \cdots j_{n-1}; j_k} + \bar{P}_{m+1} \bar{\alpha}_{j_1 \cdots j_{n-1}; m+1} \quad (3.3.4)$$

where $\bar{\alpha}_{j_1\cdots j_{n-1};j_k} \in \Omega^0(\mathbb{C}^n)_{\mu+q_{m+1}-q_{j_k}}$, $\bar{\alpha}_{j_1\cdots j_{n-1};m+1} \in \Omega^0(\mathbb{C}^n)_\mu$. Substitute (3.3.4) into (3.3.3); then from the same reasoning as in the proof of Lemma 1.2.3 (in (1.2.4), set $\beta_{j_1\cdots j_{n-1};j_k} = \beta_{j_1\cdots j_{n-1};m+1} = 0$ and apply the reasoning to our situation), it follows that the statement is true for $N = n-1$. This complete induction. □

Remark. Proposition 3.3.1 implies that $\Omega^{n-1}(\log \bar{D})_\mu = \Omega^{n-1}\langle\bar{\mathcal{D}}\rangle_\mu$ for $\mu \geq (n-1)q_L$.

Combining the results of Proposition 3.1.1 and Proposition 3.3.1, we obtain

Theorem 3.3.2 *Under Assumption* 1.1.1 *on the homogeneous parts* $\bar{P}_j$ *of* $\bar{P}_j$, $1 \leq j \leq m$, *and Assumption* 1.2.1 *on* P_j, $1 \leq j \leq m$, *if* $\mu \geq (n-1)q_L$, *then the mapping*

$$\sigma_\mu^{n-1} : \mathrm{Gr}_\mu^F \Omega^{n-1}(\log D) \longrightarrow \Omega^{n-1}(\log \bar{D})_\mu$$

is surjective and hence

$$N^{n-1}(\log D)_\mu = 0 \quad \textit{for} \quad \mu \geq (n-1)q_L.$$

We have shown in 2.3 that if $\sum_{j=1}^m (q_j+1)\alpha_j \notin \mathbb{Z}_{\geq -l}$ then

$$N^{n-1}(\log D)_\mu \simeq H^n(\mathrm{Gr}_\mu^F \Omega^{\cdot}(\log D), \mathrm{Gr}_\mu^F(\nabla_\omega)).$$

Therefore we have

$$H^n(\mathrm{Gr}_\mu^F \Omega^{\cdot}(\log D), \mathrm{Gr}_\mu^F(\nabla_\omega)) = 0 \quad \text{for} \quad \mu \geq (n-1)q_L.$$

Thus we obtain following:

Corollary 3.3.3 *Under Assumption* 1.1.1 *on the homogeneous parts* $\bar{P}_j$ *of* $\bar{P}_j$, $1 \leq j \leq m$, *and Assumption* 1.2.1 *on* P_j, $1 \leq j \leq m$, *then we have*

$$H^n(\Omega^{\cdot}(\log D), \nabla_\omega) \simeq \bigoplus_{\mu \leq (n-1)q_L - 1} H^n(\mathrm{Gr}_\mu^F(\Omega^{\cdot}(\log D), \mathrm{Gr}_\mu^F(\nabla_\omega)).$$

4. Arrangement of hyperplanes and a $\bar{\mathcal{G}}$-transverse polynomial

From this section, we consider the case that $P_j(u)$, $1 \leq j \leq m-1$ are linear polynomials and $P_m(u)$ is a polynomial of degree $q+1$. We shall assume that Assumption 1.1.1 holds for the homogeneous parts $\bar{P}_j$

of P_j for $1 \leq j \leq m$ and Assumption 1.2.1 holds for P_j, $1 \leq j \leq m$. In this section, we will show that Assumption 1.1.1 holds in our case if $\mathcal{G} = \{H_1, \ldots, H_{m-1}\}$, $H_j = \{P_j(u) = 0\}$ for $1 \leq j \leq m-1$, is in general position and $P_m(u)$ is a $\bar{\mathcal{G}}$-transverse polynomial. First, we prepare some key words about arrangements of hyperplanes which are explained explicitly in [AKOT].

4.1. A hyperplane H is an affine linear subspace of codimension one. An arrangement $\mathcal{A}$ is a finite collection of hyperplanes in $\mathbb{C}^n$. Let $|\mathcal{A}|$ denote the number of hyperplanes in $\mathcal{A}$. A subset of an arrangement is a *subarrangement.*

Let $\mathcal{A} = \{H_1, \ldots, H_N\}$ be an affine n-arrangement. We say that $\mathcal{A}$ is an affine n-arrangement in *general position* if for any subarrangement $\{H_{i_1}, \ldots, H_{i_p}\}$ of $\mathcal{A}$, we have $\operatorname{codim}(H_{i_1} \cap \cdots \cap H_{i_p}) = p$ for $p \leq n$ and $H_{i_1} \cap \cdots \cap H_{i_p} = \emptyset$ for $p > n$. Let $n \geq 2$ and $\mathcal{G} = \{H_1, \ldots, H_N\}$ be an affine n-arrangement with $|\mathcal{G}| = N$ hyperplanes in general position. The *centralization* $\bar{\mathcal{G}}$ of $\mathcal{G}$ is a central arrangement which consists of the parallel translates of H_j to the origin, $\bar{H}_j$ for $1 \leq j \leq N$.

Remark. Let $P_j(u)$, $1 \leq j \leq m$, be linear polynomials and $H_j = \{P_j = 0\}$, $1 \leq j \leq m$. Then we can easily see that the arrangement of hyperplanes $\mathcal{A} = \{H_1, \ldots, H_m\}$ is in general position if and only if Assumption 1.1.1 holds for the homogeneous parts $\bar{P}_j$ of P_j, $1 \leq j \leq m$ and Assumption 1.2.1 holds for P_j, $1 \leq j \leq m$.

4.2. Let $L(\mathcal{A})$ be the set of all intersections of elements of $\mathcal{A}$. We agree that $L(\mathcal{A})$ includes $\mathbb{C}^n$ as the intersection of the empty collection of hyperplanes. We define

$$L^+(\mathcal{A}) := \{X \in L(\mathcal{A}) \mid \dim X > 0\}.$$

Let $\mathcal{G}$ be an affine n-arrangement of N hyperplanes in general position. Suppose $n \geq 2$ and let $\bar{\mathcal{G}}$ be the centralization of $\mathcal{G}$. For $X \in L^+(\mathcal{G})$, we denote $\bar{X} \in L^+(\bar{\mathcal{G}})$ as the parallel translate of X containing the origin.

Let $n \geq 2$. We call that a polynomial $f \in S$ of positive degree is $\bar{\mathcal{G}}$-*transverse* if the restriction of the homogeneous part of maximal degree $\bar{f}$ to $\bar{X}$ is not constantly equal to 0 and has no critical point outside the origin for each $X \in L^+(\mathcal{G})$. In particular, when $X = \mathbb{C}^n$, the homogeneous part of maximal degree $\bar{f}$ has no critical point outside the origin. When $n = 1$, we agree that every nonconstant polynomial of positive degree is $\bar{\mathcal{G}}$-transverse.

Proposition 4.2.1 *Let $\mathcal{G} = \{H_1, \dots, H_{m-1}\}$ be an affine n-arrangement in general position and $H_j = \{P_j(u) = 0\}$, $1 \le j \le m-1$. Let $P_m(u)$ be a polynomial in S of degree $q+1$. Then the following conditions are equivalent:*

(1) *For $0 \le k \le n-2$, the algebraic set defined by the ideal $I = (d\bar{P}_{j_1} \wedge \cdots \wedge d\bar{P}_{j_k} \wedge d\bar{P}_m, \bar{P}_{j_1}, \dots, \bar{P}_{j_k}, \bar{P}_m)$ is either empty or the origin.*

(2) *$P_m(u)$ is $\bar{\mathcal{G}}$-transverse.*

Proof. Assume the condition (1). After a linear change of coordinates we may think that $P_{j_1} = u_1, \dots, P_{j_k} = u_k$. Then we have

$$I = \left(\frac{\partial \bar{P}_m}{\partial u_{k+1}}, \dots, \frac{\partial \bar{P}_m}{\partial u_n}, u_1, \dots, u_k \right) \tag{4.2.1}$$

(Note that $(q+1)\bar{P}_m = \sum_{i=1}^n u_i \frac{\partial \bar{P}_m}{\partial u_i}$ and hence $\bar{P}_m$ in the ideal of the right hand side above). Set $X = \{u_1 = \cdots = u_k = 0\}$. Then $2 \le \dim \bar{X} \le n$. By assumption, the algebraic set defined by the ideal I is either empty or the origin. (4.2.1) implies that $\bar{P}_m|_{\bar{X}}$ has no critical point outside the origin for each $X \in L^+(\mathcal{G})$, $2 \le \dim X \le n$. Consider the case $\dim X = 1$. Identify $\bar{P}_m|_{\bar{X}}$ with a homogeneous polynomial of positive degree in polynomial ring in one variable. Then we can easily see that $\bar{P}_m|_{\bar{X}}$ has no critical point outside the origin. Combining these facts, we have P_m is $\bar{\mathcal{G}}$-transverse.

Conversely, Assume the condition (2). After a linear change of coordinates we may assume $X = \{u_1 = \cdots = u_k = 0\}$. Let $J_{\bar{X}} \subset S$ be the ideal consisting of the polynomial vanishing on $\bar{X}$ and $S_{\bar{X}} = S/J_{\bar{X}}$ is the coordinate ring of $\bar{X}$. Since $\bar{P}_m|_{\bar{X}}$ has no critical point outside the origin, the partial derivatives of $\bar{P}_m|_{\bar{X}}$ form a regular sequence in $S_{\bar{X}}$. This means that $\frac{\partial \bar{P}_m}{\partial u_{k+1}}, \dots, \frac{\partial \bar{P}_m}{\partial u_n}, u_1, \dots, u_k$ form a regular sequence in S. Therefore the algebraic set defined by I is only the origin. □

5. Residue maps

The $\mathbb{C}$-linear map *res*, called the residue map, was studied in [RT]. The aim of this section is to apply the method of residue map to our situation.

5.1. Let $H_j = \{P_j(u) = 0\}$, $1 \le j \le m-1$, $D_m = \{P_m(u) = 0\}$, and $\mathcal{D} = \{H_1, \dots, H_{m-1}, D_m\}$. We assume that $\mathcal{D} \backslash \{D_m\} \ne \emptyset$. Fix a hyperplane

H_1. Let

$$\mathcal{D}' := \mathcal{D}\backslash\{H_1\}, \quad \mathcal{D}'' := \{H \cap H_1 \mid H \in \mathcal{D}'\}.$$

Then $\mathcal{D}'$ is an arrangement in $\mathbb{C}^n$ called the *deletion* of $\mathcal{D}$. The arrangement $\mathcal{D}''$, called the *restriction* of $\mathcal{D}$ to H_1, is an arrangement in H_1. Let $P' = P/P_1$. Then P' defines D'. Denote the quotient algebra S/P_1S by S'' and identify S'' with a polynomial algebra over $\mathbb{C}$ in $n-1$ variables. For $g \in S$, let $g'' \in S''$ denote the class represented by g. We set D'' the divisor defined by $(P')''$.

Let $\varphi \in \Omega^p(\log D)$. Choose a rational $(p-1)$-form φ' and a rational p-form φ'' such that

(1) $\varphi = \dfrac{dP_1}{P_1} \wedge \varphi' + \varphi''$

(2) neither φ' nor φ'' has a pole along H_1.

For $\varphi \in \Omega^p(\log D)$, the restriction of φ' to H_1 is called the restriction of φ and is denoted by $res(\varphi)$. Since the restriction $\varphi'|_{H_1}$ depends only on φ and H_1, $res(\varphi)$ is well-defined.

Then we can define a $\mathbb{C}$-linear map just a same way of [RT]

$$res : \Omega^p(\log D) \longrightarrow \Omega^{p-1}(\log D'').$$

Then we obtain following lemma:

Lemma 5.1.1 *The sequence*

$$0 \longrightarrow \Omega^p(\log \bar{D}')_\mu \xrightarrow{i} \Omega^p(\log \bar{D})_\mu \xrightarrow{res} \Omega^{p-1}(\log \bar{D}'')_\mu \longrightarrow 0$$

is exact for each $1 \leq p \leq n$ *and* $\mu \in \mathbb{Z}$.

Proof. The same proof of [AKOT, Proposition 6.3] works in this case. □

Let $i : \Omega^p(\log D') \longrightarrow \Omega^p(\log D)$ be the inclusion map. It is clear that both the residue map and i are compatible with the filtrations, we have the sequence

$$0 \longrightarrow \mathrm{Gr}^F_\mu \Omega^p(\log D') \xrightarrow{i} \mathrm{Gr}^F_\mu \Omega^p(\log D) \xrightarrow{res} \mathrm{Gr}^F_\mu \Omega^{p-1}(\log D'') \longrightarrow 0. \tag{5.1.1}$$

Proposition 5.1.2 *The sequence* (5.1.1) *is exact for* $1 \leq p \leq n$ *and* $\mu \in \mathbb{Z}$.

Proof. The only non-trivial case happens when $p \neq n-1$ by Lemma 2.2.1 (1) and Lemma 5.1.1. By the same reasoning of Lemma 5.1.1,

$$0 \longrightarrow \Omega^{n-1}\langle\bar{\mathcal{D}}'\rangle_\mu \xrightarrow{i} \Omega^{n-1}\langle\bar{\mathcal{D}}\rangle_\mu \xrightarrow{res} \Omega^{n-2}\langle\bar{\mathcal{D}}''\rangle_\mu \longrightarrow 0 \qquad (5.1.2)$$

is exact for each $\mu \in \mathbb{Z}$. From Lemma 2.2.1 (2) and Theorem 3.1.2, we have

$$\mathrm{Gr}^F_\mu \Omega^{n-1}(\log D) \simeq \Omega^{n-1}\langle\bar{\mathcal{D}}\rangle_\mu$$

for $\mu \in \mathbb{Z}$. This isomorphism and (5.1.2) complete the proof. □

Consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & N^p(\log D')_\mu & \xrightarrow{i} & N^p(\log D)_\mu & \xrightarrow{res} & N^{n-1}(\log D'')_\mu & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Omega^p(\log \bar{D}')_\mu & \xrightarrow{i} & \Omega^p(\log \bar{D})_\mu & \xrightarrow{res} & \Omega^{p-1}(\log \bar{D}'')_\mu & \longrightarrow & 0 \\
 & & \sigma^p_\mu\uparrow & & \sigma^p_\mu\uparrow & & \sigma^{p-1}_\mu\uparrow & & \\
0 & \longrightarrow & \mathrm{Gr}^F_\mu\Omega^p(\log D') & \xrightarrow{i} & \mathrm{Gr}^F_\mu\Omega^p(\log D) & \xrightarrow{res} & \mathrm{Gr}^F_\mu\Omega^{p-1}(\log D'') & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

for $1 \leq p \leq n$ and $\mu \in \mathbb{Z}$. We know that all the columns are exact. The following theorem is the key to obtain a basis for $H^n(\Omega^\cdot(\log D), \nabla_\omega)$.

Theorem 5.1.3 *The sequence*

$$0 \longrightarrow N^p(\log D')_\mu \xrightarrow{i} N^p(\log D)_\mu \xrightarrow{res} N^{p-1}(\log D'')_\mu \longrightarrow 0 \qquad (5.1.3)$$

is exact for $1 \leq p \leq n$ *and* $\mu \in \mathbb{Z}$.

Proof. The middle row is exact by Lemma 5.1.1 and the bottom row is exact by Proposition 5.1.2. Therefore by the 9-lemma, the top row is exact. □

6. The Poincaré series of $N^{n-1}(\log D)$

In this section we will show a Poincaré series of $N^{n-1}(\log D)$ which yields a Poincaré series of $H^n(\Omega^\cdot(\log D), \nabla_\omega)$.

6.1. First, we define a Laurent polynomial as follows. Let $n \geq 1$, $m \geq 0$, and q be non-negative integers, and let t be an indeterminate. We define

$$\begin{aligned} A(q;t) &:= \sum_{i=0}^{q} t^i = 1 + t + t^2 + \cdots + t^q, \\ T(n,m,q;t) &:= \sum_{i=n-m}^{0} \binom{i+m-1}{n-1} t^i + \sum_{i=1}^{n} \binom{m}{n-i} [A(q;t)-1]^i. \end{aligned}$$

The first summand is considered as 0 when $n - m > 0$. Direct computation gives the following lemma:

Lemma 6.1.1 *Let $n \geq 2$ and $m \geq 1$. Then*

$$T(n,m,q;t) = T(n,m-1,q;t) + T(n-1,m-1,q;t).$$

Proof. It is shown in [AKOT, Proposition 7.2]. □

6.2. By using (5.1.3) in Theorem 5.1.3, which is a very essential short exact sequence, we have the following:

Proposition 6.2.1 *Let $\mathcal{G} = \{H_1, \ldots, H_{m-1}\}$ be a general position n-arrangement with $m-1$ hyperplanes defined by $H_j = \{P_j(u) = 0\}$, $1 \leq j \leq m-1$ and P_m a $\bar{\mathcal{G}}$-transverse polynomial of degree $q+1 > 0$. Suppose that Assumption 1.2.1 holds for P_j, $1 \leq j \leq m$. If $\sum_{j=1}^{m} l_j \alpha_j \neq l, l-1, \ldots,$ then we have*

$$\mathrm{Poin}(N^{n-1}(\log D), t) = t^{-q-1} T(n, m-1, q; t). \tag{6.2.1}$$

Proof. We shall prove Proposition 6.2.1 by induction on the number m of polynomials $\bar{P}_j$, $1 \leq j \leq m$. We must check two initial conditions.

(i) $n = 1$, $m \geq 1$. Write $u = u_1$. Since we may choose $P_j(u) = u - x_j$, $1 \leq j \leq m-1$, with $x_i \neq x_j$ for $i \neq j$, we have $\bar{P}_j(u) = u$, $1 \leq j \leq m-1$. And we may assume that $\bar{P}_m = u^{q+1}$. By Theorem 3.3.2, we know that

$N^0(\log D)_\mu = 0$ for $\mu \geq 0$. If $\mu < 0$, we have

$$N^0(\log D)_\mu = \frac{u * du}{u^{m-1} \cdot u^{q+1}} \mathbb{C}[u]_{\mu+m+q-1}.$$

Therefore we have

$$\begin{aligned} \mathrm{Poin}(N^0(\log D), t) &= t^{-m-q+1} + t^{-m-q+2} + \cdots + t^{-1} \\ &= t^{-q-1}(t^{2-m} + t^{3-m} + \cdots + t^{q}) \\ &= t^{-q-1} T(1, m-1, q; t) \end{aligned}$$

which shows that Proposition 6.2.1 holds for $n = 1$, $m \geq 1$.

(ii) $n \geq 1$, $m = 1$. By (3.3.2), in the proof of Proposition 3.3.1, we know that

$$\mathrm{Poin}(N^{n-1}(\log D), t) = t^{n-q-1}(1 + t + t^2 + \cdots + t^{q-1})^n.$$

We can easily check that

$$T(n, 0, q; t) = (1 + t + t^2 + \cdots + t^{q-1})^n$$

which implies that Proposition 6.2.1 is true for $n \geq 1$, $m = 1$.

(iii) For the induction step we use deletion and restriction. Suppose $n \geq 2$ and $m \geq 2$. Recall the short exact sequence (5.1.3), in Theorem 5.1.3,

$$0 \longrightarrow N^p(\log D')_\mu \xrightarrow{i} N^p(\log D)_\mu \xrightarrow{res} N^{p-1}(\log D'')_\mu \longrightarrow 0. \tag{5.1.3}$$

The induction hypothesis applies to D' and D''. Thus we have

$$\mathrm{Poin}(N^{n-1}(\log D'), t) = t^{-q-1} T(n, m-2, q; t),$$
$$\mathrm{Poin}(N^{n-2}(\log D''), t) = t^{-q-1} T(n-1, m-2, q; t).$$

By exactness of (5.1.3), we obtain

$$\begin{aligned} &\mathrm{Poin}(N^{n-1}(\log D), t) \\ &\quad = t^{-q-1}\{T(n, m-2, q; t) + T(n-1, m-2, q; t)\}. \end{aligned} \tag{6.2.2}$$

The second factor of the right hand side of (6.2.2) is equal to $T(n, m-1, q; t)$ by Lemma 6.1.1. This completes induction. □

By setting $t = 1$ in (6.2.1), we obtain

Proposition 6.2.2

$$\dim N^{n-1}(\log D) = \sum_{i=0}^{n} \binom{m-1}{n-i} q^i.$$

7. A basis for $H^n(\Omega^{\cdot}(\log D), \nabla_\omega)$

In this section we find a basis for $H^n(\Omega \cdot (\log D), \nabla_\omega)$. To the end, we give a basis for $N^{n-1}(\log D)$. The argument to give a basis is analogous to [AKOT section 9].

7.1. Let $\mathcal{G} = \{H_1, \ldots, H_{m-1}\}$ be an affine n-arrangement in general position with $m-1$ hyperplanes defined by $H_j = \{P_j(u) = 0\}$, $1 \le j \le m-1$ and $P_m(u)$ a $\bar{\mathcal{G}}$-transverse polynomial of degree $q+1$. Suppose that Assumption 1.2.1 holds for P_j, $1 \le j \le m$. Let $X \in L^+(\mathcal{G})$. There is a unique set $\{H_{j_1}, \ldots, H_{j_p}\} \subset \mathcal{G}$ with $X = H_{j_1} \cap \cdots \cap H_{j_p}$. Then we define $Q_X = P_{j_1} \cdots P_{j_p}$.

Let $\Delta(P_m)$ denote the Jacobi ideal of P_m; $\Delta(P_m) := \left(\frac{\partial P_m}{\partial u_1}, \ldots, \frac{\partial P_m}{\partial u_n}\right)$. The quotient $S/\Delta(P_m)$is called the associated Milnor algebra.

Let $\bar{X} \in L(\bar{\mathcal{G}})$ be the parallel translate of X containing the origin and $I_{\bar{X}}$ the ideal in S consisting of the polynomials vanishing on $\bar{X}$. Then $S_{\bar{X}} := S/I_{\bar{X}}$ is the coordinate ring of $\bar{X}$. The Milnor algebra of the restriction $\bar{P}_m|_{\bar{X}}$ of $\bar{P}_m$ to $\bar{X}$ is equal to $S_{\bar{X}}/\Delta(\bar{P}_m|_{\bar{X}})$.

Lemma 7.1.1 *Let $P_m(u)$ be a $\bar{\mathcal{G}}$-transverse polynomial of degree $q+1$ and $X \in L^+(\mathcal{G})$. Then we have*

$$\mathrm{Poin}(S_{\bar{X}}/\Delta(\bar{P}_m|_{\bar{X}}), t) = A(q-1; t)^{\dim X}.$$

Proof. It is shown in [AKOT, Lemma 9.2]. □

7.2. There is a natural surjective map $\phi_X : S \longrightarrow S_{\bar{X}}/\Delta(\bar{P}_m|_{\bar{X}})$. Let $MB_X \subset S$ be a set of homogeneous polynomials on which ϕ_X is injective and so that $\phi_X(MB_X)$ is a basis for $S_{\bar{X}}/\Delta(\bar{P}_m|_{\bar{X}})$. We call MB_X a *Milnor basis* at X.

Let $X \in L^+(\mathcal{G})$. Define

$$\mathcal{P}_X = \left\{ \frac{b\sigma}{\bar{Q}_X} \;\middle|\; b \in MB_X \right\}$$

where $\sigma := \sum_{i=1}^{n} u_i * du_i$ and $\bar{Q}_X$ is the homogeneous part of maximal degree of Q_X. And We denote $\mathcal{P} = \bigcup_{X \in L^+} \mathcal{P}_X$. Let

$$\mathcal{NP} = \left\{ \frac{\sigma}{\bar{P}_{j_1} \cdots \bar{P}_{j_{n-1}} \bar{P}_k \cdots \bar{P}_{m-1}} \;\middle|\; n \le k \le m-1,\ 1 \le j_1 < \cdots < j_{n-1} \le k-1 \right\}.$$

If $n > m-1$, then we consider $\mathcal{NP} = \emptyset$. Let

$$\mathcal{B} = \mathcal{P} \cup \mathcal{NP}.$$

The notation for $\mathcal{P}$, $\mathcal{NP}$, and $\mathcal{B}$ are defined in [AKOT section 9].

We define

$$\begin{aligned}
\widetilde{\mathcal{P}}_X &= \left\{ \frac{1}{\bar{P}_m} \varphi \;\middle|\; \varphi \in \mathcal{P}_X \right\} \subset \Omega^{n-1}(\log \bar{D}), \\
\widetilde{\mathcal{P}} &= \bigcup_{X \in L^+} \widetilde{\mathcal{P}}_X, \\
\widetilde{\mathcal{NP}} &= \left\{ \frac{1}{\bar{P}_m} \varphi \;\middle|\; \varphi \in \mathcal{NP} \right\} \subset \Omega^{n-1}(\log \bar{D}), \\
\widetilde{\mathcal{B}} &= \widetilde{\mathcal{P}} \cup \widetilde{\mathcal{NP}} \subset \Omega^{n-1}(\log \bar{D}).
\end{aligned}$$

For $\varphi \in \Omega^{n-1}(\log \bar{D})$, let $[\varphi]_\mu$ denote the element of $N^{n-1}(\log D)_\mu = \Omega^{n-1}(\log \bar{D})_\mu / \operatorname{Im} \sigma_\mu^{n-1}$ represented by φ. Define

$$[\widetilde{\mathcal{P}}_X] = \left\{ \left[\frac{b\sigma}{\bar{Q}_X \bar{P}_m} \right]_{\deg b + \dim X - (q+1)} \;\middle|\; b \in MB_X \right\} \subset N^{n-1}(\log D)$$

for $X \in L^+(\mathcal{G})$ and let $[\widetilde{\mathcal{P}}] = \bigcup_{X \in L^+} [\widetilde{\mathcal{P}}_X]$. Define

$$[\widetilde{\mathcal{NP}}] = \left\{ \left[\frac{\sigma}{\bar{P}_{j_1} \cdots \bar{P}_{j_{n-1}} \bar{P}_k \cdots \bar{P}_{m-1} \bar{P}_m} \right]_{k-(m-1)-(q+1)} \;\middle|\; n \le k \le m-1,\ 1 \le j_1 < \cdots < j_{n-1} \le k-1 \right\} \subset N^{n-1}(\log D)$$

and

$$[\widetilde{\mathcal{B}}] = [\widetilde{\mathcal{P}}] \cup [\widetilde{\mathcal{NP}}] \subset N^{n-1}(\log D).$$

Proposition 7.2.1 *Let $\mathcal{G} = \{H_1, \ldots, H_{m-1}\}$ be a general position n-arrangement with $m-1$ hyperplanes defined by $H_j = \{P_j(u) = 0\}$, $1 \leq j \leq m-1$ and P_m a $\bar{\mathcal{G}}$-transverse polynomial of degree $q+1 > 0$. Suppose that Assumption* 1.2.1 *holds for P_j, $1 \leq j \leq m$. If $\sum_{j=1}^{m} l_j\alpha_j \neq l, l-1, \ldots,$ then the set $[\widetilde{\mathcal{B}}] = [\widetilde{\mathcal{P}}] \cup [\widetilde{\mathcal{NP}}]$ is a basis for $N^{n-1}(\log D)$.*

Proof. First we show that

$$\sum_{\widetilde{b}\in\widetilde{\mathcal{B}}} t^{\deg \widetilde{b}} = t^{-q-1}T(n, m-1, q; t).$$

By Lemma 7.1.1, we know $\mathrm{Poin}(S_{\bar{X}}/\Delta(\bar{P}_m|_{\bar{X}})) = A(q-1;t)^{\dim X}$. Therefore

$$\begin{aligned}\sum_{\widetilde{b}\in\widetilde{\mathcal{P}}} t^{\deg \widetilde{b}} &= \sum_{X\in L^+}\sum_{\widetilde{b}\in\widetilde{\mathcal{P}}_X} t^{\deg \widetilde{b}} \\ &= \sum_{X\in L^+} t^{-q-1+\dim X} A(q-1;t)^{\dim X} \\ &= t^{-q-1}\sum_{i=1}^{n}\binom{m-1}{n-i}[A(q;t)-1]^i.\end{aligned}$$

From the definition, we have

$$\sum_{\widetilde{b}\in\widetilde{\mathcal{NP}}} t^{\deg \widetilde{b}} = t^{-q-1}\sum_{i=n-(m-1)}^{0}\binom{i+(m-1)-1}{n-1}t^i.$$

Therefore

$$\sum_{\widetilde{b}\in\widetilde{\mathcal{B}}} t^{\deg \widetilde{b}} = \sum_{\widetilde{b}\in\widetilde{\mathcal{P}}} t^{\deg \widetilde{b}} + \sum_{\widetilde{b}\in\widetilde{\mathcal{NP}}} t^{\deg \widetilde{b}} = t^{-q-1}T(n, m-1, q; t).$$

It suffices to show that $\widetilde{\mathcal{B}}$ generates $N^{n-1}(\log D)$. We shall prove by induction on m as in Proposition 6.2.1. We must check two initial conditions.

(i) $n = 1$, $m \geq 1$. Write $u = u_1$. Then $\sigma = u * du$. We may choose $P_j(u) = u - x_j$, $1 \leq j \leq m-1$, with $x_i \neq x_j$ for $i \neq j$. Then $\bar{P}_j = u$, $1 \leq j \leq m-1$. We may choose $\bar{P}_m = u^{q+1}$. In the proof of Proposition 6.2.1 we know that

$$N^0(\log D)_\mu = \begin{cases} \dfrac{\sigma}{u^{m-1}\cdot u^{q+1}}\mathbb{C}[u]_{\mu+m+q-1} & \text{for} \quad \mu < 0, \\ 0 & \text{for} \quad \mu \geq 0. \end{cases}$$

Therefore we can choose

$$\left\{\frac{\sigma}{u^{m+q}}, \frac{\sigma}{u^{m+q-1}}, \dots, \frac{\sigma}{u^2}\right\}$$

as a basis for $N^0(\log D)$. We will show that the set above is equal to $\widetilde{\mathcal{B}}$. The element of L^+ is only $\mathbb{C}$. The Milnor algebra of $\mathbb{C}$ is $\mathbb{C}[u]/(u^q)$. It has a natural basis $MB_{\mathbb{C}} = \{1, u, \dots, u^{q-1}\}$, so

$$\widetilde{\mathcal{P}} = \left\{\frac{\sigma}{u^{q+1}}, \frac{\sigma}{u^q}, \dots, \frac{\sigma}{u^2}\right\}.$$

Simple computation shows that

$$\widetilde{\mathcal{NP}} = \left\{\frac{\sigma}{u^{m+q}}, \frac{\sigma}{u^{m+q-1}}, \dots, \frac{\sigma}{u^q}\right\}.$$

Therefore $\widetilde{\mathcal{B}} = \widetilde{\mathcal{P}} \cup \widetilde{\mathcal{NP}}$ gives a basis for $N^0(\log D)$. It shows that Theorem 7.2.1 holds for $n = 1$, $m \geq 1$.

(ii) $n \geq 1$, $m = 1$. In case $\mathcal{G}$ is empty and $X = \mathbb{C}^n$. Thus we have

$$\widetilde{\mathcal{P}} = \left\{b \cdot \frac{\sigma}{\bar{P}_m} \;\middle|\; b \in MB_{\mathbb{C}^n}\right\}, \quad \widetilde{\mathcal{NP}} = \emptyset.$$

In the proof of Proposition 3.3.1, we see that

$$\operatorname{Ann} N^{n-1}(\log D) = \left(\frac{\partial \bar{P}_1}{\partial u_1}, \dots, \frac{\partial \bar{P}_1}{\partial u_n}\right) = \Delta(\bar{P}_1).$$

Therefore $\widetilde{\mathcal{B}} = \widetilde{\mathcal{P}}$ gives a basis for $N^{n-1}(\log D)$.

(iii) Let $n \geq 2$, $m \geq 2$. The same proof of [AKOT Theorem 9.6] works in this case with the help of the short exact sequence (5.1.3) of Theorem 5.1.3. This completes induction. □

7.3. Now we will find a basis for $H^n(\mathrm{Gr}^F \Omega^{\cdot}(\log D), \mathrm{Gr}^F(\nabla_\omega))$ and for $H^n(\Omega^{\cdot}(\log D), \nabla_\omega)$. Let $\tau := du_1 \wedge \cdots \wedge du_n$ be the volume form. Define

$$\widetilde{\mathcal{P}}_X{}' = \left\{\frac{b\tau}{Q_X P_m} \;\middle|\; b \in MB_X\right\} \subset \Omega^n(\log D)$$

for $X \in L^+(\mathcal{G})$ and $\widetilde{\mathcal{P}}' = \bigcup_{X \in L^+} \widetilde{\mathcal{P}}_X$. Define

$$\widetilde{\mathcal{NP}}{}' = \left\{\frac{\tau}{P_{j_1} \cdots P_{j_{n-1}} P_k \cdots P_{m-1} P_m} \;\middle|\; n \leq k \leq m-1,\right.$$

$$1 \le j_1 < \cdots < j_{n-1} \le k-1 \Big\} \subset \Omega^n(\log D).$$

If $n > m-1$, then we consider $\widetilde{\mathcal{NP}}' = \emptyset$. Let

$$\widetilde{\mathcal{B}}' = \widetilde{\mathcal{P}}' \cup \widetilde{\mathcal{NP}}' \subset \Omega^n(\log D).$$

For $\varphi \in F_\mu \Omega^n(\log D)$, let $[\varphi]_\mu$ denote the element of $\mathrm{Gr}_\mu^F \Omega^n(\log D)$ represented by φ. Define

$$[\widetilde{\mathcal{P}}_X]' = \left\{ \left[\frac{b\tau}{Q_X P_m} \right]_{\deg b + \dim X - (q+1)} \,\middle|\, b \in MB_X \right\} \subset \mathrm{Gr}^F \Omega^n(\log D)$$

for $X \in L^+(\mathcal{G})$ and let $[\widetilde{\mathcal{P}}]' = \bigcup_{X \in L^+} [\widetilde{\mathcal{P}}_X]'$. Define

$$[\widetilde{\mathcal{NP}}]' = \left\{ \left[\frac{\tau}{P_{j_1} \cdots P_{j_{n-1}} P_k \cdots P_{m-1} P_m} \right]_{k-(m-1)-(q+1)} \,\middle|\, n \le k \le m-1,\ 1 \le j_1 < \cdots < j_{n-1} \le k-1 \right\} \subset \mathrm{Gr}^F \Omega^n(\log D)$$

and

$$[\widetilde{\mathcal{B}}]' = [\widetilde{\mathcal{P}}]' \cup [\widetilde{\mathcal{NP}}]' \subset \mathrm{Gr}^F \Omega^n(\log D).$$

In 2.3, if $\sum_{j=1}^m l_j \alpha_j \ne l,\ l-1, \ldots$ (where $l_j = \deg P_j$ and $l = \sum_{j=1}^m l_j$), we have the isomorphism:

$$\delta^{n-1} : N^{n-1}(\log D)_\mu \xrightarrow{\sim} H^n(\mathrm{Gr}_\mu^F \Omega^{\cdot}(\log D), \mathrm{Gr}_\mu^F(\nabla_\omega)). \tag{7.3.1}$$

We can easily see that

$$\delta^{n-1}[\widetilde{\mathcal{P}}] = [\widetilde{\mathcal{P}}]', \quad \delta^{n-1}[\widetilde{\mathcal{NP}}] = [\widetilde{\mathcal{NP}}]'$$

up to non-zero constant, and hence

$$\delta^{n-1}[\widetilde{\mathcal{B}}] = [\widetilde{\mathcal{B}}]'. \tag{7.3.2}$$

Therefore we obtain the following:

Theorem 7.3.1 *Let $\mathcal{G}$ be an affine n-arrangement in general position with $m-1$ hyperplanes defined by $H_j = \{P_j(u) = 0\}$, $1 \le j \le m-1$. Let $P_m(u)$ be a $\bar{\mathcal{G}}$-transverse polynomial of degree $q+1 > 0$. Suppose that Assumption 1.2.1 holds for P_j, $1 \le j \le m$. If $\sum_{j=1}^m l_j \alpha_j \ne l,\ l-1, \ldots$, then*

(1) *The set* $[\widetilde{\mathcal{B}}]'$ *gives a basis for* $H^n(\mathrm{Gr}^F\Omega^{\cdot}(\log D), \mathrm{Gr}^F(\nabla_\omega))$,
(2) *The set* $\widetilde{\mathcal{B}}'$ *gives a basis for* $H^n(\Omega^{\cdot}(\log D), \nabla_\omega)$,
(3) $\mathrm{Poin}(H^n(\mathrm{Gr}^F\Omega^{\cdot}(\log D), \mathrm{Gr}^F(\nabla_\omega)), t) = t^{-q-1}T(n, m-1, q; t)$,
(4) $\dim H^n(\Omega^{\cdot}(\log D), \nabla_\omega) = \sum_{i=0}^n \binom{m-1}{n-i} q^i$.

Proof. Since the set $[\widetilde{\mathcal{B}}]$ gives a basis for $N^{n-1}(\log D)$ by Proposition 7.2.1, we have (1) in view of (7.3.2). Assertion (1) implies (2). By Proposition 6.2.2 and (7.3.1), we obtain (3). By setting $t = 1$ in (3), we get (4). □

In case that $P_m(u)$ is a $\bar{\mathcal{G}}$-transverse polynomial of degree two and P_j, $1 \leq j \leq m$ satisfies Assumption 1.2.1, we can choose a basis for $H^n(\Omega^{\cdot}(\log D), \nabla_\omega)$ more explicitly. By setting $q = 1$ in Lemma 7.1.1, we have

$$\mathrm{Poin}(S_{\bar{X}}/\Delta(\bar{P}_m|_{\bar{X}}), t) = 1.$$

This means that $MB_X = \{1\}$ for all $X \in L^+(\mathcal{G})$. Therefore

$$\widetilde{\mathcal{P}}' = \left\{ \frac{\tau}{P_{j_1} \cdots P_{j_r} P_m} \,\middle|\, 1 \leq j_1 < \cdots < j_r \leq m-1,\ 0 \leq r \leq n-1 \right\}.$$

Next consider $\widetilde{\mathcal{NP}}'$. Since $\mathcal{G} = \{H_1, \ldots, H_{m-1}\}$ is in general position, 1 is written as $1 = \sum_{\nu=1}^{n+1} c_\nu P_{j_\nu}$ for some constants $c_\nu \in \mathbb{C}$. By using the trick of partial fractional decomposition (see [K, pp. 74–75]), we can rewrite $\widetilde{\mathcal{NP}}'$ as

$$\left\{ \frac{\tau}{P_{j_1} \cdots P_{j_n} P_m} \,\middle|\, 1 \leq j_1 < \cdots < j_n \leq m-1 \right\}.$$

Therefore we obtain following:

Theorem 7.3.2 *Let* $\mathcal{G}$ *be an affine* n*-arrangement in general position with* $m-1$ *hyperplanes defined by* $H_j = \{P_j(u) = 0\}$, $1 \leq j \leq m-1$. *Let* $P_m(u)$ *be a* $\bar{\mathcal{G}}$*-transverse polynomial of degree two. Suppose that Assumption* 1.2.1 *holds for* P_j, $1 \leq j \leq m$. *If* $\sum_{j=1}^m l_j \alpha_j \neq l, l-1, \ldots$, *then we can choose*

$$\left\{ \frac{\tau}{P_{j_1} \cdots P_{j_r} P_m} \,\middle|\, 1 \leq j_1 < \cdots < j_r \leq m-1,\ 0 \leq r \leq n \right\}$$

as a basis for $H^n(\Omega^{\cdot}(\log D), \nabla_\omega)$, *and hence*

$$\dim H^n(\Omega^{\cdot}(\log D), \nabla_\omega) = \sum_{i=0}^{n} \binom{m-1}{n-i}.$$

Example 7.3.3 Consider the case $n = 2$, $m = 4$, and $q = 1$. Then $\tau = du_1 \wedge du_2$. Let $P_1 = u_1$, $P_2 = u_2$, $P_3 = u_1 + u_2 - 1/2$, $P_4 = u_1^2 + u_2^2 - 1$. Let $\mathcal{G} = \{H_1, H_2, H_3\}$, where $H_j = \{P_j = 0\}$, $1 \leq j \leq 3$. Then P_4 is a $\bar{\mathcal{G}}$-transverse and $P_1, \ldots, P_4$ satisfies Assumption 1.2.1. It follows from Theorem 7.3.1 (3) that $\dim H^2(\Omega^{\cdot}(\log D), \nabla_\omega) = 7$ and from Theorem 7.3.1 (2) that

$$\begin{aligned} \mathrm{Poin}(H^2(\mathrm{Gr}^F \Omega^{\cdot}(\log D), \mathrm{Gr}^F(\nabla_\omega)), t) &= t^{-2} T(2,3,1;t) \\ &= t^{-3} + 2t^{-2} + 3t^{-1} + 1. \end{aligned}$$

Then we can choose the following basis for $H^2(\Omega^{\cdot}(\log D), \nabla_\omega)$ by Theorem 7.3.1 (1):

$$\begin{aligned} \widetilde{\mathcal{P}}_{\mathbb{C}^2}{}' &= \left\{ \frac{\tau}{P_4} \right\} \\ \widetilde{\mathcal{P}}_{H_1}{}' &= \left\{ \frac{\tau}{P_1 P_4} \right\} \\ \widetilde{\mathcal{P}}_{H_2}{}' &= \left\{ \frac{\tau}{P_2 P_4} \right\} \\ \widetilde{\mathcal{P}}_{H_3}{}' &= \left\{ \frac{\tau}{P_3 P_4} \right\} \\ \widetilde{\mathcal{NP}}{}' &= \left\{ \frac{\tau}{P_1 P_3 P_4}, \frac{\tau}{P_2 P_3 P_4}, \frac{\tau}{P_1 P_2 P_3 P_4} \right\}. \end{aligned}$$

In view of Theorem 7.3.2, we can choose

$$\begin{aligned} &\frac{\tau}{P_4}, \frac{\tau}{P_1 P_4}, \frac{\tau}{P_2 P_4}, \frac{\tau}{P_3 P_4}, \\ &\frac{\tau}{P_1 P_2 P_4}, \frac{\tau}{P_1 P_3 P_4}, \frac{\tau}{P_2 P_3 P_4} \end{aligned}$$

as a basis for $H^2(\Omega^{\cdot}(\log D), \nabla_\omega)$.

Remark. Consider the case $q = 0$, that is, $P_j(u)$, $1 \leq j \leq m$ are all linear and the arrangement $\{P_j = 0\}_{1 \leq j \leq m}$ is in general position. By setting $q = 0$

in Lemma 7.1.1, we have

$$\mathrm{Poin}(S_{\bar{X}}/\Delta(\bar{P}_m|_{\bar{X}}), t) = 0.$$

This means that $\widetilde{\mathcal{P}}' = \emptyset$. By the same reasoning in case $q = 1$, we have

$$\widetilde{\mathcal{NP}}' = \left\{ \frac{\tau}{P_{j_1} \cdots P_{j_n} P_m} \,\middle|\, 1 \le j_1 < \cdots < j_n \le m-1 \right\}.$$

Since $\{H_{j_1}, \dots, H_{j_n}, H_m\}$ is in general position, 1 is written as $1 = \sum_{\nu=1}^{n} c_\nu P_{j_\nu} + c_m P_m$ for some constants $c_\nu, c_m \in \mathbb{C}$. Then

$$\begin{aligned} \frac{\tau}{P_{j_1} \cdots P_{j_n} P_m} &= \sum_{\nu=1}^{n} \frac{c_\nu \tau}{P_{j_1} \cdots \widehat{P_{j_\nu}} \cdots P_{j_n} P_m} + \frac{c_m \tau}{P_{j_1} \cdots P_{j_n}} \\ &= \sum_{\nu=1}^{n} \frac{c_\nu}{d_\nu} \varphi\langle j_1, \dots, \widehat{j_\nu}, \dots, j_n, m\rangle + \frac{c_m}{d_m} \varphi\langle j_1, \dots, j_n\rangle \end{aligned}$$

where d_ν, d_m are non-zero constants as

$$d_\nu = \left| \frac{\partial(P_{j_1}, \dots, \widehat{P_{j_\nu}}, \dots, P_{j_n}, P_m)}{\partial(u_1, \dots, u_n)} \right|, \quad d_m = \left| \frac{\partial(P_{j_1}, \dots, P_{j_n})}{\partial(u_1, \dots, u_n)} \right|$$

and denote

$$\varphi\langle j_1, \dots, j_p\rangle := \frac{dP_{j_1}}{P_{j_1}} \wedge \cdots \wedge \frac{dP_{j_p}}{P_{j_p}}.$$

Notice that

$$\nabla_\omega \varphi\langle j_1, \dots, j_{n-1}\rangle = \sum_{k=1}^{m} \alpha_k \varphi\langle k, j_1, \dots, j_{n-1}\rangle$$

and hence

$$\varphi\langle m, j_1, \dots, j_{n-1}\rangle = -\sum_{k=1}^{m-1} \frac{\alpha_k}{\alpha_m} \varphi\langle k, j_1, \dots, j_{n-1}\rangle$$

in $H^n(\Omega^{\cdot}(\log D), \nabla_\omega)$. Therefore we can choose

$$\left\{ \frac{dP_{j_1}}{P_{j_1}} \wedge \cdots \wedge \frac{dP_{j_n}}{P_{j_n}} \,\middle|\, 1 \le j_1 < \cdots < j_n \le m-1 \right\}$$

as a basis for $H^n(\Omega^{\cdot}(\log D), \nabla_\omega)$.

This result agrees with the formula for $H^n(\Omega^{\cdot}(\log D), \nabla_\omega)$ shown in [A1], [K].

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