Hokkaido Mathematical Journal Vol. 27 (1998) p. 553–565

Point separation of a two-sheeted disc by bounded analytic functions

(To Professor Mitsuru Nakai on his sixtieth birthday)

Mikihiro HAYASHI and Takao KATO

(Received May 7, 1997)

Abstract. Let $\pi:\widetilde{\Delta}\to\Delta$ be a two-sheeted unlimited covering of the open unit disc Δ, and a_n be its ramification points. Taking mutually disjoint closed discs $\Delta_n=\{z:|z-a_n|\leq r_n\}$ in Δ, we let $\widetilde{D}=\pi^{-1}(\Delta\setminus\cup\Delta_n)$. For $a_n=1-\frac{1}{n}$, the points of Riemann surface $\widetilde{D}$ are separated or *not* separated by bounded analytic functions depending on the sizes of radii r_n. We show the sharpness of such a condition on r_n. We also obtain a similar result for the case $a_{nj}=(1-2^{-n})e^{2\pi ij/2^n}$.

Key words: point separation, bounded analytic function, two-sheeted disc, Riemann surface.

1. Introduction and Main Results

Let $\Delta=\{z:|z|<1\}$ be the open unit disc in the complex plane $\mathbb{C}$. Let $\pi:\widetilde{\Delta}\to\Delta$ be a two-sheeted unlimited covering of Δ whose ramification points, say a_n, does not accumulate in Δ. Throughout this paper, we denote $\widetilde{D}=\pi^{-1}(D)$ for any subdomain D of Δ. In this paper, we consider the problem when the points of $\widetilde{D}$ are **separated** by bounded analytic functions, that is, for any pair of distinct points a, b in $\widetilde{D}$, there is a bounded analytic function f on $\widetilde{D}$ such that $f(a)\neq f(b)$. The purpose of this paper is to show a sharpness of the non-separating conditions given in [6] by improving the separating conditions given there.

We are interested in this problem from several aspects. First, if a given Riemann surface admits a non constant bounded analytic function, then the next problem to be asked is the point separation. Here, we have some general theories (e.g. [1], [7]). However, our knowledge in this area is still

1991 Mathematics Subject Classification : Primary 30D50 30H05 32E25, Secondary 46J15.

The first author was partially supported by Grant-in-Aid for Scientific Research, No. 07640154, Japanese Ministry of Education, Science and Culture.

The second author was partially supported by Grant-in-Aid for Scientific Research, No. 06302011, Japanese Ministry of Education, Science and Culture.

quite limited. In fact, we have to solve the problem directly by constructing a bounded analyitc function. Second, the construction of a bounded analytic function on a Riemann surfece, is not a matter of triviality, even in a simple case like the two-sheeted domain $\widetilde{D}$. Our construction, which we have found by "trial and success," may have own interests. Third, the points of the domain $\widetilde{D}$ are separated by bounded analytic function, then the pole set of domain $\widetilde{D}$ turns out to be the whole $\widetilde{D}$, which incereses the number of examples to which the theory in [2] can apply.

It is known [8] that the points of $\widetilde{\Delta}$ is separated by bounded analytic functions if and only if $\{a_n\}$ satisfies the Blaschke condition, that is, $\sum(1-|a_n|)<\infty$.

Now, in what follows we always assume that $\sum(1-|a_n|)=\infty$. In this case, choosing a sequence of closed discs $\Delta_n=\{z:|z-a_n|\le r_n\}$ satisfying the condition

$$\Delta_n \subset \Delta \quad \text{and} \quad \Delta_n \cap \Delta_m = \emptyset \quad (n \neq m), \tag{1}$$

we define a subdomain D of Δ by

$$D = \Delta \setminus \bigcup_n \Delta_n. \tag{2}$$

Then, it may happen that the points of $\widetilde{D}$ are separated by bounded analytic functions, while it depends on the distribution of ramificaiton points $\{a_n\}$ and the sizes of radii $\{r_n\}$ ([3], [4], [5], [6]). We are interested in studying how the sizes of r_n are related with the ramification points a_n. A sequence of closed discs Δ_n (more simply, a sequence of the radii r_n) is called **admissible** if Δ_n satisfies condition (1).

The following two theorems were obtained in [6].

Theorem A *Let $a_n = 1-\frac{1}{n}$ and radii r_n be admissible.*

(a) *If there is a positive sequence $\{\eta_n\}$ such that $\sum_{n=1}^{\infty}\frac{1}{\eta_n}<\infty$ and $r_n \le (\frac{1}{n})^{\eta_n}$, then the points of $\widetilde{D}$ are not separated by bounded analytic functions.*

(b) *It is possible to choose r_n so that the points of the covering domain $\widetilde{D}$ are separated by bounded analytic functions.*

Theorem B *Let $a_{nj}=(1-2^{-n})e^{2\pi i j/2^n}$ $(0\le j<2^n,\ n\ge 1)$ and radii r_{nj} be admissible.*

(a) *If there is a positive sequence $\{\eta_n\}$ such that $\sum_{n=1}^{\infty}\frac{1}{\eta_n}<\infty$ and*

$r_{nj} \leq 2^{-\eta_n}$, *then the points of* $\widetilde{D}$ *are not separated by bounded analytic functions.*

(b) *It is possible to choose* r_{nj} *so that the points of the covering domain* $\widetilde{D}$ *are separated by bounded analytic functions.*

The parts (b) were proved there in the following way: First, one can find a sequence of discs Δ'_k with center a_k such that the interiors of these discs are mutually disjoint and such that Δ'_{2k-1} touches with Δ'_{2k} at a point, where one may consider a suitable renumbering for the sequence a_{nj}. Second, one replaces Δ'_k by a smaller ones Δ_k so as to be admissible. Finally, if one choose Δ_k's so large that Δ_{2k-1} almost touches with Δ_{2k}, the points of the covering domain $\widetilde{D}$ can be separated by bounded analytic functions.

Because of this way of proof, it was not able to determine whether the conditions stated in the parts (a) are sharp or not. Our aim is to show that these conditions are acutually sharp in a sense.

Meanwhile, our answer is not complete by any means. We shall leave some problems which may improve our results. This sort of problems could be completely answered if one would give a necessary and sufficient condition. However, a consideration shows that it seems to be quite difficult, or might be impossible to do so; for instance, sizes of r_{n_k} have no importance for any subsequence a_{n_k} satisfying the Blaschke condition.

Throughout this paper, we denote by $\{\sigma_n\}$ an arbitrary sequence of positive numbers satisfying

$$0 < \liminf_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} \sigma_n < \infty.$$

Theorem 1 *Let* $a_n = 1-\frac{1}{n}$ *and let* $r_n = e^{-\sigma_n n^p}$ *be an admissible sequence. Then, the points of the covering domain* $\widetilde{D}$ *are separated by bounded analytic functions if* $0 < p \leq 1$, *and not separeated if* $p > 1$.

Theorem 2 *Let* $a_{nj} = (1 - 2^{-n})e^{2\pi i j/2^n}$ $(0 \leq j < 2^n,\ n \geq 1)$ *and let* $r_{nj} = \sigma_n 2^{-n^p}$ *be an admissible sequence. Then, the points of the covering domain* $\widetilde{D}$ *are separated by bounded analytic functions if* $p = 1$, *and not separeated if* $p > 1$.

In Theorem 2, the case $p < 1$ is excluded because the sequence r_{nj} is then not admissible.

The "not separated" parts of the theorems immediately follows from the parts (a) of Theorem A and B, respectively. We only have to show the "separated" parts.

2. Proof of Theorem 1

Set

$$Q_{k,N}(z) = \prod_{j=k}^{N} q_{4j-3}(z), \quad q_n(z) = \frac{(n-z)(n+3-z)}{(n+1-z)(n+2-z)}.$$

First, we estimate the growth of functions $Q_{k,N}$ on the domain

$$G = \mathbb{C} \setminus \bigcup_{n=1}^{\infty} G_n, \quad G_n = \{z : |z-n| \leq \rho_n\},$$

where $0 < \rho_n < \frac{1}{2}$.

The following estimate implies that the infinite product $Q = Q_{1,\infty}$ converges to a meromorphic function on the complex plane $\mathbb{C}$.

Estimate 2.1 *If* $z \in \partial G_{4k-2} \cup \partial G_{4k-1}$, *then*

$$|Q_{1,k-1}(z)| \leq e^3 \tag{3}$$

$$|q_{4k-3}(z)| \leq \frac{8}{\min(\rho_{4k-2}, \rho_{4k-1})} \tag{4}$$

$$|Q_{k+1,N}(z)| \leq e^3 \tag{5}$$

Proof. If $|z-1| = \rho_1$, then

$$|q_0(z)| \leq \frac{(1+\rho_1)(2+\rho_1)}{\rho_1(1-\rho_1)} \leq \frac{15}{2\rho_1} < \frac{8}{\rho_1}.$$

In the same way, if $|z-2| = \rho_2$, then

$$|q_0(z)| \leq \frac{8}{\rho_2}.$$

Since $q_n(z) = q_0(z-n)$, this proves (4). Next, note that

$$1 - q_n(z) = \frac{2}{(n+1-z)(n+2-z)}.$$

If $|m-n| \geq 4$ and $z \in \partial G_{m+1} \cup \partial G_{m+2}$, then

$$\operatorname{dist}(\{n+1, n+2\}, z) \geq |n-m| - 3.$$

Therefore,

$$|1 - q_n(z)| \leq \frac{2}{(|m-n|-3)^2}.$$

If $z \in \partial G_{4k-2} \cup \partial G_{4k-1}$, then

$$\begin{aligned}\sum_{j=k+1}^{N} |1 - q_{4j-3}(z)| &\leq \sum_{j=k+1}^{\infty} \frac{2}{(4j-4k-3)^2} = \sum_{j=1}^{\infty} \frac{2}{(4j-3)^2} \\ &\leq 2 + \frac{1}{8}\sum_{j=2}^{\infty} \frac{1}{(j-1)^2} < 3.\end{aligned}$$

Since $|x| \leq e^{|x|-1} \leq e^{|1-x|}$,

$$|Q_{k+1,N}(z)| \leq \exp\left(\sum_{j=k+1}^{N} |1 - q_{4j-3}(z)|\right) < e^3.$$

This proves (5). In a similar way, we have

$$\sum_{j=1}^{k-1} |1 - q_{4j-3}(z)| \leq \sum_{j=1}^{k-1} \frac{2}{(4j-4k-3)^2} = \sum_{j=1}^{\infty} \frac{2}{(4j-3)^2} < 3,$$

and we see (3). □

Lemma 2.2 *Let σ be a positive constant. Then, the function $g(z) = e^{-\sigma z}Q(z)$ is bounded on the domain*

$$G' = \{z : \Re z > 0\} \setminus \bigcup_{n=1}^{\infty} G_n, \quad G_n = \{z : |z-n| \leq \rho_n\}$$

whenever $\limsup_{n\to\infty} \frac{\log 1/\rho_n}{n} < \sigma$.

Proof. Since it suffices to prove the boundedness of function g on a larger domain than given domain G', we may replace positive numbers ρ_n by smaller ones so that $0 < \rho_n < \frac{1}{2}$ and $\rho_n \geq \rho_{n+1}$. Set $g_N(z) = e^{-\sigma z}Q_{1,N}$, If $z \in \partial G_{4k-2} \cup \partial G_{4k-1}$, then by Estimate 2.1

$$|g_N(z)| \leq e^{-\sigma(4k-3)} \cdot \frac{8e^6}{\rho_{4k-1}} = 8e^{-\sigma(4k-1)+\log(1/\rho_{4k-1})}e^{2\sigma+6}.$$

By the assumption, we find a number k_0, independent of N, such that $-\sigma(4k-1)+\log(1/\rho_{4k-1})<0$ when $k>k_0$. By the maximum modulus principle, we see that g_N is bounded on the domain G'. Letting $N\to\infty$, we are done. $\square$

Proof of Theorem 1. As we noted after the statement of the theorems, we only consider the case $p\le 1$. By the same reason as in the proof of the preceding lemma, it is sufficient to prove the theorem for $p=1$. The transformation $\psi(z)=\frac{1}{1-z}$ maps the closed disc Δ_n onto the closed disc with diameter, the line segment $\left[n-\frac{n^2r_n}{1+nr_n}, n+\frac{n^2r_n}{1-nr_n}\right]$. If we set $\rho_n=\frac{n^2r_n}{1+nr_n}$, then the domain G' defined in Lemma 2.2 includes the set $\psi(D)$. Since

$$\begin{aligned}\limsup_{n\to\infty}\frac{\log 1/\rho_n}{n} &= \limsup_{n\to\infty}\frac{\log(1+ne^{-\sigma_n n})+\sigma_n n-2\log n}{n}\\ &= \limsup_{n\to\infty}\sigma_n,\end{aligned}$$

it follows that $e^{-\sigma z}Q(z)$ is bounded on $\phi(D)$ if we choose the number σ so large that $\limsup_{n\to\infty}\sigma_n<\sigma$. Now, any pair of distinct points of $\widetilde{D}$ can be separated by one of two functions $\sqrt{e^{-\sigma\psi(z)}Q(\psi(z))}$ and $z\circ\pi$ on $\widetilde{D}$. $\square$

We close this section with two comments.

Using $P(z)=\prod_{n=1}^{\infty}\frac{z-a_{4n-3}}{z-a_{4n-2}}\frac{z-a_{4n}}{z-a_{4n-1}}$ in place of $Q(\psi(z))$, one can prove Theorem 1 rather directly without using domain transformation $\psi(z)$, while one may need a little more efforts to estimate the function $P(z)$.

It might be an interesting problem to determine whether the points of $\widetilde{D}$ are separated by bounded analytic functions or not, when $r_n=(\frac{1}{n})^{\sigma_n n}$ in Theorem 1.

3. Proof of Theorem 2

For given real number $0<\rho<1$ and positve integer m, we put $\omega_m=e^{2\pi i/2m}$, $a_j=\rho\omega_m^j$ and

$$\begin{aligned}P_m^{\rho}(z) &= \prod_{j=0}^{m-1}\frac{a_{2j}}{|a_{2j}|}\frac{1-\bar{a}_{2j}z}{a_{2j}-z}\frac{|a_{2j+1}|}{a_{2j+1}}\frac{a_{2j+1}-z}{1-\bar{a}_{2j+1}z}\\ &= \frac{1-\rho^m z^m}{\rho^m-z^m}\frac{\rho^m+z^m}{1+\rho^m z^m}.\end{aligned} \tag{6}$$

It follows that

$$P_m^\rho(z) - 1 = \frac{2(1-\rho^{2m})z^m}{(\rho^m - z^m)(1+\rho^m z^m)} \tag{7}$$

$$|P_m^\rho(z)|^2 - 1 = \frac{4(1-\rho^{2m})(1-|z|^{2m})\rho^m \Re z^m}{|\rho^m - z^m|^2|1+\rho^m z^m|^2}. \tag{8}$$

Lemma 3.1 *If $0 < \sigma < 1$, then there is an integer N_σ and a positive constant c_σ such that*

$$c_\sigma < \left|1 - \left(1 + \frac{z}{m}\right)^m\right| \leq 4$$

for all complex numbers z and intergers m with $\sigma \leq |z| \leq 1$ and $m \geq N_\sigma$.

Proof. Since

$$\left|1 + \frac{z}{m}\right|^m \leq \left(1 + \frac{|z|}{m}\right)^m \leq \left(1 + \frac{1}{m}\right)^m < e < 3,$$

the second inequality holds. To prove the first inequality, we note that $(1+\frac{z}{m})^m$ converges to e^z uniformly in z for $|z| \leq 1$. Let $c_\sigma = \frac{1}{2}\inf\{|1-e^z| : \sigma \leq |z| \leq 1\}$. Then, $c_\sigma > 0$. Hence, there exists an integer N_σ such that

$$\left|\left(1 + \frac{z}{m}\right)^m - e^z\right| \leq c_\sigma$$

for $|z| \leq 1$ and $m \geq N_\sigma$. This proves the first inequality. □

The next lemma is elementary.

Lemma 3.2 *If $0 < x < m$, then $(1 - \frac{x}{m})^m$ is decreasing in x and increasing in m. In particular, $(1-\frac{x}{m})^m \leq e^{-c}$ if $0 < c \leq x < m$.*

Estimate 3.3 *If $|z| < \rho$, then*

$$|P_m^\rho(z) - 1| < \frac{2|z|^m}{(\rho^m - |z|^m)(1-|z|^m)}$$

Proof. From (7),

$$|P_m^\rho(z) - 1| \leq \frac{2(1-\rho^{2m})|z|^m}{(\rho^m - |z|^m)(1-|z|^m)} < \frac{2|z|^m}{(\rho^m - |z|^m)(1-|z|^m)}.$$

□

Estimate 3.4 *Let* $0 < \sigma < 1$, $0 < \rho < 1$ *and let* $\Delta_j = \{z : |z - a_j| \le \frac{\rho\sigma}{m}\}$. *If* $m \ge N_\sigma$, *then*

$$|P^\rho_m(z)| \le \frac{8}{c_\sigma} \cdot \frac{1}{1-\rho^m} \quad on \quad \Delta \setminus \bigcup_{j=0}^{m-1} \Delta_{2j}$$

Proof. Since $|P^\rho_m(z)| = 1$ on $|z| = 1$, we have only to estimate P^ρ_m on $\partial\Delta_{2j}$, $0 \le j < m$. Noting a rotation invariance by angle ω_m, we may consider only the case $j = 0$. Let $|z - \rho| = \frac{\rho\sigma}{m}$, or $z = \rho(1 + \frac{\sigma}{m}e^{i\theta})$. By the sake of Lemma 3.1, it follows from (6) that

$$\begin{aligned} |P^\rho_m(z)| &\le \frac{2}{\rho^m|1 - (1 + \frac{\sigma}{m}e^{i\theta})^m|} \cdot \frac{\rho^m|1 + (1 + \frac{\sigma}{m}e^{i\theta})^m|}{1-\rho^m} \\ &\le \frac{2 \cdot 4}{c_\sigma} \cdot \frac{1}{1-\rho^m} \end{aligned}$$

□

Estimate 3.5 *If* $\rho < |z| < 1$, *then*

$$||P^\rho_m(z)|^2 - 1| \le \frac{8(1 - |z|^{2m})|\frac{\rho}{z}|^m}{(1 - |\frac{\rho}{z}|^m)^2|1-\rho^m|}.$$

Proof. From (8),

$$\begin{aligned} ||P^\rho_m(z)|^2 - 1| &\le \frac{4(1-\rho^{2m})(1-|z|^{2m})\rho^m|\Re z^m|}{(|z^m| - \rho^m)^2(1-\rho^m)^2} \\ &\le \frac{4(1+\rho^m)(1-|z|^{2m})|\frac{\rho}{z}|^m}{(1-|\frac{\rho}{z}|^m)^2(1-\rho^m)} \\ &\le \frac{8(1-|z|^{2m})|\frac{\rho}{z}|^m}{(1-|\frac{\rho}{z}|^m)^2|1-\rho^m|}. \end{aligned}$$

□

Now, let $\{\rho_n\}$ be a strictly increasing sequence of positive numbers with $\rho_n \to 1$, and let $\{m_n\}$ be an increasing sequence of positive integers. Put

$$a_{nj} = \rho_n \omega^j_{m_n} \quad (0 \le j < m_n)$$

$$P(z) = \prod_{n=1}^{\infty} P_n(z), \quad P_n(z) = P^{\rho_n}_{m_n}(z)$$

$$D = \Delta \setminus \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{m_n} \Delta_{nj}, \quad \Delta_{nj} = \{z : |z - a_{nj}| \le r_n\}, \quad r_n = \frac{\sigma_n}{m_n}$$

Proposition 3.6 *Let $\{\Delta_{nj}\}$ be an admissible sequence of closed discs, defined as above. If there are constants δ, M_1 and M_2 (independent of k) such that*

$$\lim_{k \to \infty} m_k = \infty \tag{9}$$

$$\left(\frac{\rho_k}{\rho_{k+1}}\right)^{m_k} \le \delta < 1 \quad (\textit{for sufficiently large } k) \tag{10}$$

$$\sum_{n=1}^{k-1} (1 - \rho_k^{2m_n}) \left(\frac{\rho_n}{\rho_k}\right)^{m_n} \le M_1 \tag{11}$$

$$\sum_{n=k+1}^{\infty} \left(\frac{\rho_k}{\rho_n}\right)^{m_n} \le M_2. \tag{12}$$

Then, the points of the covering domain $\widetilde{D}$ is separated by bounded analytic functions.

Proof. Since any pair of distinct points of $\widetilde{D}$ can be separeted by one of two functions $\sqrt{P(z)}$ and $z \circ \pi$ on $\widetilde{D}$, it is enough to prove the boundedness of function $P(z)$. Noting $\liminf \sigma_n/\rho_n = \liminf \sigma_n > 0$, we find a constant σ such that $0 < \sigma < 1$ and $\sigma\rho_n < \sigma_n$ for all n. It suffices to prove the boundedness of $P(z)$ for $r_n = \frac{\sigma\rho_n}{m_n}$. Also, replacing integer N_σ in Lemma 3.1 to a larger one, if necessary, we may assume that (10) holds for $k \ge N_\sigma$. Let $z \in \partial\Delta_{kj}$, $k > N_\sigma$. Note that $\rho_{k-1} \le |z| \le \rho_{k+1}$. If $N_\sigma \le n < k-1$, then

$$\rho_n^{m_n} \le \left(\frac{\rho_n}{\rho_{k-1}}\right)^{m_n} \le \left(\frac{\rho_n}{\rho_{n+1}}\right)^{m_n} \le \delta.$$

By Estimate 3.5,

$$\begin{aligned} ||P_n(z)| - 1| \le \left||P_n(z)|^2 - 1\right| &\le \frac{8(1 - \rho_{k-1}^{2m_n})(\frac{\rho_n}{\rho_{k-1}})^{m_n}}{(1 - (\frac{\rho_n}{\rho_{k-1}})^{m_n})^2(1 - \rho_n^{m_n})} \\ &\le \frac{8}{(1-\delta)^3}(1 - \rho_{k-1}^{2m_n}) \left(\frac{\rho_n}{\rho_{k-1}}\right)^{m_n} \end{aligned}$$

and hence, (11) yields

$$\sum_{n=N_\sigma}^{k-2} ||P_n(z)| - 1| \leq \frac{8M_1}{(1-\delta)^3}.$$

If $n = k-1, k, k+1$, then

$$\rho_n^{m_n} \leq \left(\frac{\rho_n}{\rho_{n+1}}\right)^{m_n} \leq \delta.$$

By Estimate 3.4,

$$|P_n(z)| \leq \frac{8}{c_\sigma} \cdot \frac{1}{1-\rho_n^{m_n}} \leq \frac{8}{c_\sigma(1-\delta)}.$$

If $k+1 < n$, then

$$\rho_{k+1}^{m_n} \leq \left(\frac{\rho_{k+1}}{\rho_n}\right)^{m_n} \leq \left(\frac{\rho_{n-1}}{\rho_n}\right)^{m_{n-1}} \leq \delta,$$

where we use $m_{n-1} \leq m_n$. By Estimate 3.3,

$$|P_n(z) - 1| \leq \frac{2(\frac{\rho_{k+1}}{\rho_n})^{m_n}}{(1-(\frac{\rho_{k+1}}{\rho_n})^{m_n})(1-\rho_{k+1}^{m_n})} \leq \frac{2}{(1-\delta)^2}\left(\frac{\rho_{k+1}}{\rho_n}\right)^{m_n}$$

and hence, (12) yields

$$\sum_{n=k+2}^{\infty} |P_n(z) - 1| \leq \frac{2M_2}{(1-\delta)^2}.$$

Combining these inequalities, we have

$$\begin{aligned} |P| &\leq \left|\prod_{n<N_\sigma} P_n\right| \cdot \prod_{n=N_\sigma}^{k-2} |P_n| \cdot |P_{k-1}| \cdot |P_k| \cdot |P_{k+1}| \cdot \prod_{n=k+2}^{\infty} |P_n| \\ &\leq \left|\prod_{n<N_\sigma} P_n\right| \cdot \exp\left(\frac{8M_1}{(1-\delta)^3}\right) \cdot \left\{\frac{8}{c_\sigma(1-\delta)}\right\}^3 \cdot \exp\left(\frac{2M_2}{(1-\delta)^2}\right). \end{aligned}$$

This completes the proof. □

Proof of Theorem 2. Set $\rho_n = 1 - 2^{-n}$ and $m_n = 2^n$. It follows from Lemma 3.2 that

$$\left(\frac{\rho_k}{\rho_{k+1}}\right)^{m_k} = \left(1 - \frac{1}{2^{k+1}}\frac{1}{1-2^{-k-1}}\right)^{2^k} \leq e^{-\frac{1}{2}} = \delta < 1.$$

If $n < k$, then $(\frac{\rho_n}{\rho_k})^{m_n} \leq (\frac{\rho_n}{\rho_{n-1}})^{m_n} \leq \delta$. Since $1-nx \leq (1-x)^n$ for $0 < x < 1$, we have $1 - \rho_k^{2m_n} = 1 - (1 - \frac{1}{2^k})^{2^{n+1}} \leq \frac{2^{n+1}}{2^k} = 2^{n-k+1}$. Hence,

$$\sum_{n=1}^{k-1}(1 - \rho_k^{2m_n})\left(\frac{\rho_n}{\rho_k}\right)^{m_n} \leq \delta \sum_{n=1}^{k-1} 2^{n-k+1} < 2\delta = M_1.$$

If $n > k$, then it follows from Lemma 3.2 that

$$\left(\frac{\rho_k}{\rho_n}\right)^{m_n} = \left(1 - \frac{1}{2^n}\frac{2^{n-k} - 1}{1 - 2^{-n}}\right)^{2^n} \leq e^{-2^{n-k}+1}.$$

Hence,

$$\sum_{n=k+1}^{\infty} \left(\frac{\rho_k}{\rho_n}\right)^{m_n} \leq \sum_{\ell=1}^{\infty} e^{-2^{\ell}+1} = M_2 < \infty.$$

The theorem now follows from the preceding proposition. □

A variety of versions of Theorem 2 may be derived from Proposition 3.6. Here, we state two such versions.

Theorem 3 (a) *Let $a_{nj} = (1-\frac{1}{n})e^{2\pi i j/n^2}$ $(0 \leq j < n^2)$ and let $r_{nj} = \frac{\sigma_n}{n^2}$ be an admissible sequence. Then, the points of the covering domain $\widetilde{D}$ are separated by bounded analytic functions.*

(b) *Let $\{\rho_n\}$ be a strictly increasing positive sequence with $\rho_n \to 1$. Let $a_{nj} = \rho_n e^{2\pi i j/m_n}$ $(0 \leq j < m_n)$ and $r_{nj} = \frac{\sigma_n}{m_n}$ be an admissible sequence. If intergers m_n increases sufficiently rapidly, then the points of the covering domain $\widetilde{D}$ are separated by bounded analytic functions.*

Proof. (a) Set $\rho_n = 1 - \frac{1}{n}$ and $m_n = n^2$. It follows from Lemma 3.2 that

$$\left(\frac{\rho_n}{\rho_k}\right)^{m_n} = \left(1 - \frac{1}{nk}\frac{k-n}{1-\frac{1}{k}}\right)^{n^2} \leq e^{-\frac{(k-n)n}{k}} \qquad (n < k),$$

$$\left(\frac{\rho_k}{\rho_n}\right)^{m_n} = \left(1 - \frac{1}{nk}\frac{n-k}{1-\frac{1}{n}}\right)^{n^2} \leq e^{-\frac{(n-k)n}{k}} \leq e^{-(n-k)} \quad (n > k).$$

In particular,

$$\left(\frac{\rho_k}{\rho_{k+1}}\right)^{m_k} = \left(1 - \frac{1}{k^2}\right)^{k^2} \leq e^{-1} = \delta < 1.$$

Noting $1 - \rho_k^{2m_n} < 1$, we have

$$\begin{aligned}\sum_{n=1}^{k-1}(1 - \rho_k^{2m_n})\left(\frac{\rho_n}{\rho_k}\right)^{m_n} &\leq \sum_{n=1}^{k-1} e^{-\frac{(k-n)n}{k}} \\ &\leq \sum_{n\leq\frac{k}{2}} e^{-\frac{n}{2}} + \sum_{\frac{k}{2}<n\leq k-1} e^{-\frac{(k-n)}{2}} \\ &\leq 2\sum_{n=1}^{\infty} e^{-\frac{n}{2}} = M_1 < \infty.\end{aligned}$$

Also,

$$\sum_{n=k+1}^{\infty}\left(\frac{\rho_k}{\rho_n}\right)^{m_n} \leq \sum_{n=k+1}^{\infty} e^{-(n-k)} = \sum_{n=1}^{\infty} e^{-n} = M_2.$$

Now we apply Proposition 3.6.

(b) Choose m_k so large that $(\frac{\rho_k}{\rho_{k+1}})^{m_k} \leq 2^{-k}$. Then, (10) holds. Since $1 - \rho_k^{2m_n} < 1$ and since $(\frac{\rho_n}{\rho_k})^{m_n} \leq (\frac{\rho_n}{\rho_{n-1}})^{m_n} \leq 2^{-n}$ for $n < k$, we have (11). Also, $(\frac{\rho_k}{\rho_n})^{m_n} \leq (\frac{\rho_{n-1}}{\rho_n})^{m_{n-1}} \leq 2^{-(n-1)}$ for $n > k$. This implies (12), and we apply Proposition 3.6, again. $\square$

It might be an interesting problem to determine whether the points of $\widetilde{D}$ are separated by bounded analytic functions or not, when $r_{nj} = 2^{-\sigma_n n}$ in Theorem 2.

References

[1] Forelli F., *A note on divisibility in $H^\infty(X)$*. Canadian J. Math. **36** (1984), 458–469.

[2] Gamelin T.W. and Hayashi M., *The algebra of bounded analytic functions on a Riemann surface*. J. Reine Angew Math. **382** (1987), 49–73.

[3] Hayashi M. and Nakai M., *Point separation by bounded analytic functions of a covering Riemann surface*. Pacific J. Math. **134** (1988), 261–273.

[4] Hayashi M. and Nakai M., *On the Myrberg type phenomenon*. Analytic Fuction Theory of One Complex Variable (C.C. Yang, Y. Komatsu and K. Niino, eds.), Pitman Research Notes in Mathematics Series (Longman Scientific & Technical), Vol. 212, Pitman, New York, 1989, 1–12.

[5] Hayashi M., Nakai M. and Segawa S., *Bounded analytic functions on two sheeted discs.* Trans. Amer. Math. Soc. **333** (1992), 799–819.
[6] Hayashi M., Nakai M. and Segawa S., *Two sheeted discs and bounded analytic functions.* J. D'Anal. Math. **61** (1993), 293–325.
[7] Royden H.L., *Algebras of bounded analytic functions of Riemann surfaces.* Acta Math. **114** (1965), 113–142.
[8] Selberg H.L., *Ein Satz über beschränkte endlichvieldeutig analytische Funktionen.* Comm. Math. Helv. **9** (1937), 104–108.

Mikihiro Hayashi
Department of Mathematics
Hokkaido University
Sapporo 060-0810, Japan
E-mail: hayashi@math.sci.hokudai.ac.jp

Takao Kato
Department of Mathematics
Yamaguchi University
Yamaguchi 753-8512, Japan
E-mail: kato@po.cc.yamaguchi-u.ac.jp