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Algebraic descriptions of non-isolated singularities

Guangfeng JIANG*

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Abstract. For isolated singularities, there exist some algebraic characterizations called Mather-Yau statements. In this article, we generalize these to non-isolated singularities.

Key words: non-isolated singularity, $\mathcal{R}_I$-equivalence, right-left equivalence, isomorphism of algebras.

1. Introduction

Many authors have been trying to characterize singularities algebraically. Benson [B] and independently Shoshitaishvili [Sh2] have proved that, for (weighted) homogeneous hypersurface with isolated singularity, the Jacobian ideal of the defining polynomial determined completely its analytic equivalence class. Mather and Yau [MY] have proved that the moduli algebra of a hypersurface determined its analytic equivalence class. Scherk [Sc] and Yau [Y] have considered the $\mathcal{O}_1$ and respectively, $\mathbb{C}\{t\}/(t^{n+1})$-algebra structures on the Jacobian algebra $\frac{\mathcal{O}}{J(f)}$, and proved that this algebra determined completely the right-left equivalence class of function f with isolated singularities. This result has been generalized to functions on analytic varieties with isolated singularities by Matsuoka [M]. Dimca [Di] has considered whether the singular subspace of a complete intersection with isolated singularity can determine the analytic equivalence class of the whole space. Gaffney and Hauser [GH] and later Hauser and Müller [HM] have considered the singularities with isolated singularity type and so called harmonic singularities. For these singularities, the singular subspace, which may be non-isolated, determined completely the singularities. Martin [Ma] also gave some cohomology characterizations for some singularities.

We consider, in this paper, mainly non-isolated singularities. We find that some isomorphism between the ideals of algebras related to singularities can be lifted to an isomorphism between the algebras.

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1.1 We denote the ring of germs of analytic functions from $(\mathbb{C}^{n+1}, 0)$ to $\mathbb{C}$ by $\mathcal{O}_{\mathbb{C}^{n+1}}$ or $\mathcal{O}_{n+1}$, or simply by $\mathcal{O}$. Denote the maximal ideal of $\mathcal{O}_{n+1}$ by $\mathfrak{m}_{n+1}$ or $\mathfrak{m}$. Let I be an ideal of $\mathcal{O}$. Let Σ be the analytic space defined by I. The collection of all functions having Σ in their singular loci is denoted by $\int I$. Let $\mathcal{R}$ be the group of all germs of local analytic automorphisms of $(\mathbb{C}^{n+1}, 0)$. If I is radical, then (see [P1] (2.14)) $\mathcal{R}_I := \{\varphi \in \mathcal{R} \mid \varphi^* I = I\} = \mathcal{R}_{\int I}$.

For the definitions of $\mathcal{K}$, $\mathcal{A}$, $\mathcal{C}$, $\mathcal{R}$ and $\mathcal{L}$ see Mather [M1]. Notations and definitions which are not defined here can be found in [P1], [M1] and [M2]. Denote ${}_I\mathcal{K} := \mathcal{R}_I \rtimes \mathcal{C}$, $\mathcal{A}_I := \mathcal{R}_I \times \mathcal{L}$.

Two germs $f, g \in \int I$ are called *$\mathcal{G}$-equivalent* if there exists a $\Phi \in \mathcal{G}$ such that $g = \Phi \cdot f$, where $\mathcal{G}$ is one of the above groups. For $\mathcal{G} = \mathcal{R}$ or $\mathcal{R}_I$, two hypersurface germs $(f^{-1}(0), 0), (g^{-1}(0), 0)$ are called *$\mathcal{G}$-equivalent* if there exists $\phi \in \mathcal{G}$ such that $(g) = (f \circ \phi)$ as ideals. In this case we also say that $(f^{-1}(0), 0)$ is analytically equivalent to $(g^{-1}(0), 0)$ by a $\phi \in \mathcal{G}$.

1.2 Let Der $=$ $\mathrm{Der}_{\mathbb{C}}(\mathcal{O})$ $=$ the $\mathcal{O}$-module of $\mathbb{C}$-derivations of $\mathcal{O}$. $\mathrm{Der}_I = \{\eta \in \mathrm{Der} \mid \eta(I) \subset I\}$. Write $J(f) = (\frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n})$, the Jacobian ideal, and (see [P1,2])

$$T\mathcal{R}_I = \left\{\eta \mid \eta = \sum_{j=0}^{n} \eta_j \frac{\partial}{\partial z_j} \in \mathrm{Der}_I,\ \eta_j \in \mathfrak{m},\ j = 0, \ldots, n\right\}$$

$$\tau_{I,e}(f) = \{\eta(f) \mid \eta \in \mathrm{Der}_I\} \qquad \tau_I(f) = \{\eta(f) \mid \eta \in T\mathcal{R}_I\}$$

1.3 Let K, K' be ideals of $\mathcal{O}$ and assume we are given an isomorphism of $\mathbb{C}$-algebras $\varphi : \frac{\mathcal{O}}{K} \longrightarrow \frac{\mathcal{O}}{K'}$. Then φ induces an $\mathcal{O}$-module structure on $\frac{\mathcal{O}}{K'}$ as follows: for any $a \in \mathcal{O}$, $[b]' \in \frac{\mathcal{O}}{K'}$, define $a \cdot [b]' := (\varphi[a])[b]'$, then $\frac{\mathcal{O}}{K'}$ is a module over $\mathcal{O}$. Moreover φ is an isomorphism of $\mathcal{O}$-modules if $\frac{\mathcal{O}}{K'}$ is given the induced $\mathcal{O}$-module structure.

1.4 We call $Q(f) = \frac{\mathcal{O}}{J(f)}$ the Jacobian algebra of f. Let I be an ideal of $\mathcal{O}$, $f \in \mathcal{O}$. If $J(f) \subset I$, then $\frac{I}{J(f)}$ is called the Jacobi module of f ([P2] (5.1)).

We call $\mathbb{C}$-algebra $A_I := \frac{\mathcal{O}}{\tau_I(f)}$ the *generalized Jacobian algebra* of f, and $M_I(f) := \frac{\mathcal{O}}{\tau_I(f)+(f)}$ the *generalized moduli algebra* of f. Under the canonical projection, every ideal of $\mathcal{O}$ gives an ideal of the generalized Jacobian algebra or the generalized moduli algebra. We call $N_I(f) := \frac{\int I}{\tau_I(f)}$ the *normal space* of $\mathcal{R}_I(f)$ at f, or the *right normal space* of f, and $\widetilde{N}_I(f) := \frac{\int I}{\tau_I(f)+(f)}$ the *normal space* of ${}_I\mathcal{K}(f)$ at f, or the *contact normal space* of f.

Theorem 1.5 (For ${}_I\mathcal{K}$-equivalence) 1) *Let $f, g \in \int I$. If germ $(f^{-1}(0), 0)$ is analytically equivalent to $(g^{-1}(0), 0)$ by a $\varphi \in \mathcal{R}_I$, then φ induces an isomorphism of $\mathcal{O}$-modules $\varphi_r^* : \widetilde{N}_I(f) \longrightarrow \widetilde{N}_I(g)$ which can be lifted to an isomorphism of $\mathbb{C}$-algebras $\varphi^* : M_I(f) \longrightarrow M_I(g)$;*

2) *Let I be a radical ideal. If we are given an isomorphism of $\mathcal{O}$-modules $\alpha_r : \widetilde{N}_I(f) \longrightarrow \widetilde{N}_I(g)$ which can be lifted to an isomorphism of $\mathbb{C}$-algebras $\alpha : M_I(f) \longrightarrow M_I(g)$, then germ $(f^{-1}(0), 0)$ is analytically equivalent to $(g^{-1}(0), 0)$ by a $\varphi \in \mathcal{R}_I$.*

Theorem 1.5* (For $\mathcal{K}$-equivalence) 1) *Let I be an ideal, $f \in \int I$, $g \in \mathcal{O}$. If germ $(f^{-1}(0), 0)$ is analytically equivalent to $(g^{-1}(0), 0)$ by a $\varphi \in \mathcal{R}$, then $g \in \varphi^* \int I = \int \varphi^* I$ and φ induces an isomorphism of $\mathcal{O}$-modules $\varphi_r^* : \widetilde{N}_I(f) \longrightarrow \widetilde{N}_{\varphi^*(I)}(g)$ which can be lifted to an isomorphism of $\mathbb{C}$-algebras $\varphi^* : M_I(f) \longrightarrow M_{\varphi^*(I)}(g)$,*

2) *Let I, I' be radical ideals. Let $f \in \int I$, $g \in \int I'$. If we are given an isomorphism of $\mathcal{O}$-modules $\alpha_r : \widetilde{N}_I(f) \longrightarrow \widetilde{N}_{I'}(g)$ which can be lifted to an isomorphism of $\mathbb{C}$-algebras $\alpha : M_I(f) \longrightarrow M_{I'}(g)$, then there exists a $\phi \in \mathcal{R}$ such that $(f^{-1}(0), 0)$ is analytically equivalent to $(g^{-1}, 0)$ by ϕ and $\phi^* I = I'$.*

Theorem 1.6 (For left-right equivalence and weighted homogeneous polynomials) *Let I, I' be radical ideals generated by weighted homogeneous polynomials.*

1) *Two weighted homogeneous polynomial germs $f, g \in \int I$ are $\mathcal{R}_I$ equivalent if and only if there exists an isomorphism of $\mathcal{O}$-modules $\alpha_r : N_I(f) \longrightarrow N_I(g)$ which can be lifted to an isomorphism of $\mathbb{C}$-algebras $\alpha : A_I(f) \longrightarrow A_I(g)$.*

2) *Two weighted homogeneous polynomial germs $f \in \int I$, $g \in \int I'$ are $\mathcal{R}$-equivalent if and only if there exists an isomorphism of $\mathcal{O}$-modules $\alpha_r : N_I(f) \longrightarrow N_{I'}(g)$ which can be lifted to an isomorphism of $\mathbb{C}$-algebras $\alpha : A_I(f) \longrightarrow A_{I'}(g)$.*

Theorem 1.7 (For $\mathcal{A}_I$-equivalence) 1) *Let I be an ideal, $f, g \in \int I$. If there exists a $\varphi \in \mathcal{R}_I$, $\psi \in \mathcal{L}$ such that $g = \psi \circ f \circ \varphi$, then φ induces an isomorphism of $\mathcal{O}_1$-modules $\varphi_r^* : N_I(f) \longrightarrow N_I(g)$ over $(\psi^{-1})^* : \mathcal{O}_1 \longrightarrow \mathcal{O}_1$, such that φ_r^* can be lifted to an $\mathcal{O}_1$-algebra isomorphism $\varphi^* : A_I(f) \longrightarrow$*

$A_I(g)$ over $(\psi^{-1})^*$

2) *Let I be a radical ideal. If we are given an isomorphism of $\mathcal{O}_1$-modules: $\alpha_r : N_I(f) \longrightarrow N_I(g)$ over a $\mathbb{C}$-algebra isomorphism $\sigma : \mathcal{O}_1 \longrightarrow \mathcal{O}_1$ such that α_r can be lifted to an $\mathcal{O}_1$-algebra isomorphism $\alpha : A_I(f) \longrightarrow A_I(g)$ over σ, then f and g are $\mathcal{A}_I$-equivalent.*

Theorem 1.8 (For $\mathcal{R}_I$-equivalence) 1) *Let I be an ideal, $f, g \in \int I$. If there exists a $\varphi \in \mathcal{R}_I$ such that $g = f \circ \varphi$, then φ induces an isomorphism of $\mathcal{O}_1$-modules $\varphi_r^* : N_I(f) \longrightarrow N_I(g)$ over* id $: \mathcal{O}_1 \longrightarrow \mathcal{O}_1$ *such that φ_r^* can be lifted to an $\mathcal{O}_1$-algebra isomorphism $\varphi^* : A_I(f) \longrightarrow A_I(g)$ over* id;

2) *Let I be a radical ideal, and $f, g \in \int I$. If we are given an isomorphism of $\mathcal{O}_1$-modules: $\alpha_r : N_I(f) \longrightarrow N_I(g)$ over $\mathbb{C}$-algebra isomorphism* id $: \mathcal{O}_1 \longrightarrow \mathcal{O}_1$ *such that α_r can be lifted to an isomorphism of $\mathcal{O}_1$-algebras $\alpha : A_I(f) \longrightarrow A_I(g)$ over* id, *then f and g are $\mathcal{R}_I$-equivalent.*

Theorem 1.9 (Hauser [H1]) (For $\mathcal{A}$-equivalence) *Two germs $f, g \in \mathcal{O}$ are right-left equivalent if and only if there is an $\mathcal{O}_1$-algebra isomorphism $\alpha : Q(f) \longrightarrow Q(g)$ over some $\mathbb{C}-$algebra isomorphism $\sigma : \mathcal{O}_1 \longrightarrow \mathcal{O}_1$.*

Remark 1.10 1) Although the theorems are stated for non-isolated singularities, they are true and known (see *e.g.* [B], [GH], [HM], [H1], [H2], [MY], [Sc], [Sh2] and [Y]) for isolated singularities if we take I to be the maximal ideal of $\mathcal{O}$;

2) If the σ in theorem 1.9 is an identity, then we can get a similar conclusion about right equivalence. Since $f^k \in J(f)$ for $k >> 0$, we can get similar conclusions to those in [Y].

Example 1.11 In $(\mathbb{C}^3, 0)$, Let $I = (y, z)$, $f = y^2 + z^2$, $(X, 0) = (f^{-1}(0), 0)$; $g = xy^2 + z^2$, $(Y, 0) = (g^{-1}(0), 0)$. we have $\tau_I(f) = \tau_{I,e}(f) = \tau_{I,e}(g) = I^2$, but $\tau_I(g) = (xy^2, yz, z^2, y^3, xyz)$. Hence $A(f) = \frac{\mathcal{O}}{\tau_I(f)}$ and $A(g) = \frac{\mathcal{O}}{\tau_I(g)}$ are not isomorphic as algebras or modules. But $\frac{\mathcal{O}}{\tau_{I,e}(f)} = \frac{\mathcal{O}}{I^2} = \frac{\mathcal{O}}{\tau_{I,e}(g)}$ and $\frac{I^2}{\tau_{I,e}(f)} = 0 = \frac{I^2}{\tau_{I,e}(g)}$. This example shows $\frac{\mathcal{O}}{\tau_{I,e}(f)}$ cannot characterize singularities. For $(Y, 0)$, in [GH], $(g) + J(g)$ was used to describe the hypersurface. We here use $(g) + \tau_I(g)$ to do the job for non-isolated singularities.

2. Equivalence and Triviality

2.1 It is easy to prove the following

Lemma *Two germs $f, g \in \int I$ are ${}_I\mathcal{K}$-equivalent if and only if $(\mathcal{V}(f), 0)$ and $(\mathcal{V}(g), 0)$ are analytically equivalent by a $\varphi \in \mathcal{R}_I$, where $\mathcal{V}(f) = f^{-1}(0)$, $\mathcal{V}(g) = g^{-1}(0)$.*

2.2 Let I be generated by weighted homogeneous polynomials, and $f, g \in \int I$ weighted homogeneous polynomials. The following lemma is a generalization of a result due to Durfee [Du] and the proof is similar.

Lemma *Germs $(\mathcal{V}(f), 0)$ and $(\mathcal{V}(g), 0)$ are analytically equivalent by a $\varphi \in \mathcal{R}_I$ if and only if f, g are $\mathcal{R}_I$-equivalent.*

Definition 2.3 (cf. [J]) 1) Let $\mathcal{G}$ be a subgroup of $\mathcal{K}$, and $a \in \mathbb{C}$. A $(\mathbb{C}, a)$-level-preserving map-germ $G : (\mathbb{C}^n \times \mathbb{C}, 0 \times a) \longrightarrow (\mathbb{C}^p \times \mathbb{C}, 0 \times a)$ is said to be $\mathcal{G}$-trivial at a if there exist $(\mathbb{C}, a)$-level preserving map-germs

$$H' : (\mathbb{C}^n \times \mathbb{C}^p \times \mathbb{C}, 0 \times 0 \times a) \longrightarrow (\mathbb{C}^n \times \mathbb{C}^p \times \mathbb{C}, 0 \times 0 \times a)$$

and

$$H : (\mathbb{C}^n \times \mathbb{C}, 0 \times a) \longrightarrow (\mathbb{C}^n \times \mathbb{C}, 0 \times a)$$

such that

$$H'^{-1} \circ (\pi_1, G) \circ H = (\pi_1, G_a) \times 1_{(\mathbb{C},a)} \tag{2.1}$$

where $\pi_1 : (\mathbb{C}^n \times \mathbb{C}, 0 \times a) \longrightarrow (\mathbb{C}^n, 0)$ is the germ of the projection, and if $\pi_2 : (\mathbb{C}^n \times \mathbb{C}^p \times \mathbb{C}, 0 \times 0 \times a) \longrightarrow (\mathbb{C}^n \times \mathbb{C}^p, 0 \times 0)$, $\pi_3 : (\mathbb{C}^n \times \mathbb{C}^p, 0 \times 0) \longrightarrow (\mathbb{C}^n, 0)$ are the germs of projections, then $H'_t =: \pi_2 \circ H'(-,-,t) \in \mathcal{G}$ for each $t \in (\mathbb{C}, 0)$, and $H_t =: \pi_1 \circ H(-,t) = \pi_3 \circ H'_t$ for each $t \in (\mathbb{C}, a)$.

2) Let $T \subset \mathbb{C}$ be an open domain. A $(\mathbb{C}, T)$-level-preserving map-germ $G : (\mathbb{C}^n \times \mathbb{C}, 0 \times T) \longrightarrow (\mathbb{C}^p \times \mathbb{C}, 0 \times T)$ is said to be locally $\mathcal{G}$-trivial if the restricted germ

$$G^a : (\mathbb{C}^n \times \mathbb{C}, 0 \times a) \longrightarrow (\mathbb{C}^p \times \mathbb{C}, 0 \times a)$$

of G at $a \in T$ is $\mathcal{G}$-trivial at each $a \in T$.

Lemma 2.4 (cf. [J] or [dPW]) *Let $T \subset \mathbb{C}$ be a path connected open domain. If $(\mathbb{C}, T)$-level-preserving map-germ $G : (\mathbb{C}^n \times \mathbb{C}, 0 \times T) \longrightarrow (\mathbb{C}^p \times \mathbb{C}, 0 \times T)$ is locally $\mathcal{G}$-trivial, then G_u and G_v are $\mathcal{G}$-equivalent for any $u, v \in T$, where $G_w =: \pi_4 \circ G(-, w)$ and $\pi_4 : (\mathbb{C}^p \times \mathbb{C}, 0 \times T) \longrightarrow (\mathbb{C}^p, 0)$ is the germ of projection.*

Thom-Levine Type Lemma 2.5 *Let T be a domain in $\mathbb{C}$. If $F : (\mathbb{C}^n \times \mathbb{C}, 0 \times T) \longrightarrow (\mathbb{C}^p \times \mathbb{C}, 0 \times T)$ is a $(\mathbb{C}, T)$-level-preserving map-germ.*

1) *Germ F is locally $\mathcal{R}_I$-trivial at $a \in T$ if and only if*

$$\frac{\partial \pi_4 \circ F^a}{\partial t} \in \bar{T}\mathcal{R}_I F^a$$
$$=: \left\{ \eta(\pi_4 \circ F^a) \mid \eta = \sum_{j=0}^{n} \eta_j \frac{\partial}{\partial z_j}, \eta(I) \subset I\mathcal{O}_{n+1}, \eta_j \in \mathfrak{m}_n \mathcal{O}_{n+1} \right\}$$

2) *Germ F is locally ${}_I\mathcal{K}$-trivial at $a \in T$ if and only if*

$$\frac{\partial \pi_4 \circ F^a}{\partial t} \in \bar{T}\mathcal{R}_I F^a + ((\pi_4 \circ F^a)^* \mathfrak{m}_p) \mathcal{O}_{n+1}^{\times p}$$

3) *Germ F is locally $\mathcal{A}_I$-trivial at $a \in T$ if and only if*

$$\frac{\partial \pi_4 \circ F^a}{\partial t} \in \bar{T}\mathcal{R}_I F^a + ((\pi_4 \circ F^a)^* (\mathfrak{m}_p \mathcal{O}_{p+1}))^{\times p}$$

This lemma can be proved by the same way as in [M1], [J], or [P1].

3. Proofs of Theorems

Lemma 3.1 *Let $I \subset I'$, $J \subset J'$ be ideals of $\mathcal{O}$. If an isomorphism of $\mathcal{O}$-submodules $\alpha_r : \frac{I'}{I} \longrightarrow \frac{J'}{J}$ can be lifted to a $\mathbb{C}$-algebra isomorphism $\alpha : \frac{\mathcal{O}}{I} \longrightarrow \frac{\mathcal{O}}{J}$, there exists an analytic automorphism $\varphi : (\mathbb{C}^{n+1}, 0) \longrightarrow (\mathbb{C}^{n+1}, 0)$ with $\varphi^* I = J$, $\varphi^* I' = J'$, such that φ induces α and α_r.*

Proof. It is obvious that diagram A is commutative, where the two horizontal sequences are exact and $\bar{\alpha}$ is an isomorphism of $\mathcal{O}$-modules determined uniquely by α (which is also an $\mathcal{O}$-module isomorphism in a canonical way) and α_r, and p_1, p_2 are canonical projections.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \frac{I'}{I} & \longrightarrow & \frac{\mathcal{O}}{I} & \xrightarrow{p_1} & \frac{\mathcal{O}}{I'} & \longrightarrow & 0 \\
 & & \downarrow \alpha_r & & \downarrow \alpha & & \downarrow \bar{\alpha} & & \\
0 & \longrightarrow & \frac{J'}{J} & \longrightarrow & \frac{\mathcal{O}}{J} & \xrightarrow{p_1} & \frac{\mathcal{O}}{J'} & \longrightarrow & 0
\end{array}$$

Diagram A

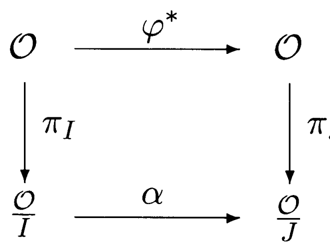

Diagram B

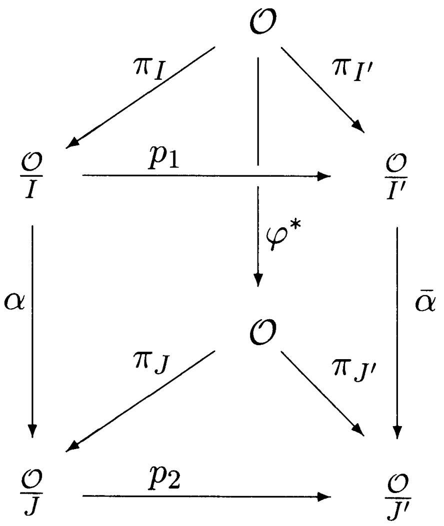

Diagram C

By [Lo] Lemma (1.7), there exists an analytic automorphism $\varphi : (\mathbb{C}^{n+1}, 0) \longrightarrow (\mathbb{C}^{n+1}, 0)$ with $\varphi^* I = J$ such that φ^* induces α, namely diagram B is commutative. In diagram C, we have $\alpha \circ \pi_I = \pi_J \circ \varphi^*$, $p_2 \circ \alpha = \bar{\alpha} \circ p_1$, $p_1 \circ \pi_I = \pi_{I'}$, $p_2 \circ \pi_J = \pi_{J'}$. Hence all the faces of Diagram C are commutative. This implies $\varphi^* I' = J'$, and φ induces $\bar{\alpha}$ by the uniqueness of $\bar{\alpha}$. □

Hauser Lemma 3.2 ([H1] §2) Let T be an analytic manifold, $t_0 \in T$, and $(M_t)_{t \in T}$ an analytic family of $\mathcal{O}_k$ (for some $k = 1, \dots, n+1$) modules in $\mathcal{O}_{n+1}$. If $M_t \subset M_{t_0}$ pointwise for any $t \in T$, then $M_t = M_{t_0}$ holds analytically for all t in a Zariski open subset T' of T.

Lemma 3.3 *Let I be an ideal of $\mathcal{O}$, $f, g \in \int I$.*

1) *If $g - f \in \tau_I(f) = \tau_I(g)$, then f is $\mathcal{R}_I$-equivalent to g;*
2) *If $(f) + \tau_I(f) = (g) + \tau_I(g)$, then f is ${}_I\mathcal{K}$-equivalent to g;*
3) *If $f^*\mathfrak{m}_1 + \tau_I(f) = g^*\mathfrak{m}_1 + \tau_I(g)$, then f is $\mathcal{A}_I$-equivalent to g.*

Proof. Let $T = \mathbb{C}$, and $G : (\mathbb{C}^{n+1} \times \mathbb{C}, 0 \times T) \longrightarrow (\mathbb{C} \times \mathbb{C}, 0 \times T)$ be a $(\mathbb{C}, T)$-level preserving map germ defined by $G(x,t) = (f(x) + t(g(x) - f(x)), t)$. We are going to prove that for any $a \in T$, the restricted germ G^a is locally trivial at a with respect to any of the three groups. We only give the detailed proof of 2), the reader can follow the same way to give the proofs of the other conclusions.

Let

$$\begin{aligned} M_t &= \bar{T}\mathcal{R}_I G^a + ((\pi_4 \circ G^a)^* \mathfrak{m}_1)\mathcal{O}_{n+2} \\ &= \left\{\eta(f + t(g-f)) \mid \eta = \sum_{j=0}^{n} \eta_j \frac{\partial}{\partial z_j}, \eta(I) \subset I\mathcal{O}_{n+2}, \eta_j \in \mathfrak{m}\mathcal{O}_{n+2}\right\} \\ &\quad + ((\pi_4 \circ G^a)^* \mathfrak{m}_1)\mathcal{O}_{n+2} \end{aligned}$$

Then $(M_t)_{t\in T}$ is an analytic family of $\mathcal{O}_{n+1}$-modules. From 2) we have $M_t \subset M_0 = M_1$ for every $t \in T$. By Hauser Lemma, $M_t = M_0 = M_1$ for all $t \in T_0 =: \mathbb{C} - \{\text{finitepoints} \neq 1, 0\}$. Hence for any $a \in T_0$

$$\frac{\partial G^a}{\partial t} = g - f \in M_0 = M_t$$

this proves that G^a is locally ${}_I\mathcal{K}$-trivial at every $a \in T_0$. By lemma 2.4, $f = G^a_0 = \pi_4 \circ G^a(-, 0)$ and $g = G^a_1 = \pi_4 \circ G^a(-, 1)$ are ${}_I\mathcal{K}$-equivalent. □

3.4 *Proof of* 1.5 1) Let $g = uf \circ \varphi$, $\varphi \in \mathcal{R}_I$, $u \in \mathcal{O}$, $u(0) \neq 0$. Notice that for any $\phi \in \mathcal{R}_I$, $\phi^*(\tau_I(f)) = \tau_I(f \circ \phi)$. It follows that $\varphi^*((f) + \tau_I(f)) = (g) + \tau_I(g)$.

Hence φ^* induces an isomorphism of $\mathbb{C}$-algebras $\bar{\varphi}^* : M_I(f) \longrightarrow M_I(g)$, and the restriction of $\bar{\varphi}^*$ gives an isomorphism of $\mathcal{O}$-modules $\bar{\varphi}^*_r : \widetilde{N}_I(f) \longrightarrow \widetilde{N}_I(g)$. □

3.5 *Proof of* 1.5 2) By lemma 3.1, there exists an analytic automorphism $\varphi : (\mathbb{C}^{n+1}, 0) \longrightarrow (\mathbb{C}^{n+1}, 0)$ with $\varphi^*(\tau_I(f) + (f)) = \tau_I(g) + (g)$ and $\varphi^*(\int I) = \int I$. By [P1] (2.14), $\varphi^* I = I$, namely $\varphi \in \mathcal{R}_I$.

Hence we have $(f \circ \varphi) + \tau_I(f \circ \varphi) = \varphi^*((f) + \tau_I(f)) = (g) + \tau_I(g)$. From this we can assume that $(f) + \tau_I(f) = (g) + \tau_I(g)$.

In order to prove that $(f^{-1}(0), 0)$ is analytically equivalent to $(g^{-1}(0), 0)$ by an automorphism $\varphi : (\mathbb{C}^{n+1}, 0) \longrightarrow (\mathbb{C}^{n+1}, 0)$ with $\varphi^* I = I$, By lemma 2.1, it is enough to prove f and g are ${}_I\mathcal{K}$-equivalent. Lemma 3.3 2) gives this conclusion. □

3.6 *Proof of* 1.5* 1) For any automorphism $\varphi : (\mathbb{C}^{n+1}, 0) \longrightarrow (\mathbb{C}^{n+1}, 0)$, we have $\varphi^* \int I = \int \varphi^* I$ and $\varphi^*((f) + \tau_I(f)) = (g) + \tau_{\varphi^*(I)}(g)$.

2) By Lemma 3.1, there exists an automorphism $\varphi : (\mathbb{C}^{n+1}, 0) \longrightarrow (\mathbb{C}^{n+1}, 0)$ such that $\varphi^*((f) + \tau_I(f)) = (g) + \tau_{I'}(g)$ and $\varphi^* \int I = \int I'$ which

gives $\varphi^* I = I'$ (see [P1] (2.14)). We also have

$$\begin{aligned}(g) + \tau_{I'}(g) &= \varphi^*((f) + \tau_I(f)) = (f \circ \varphi) + \tau_{\varphi^*(I)}(f \circ \varphi) \\ &= (f \circ \varphi) + \tau_{I'}(f \circ \varphi)\end{aligned}$$

By lemma 3.3 2), we know that $f \circ \varphi$ and g are ${}_{I'}\mathcal{K}$-equivalent. □

3.7 *Proof of* 1.6 If we notice the fact that Euler derivation is a generator of $T\mathcal{R}_I$, then $f \in \tau_I(f)$, $g \in \tau_I(g)$. Theorem 1.6 follows from theorem 1.5, 1.5* and lemma 2.2. □

3.8 *Proof of* 1.7 1) If $g = \psi \circ f \circ \varphi$, $\varphi \in \mathcal{R}_I$, $\psi \in \mathcal{L}$, then $\varphi^* \tau_I(f) = \tau_I(g)$ (since $\frac{\partial \psi^{-1}}{\partial t} \circ g \in \mathcal{O}_{n+1}$ is a unit). Hence we have an $\mathcal{O}_1$-algebra isomorphism $\varphi^* : A_I(f) \longrightarrow A_I(g)$ over $(\psi^{-1})^* : \mathcal{O}_1 \longrightarrow \mathcal{O}_1$. Of course, φ^* is a $\mathbb{C}$-algebra isomorphism which induces an isomorphism of $\mathcal{O}$-modules. Since $\varphi^* \int I = \int I$, so φ^* restricts to an $\mathcal{O}_1$-module isomorphism $\varphi_r^* : N_I(f) \longrightarrow N_I(g)$ which is also an $\mathcal{O}$-module isomorphism induced by φ^*. □

3.9 *Proof of* 1.7 2) Let $\alpha : A_I(f) \longrightarrow A_I(g)$ be an isomorphism of $\mathcal{O}_1$-algebras over $\mathbb{C}$-algebra isomorphism $\sigma : \mathcal{O}_1 \longrightarrow \mathcal{O}_1$, then α is also a $\mathbb{C}$-algebra isomorphism and induces α_r which is an $\mathcal{O}$-module isomorphism. By lemma 3.1 we have an analytic automorphism $\varphi : (\mathbb{C}^{n+1}, 0) \longrightarrow (\mathbb{C}^{n+1}, 0)$ with $\varphi^* \tau_I(f) = \tau_I(g)$, $\varphi^* \int I = \int I$ such that $\varphi*$ induces α as $\mathbb{C}$-algebra isomorphism. It is easy to check that φ^* is also an $\mathcal{O}_1$-algebra isomorphism over σ. Since $\mathcal{R}_I = \mathcal{R}_{\int I}$, $\varphi^* I = I$ and $\tau_I(g) = \varphi^* \tau_I(f) = \tau_I(f \circ \varphi)$. So in the following, we assume $\tau_I(f) = \tau_I(g)$, $\varphi = \mathrm{id}$.

For $t \in \mathcal{O}_1$, $[1] \in A(f)$, and $[1]' \in A(g)$, we have $\varphi^*(t \cdot [1]) = \sigma(t) \cdot \varphi^*[1] = \sigma(t) \cdot [1]'$ while $\varphi^*(t \cdot [1]) = [(t \circ f) \cdot 1]' = [f]'$. Let $a(t) = \sigma(t) \in \mathcal{O}_1$, then $\sigma(t) \cdot [1]' = [a \circ g]'$. Hence

$$f - a \circ g \in \tau_I(g) = \tau_I(f)$$

Set $b = a^{-1}$ (since σ is an isomorphism), then

$$g - b \circ f \in \tau_I(f)$$

These tell us that

$$g^* m_1 + \tau_I(g) = f^* m_1 + \tau_I(f)$$

By lemma 3.3 3), f and g are $\mathcal{A}_I$-equivalent. □

3.10 *Proof of* 1.8 Replace in 3.8 and 3.9 ψ by id and σ by id. □

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Department of Mathematics
Jinzhou Normal University
Jinzhou, Liaoning, 121000
P. R. China

Present Address (till March. 1, 1998):
Department of Mathematics
Utrecht University
P.O. Box 80.010, 3508 TA Utrecht
The Netherlands
E-mail: jiang@math.ruu.nl