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# Sublinear operators with rough kernel on generalized Morrey spaces

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**Abstract.** In this paper, we establish the boundedness of rough operators and their commutators with BMO functions in generalized Morrey spaces.

*Key words*: sublinear operator, Morrey space, commutator, BMO function.

The classical Morrey spaces were introduced in [6] by Morrey to study the local behaviour of solutions to second order elliptic partial differential equations. Since then, these spaces play an important role in studying the regularity of solutions to partial differential equations; see [2, 3]. In [5], Mizuhara introduced the following generalized Morrey spaces and discussed the boundedness of Calderón-Zygmund operators on these spaces.

Let $\phi$ be a positively growth function on $(0,\infty)$ and satisfy that for all $r>0$,

$$\phi(2r) \leq D\phi(r), \tag{1}$$

where $D \geq 1$ is a constant independent of $r$.

**Definition** ([5]) Let $1 \leq p < \infty$. We denote by $L^{p,\phi} = L^{p,\phi}(\mathbf{R}^n)$ the space of locally integrable function $f$ for which,

$$\int_{B_r(x_0)} |f(x)|^p \, dx \leq C^p \phi(r) \tag{2}$$

for all $x_0 \in \mathbf{R}^n$ and every $r>0$, where $B_r(x_0) = \{x \in \mathbf{R}^n : |x-x_0| \leq r\}$; and we denote the smallest constant $C$ satisfying (2) by $\|f\|_{L^{p,\phi}}$.

Obviously, when $\phi(r) = r^\lambda$, $0<\lambda<n$, $L^{p,\phi}$ is just the classical Morrey spaces in [6].

The purpose of this paper is to establish the boundedness on the generalized Morrey spaces for a large class of sublinear operators with rough

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kernel and their commutators with BMO functions. For convenience, we write $p' = p/(p-1)$ for $p \in (1, \infty)$.

Let us first establish a key lemma on rough maximal operator defined by

$$M_\Omega f(x) = \sup_{r>0} \frac{1}{r^n} \int_{|y|\le r} |\Omega(y) f(x-y)| dy,$$

where $\Omega$ is homogeneous of degree zero on $\mathbf{R}^n \backslash \{0\}$.

In what follows, for any $x_0 \in \mathbf{R}^n$, $r > 0$, and any complex-valued measurable function $f(y)$ on $\mathbf{R}^n$, we write

$$f(y) = f\chi_{B_{2r}(x_0)}(y) + \sum_{k=1}^{\infty} f\chi_{B_{2^{k+1}r}(x_0)\backslash B_{2^k r}(x_0)}(y) \equiv \sum_{k=0}^{\infty} f_k(y). \tag{3}$$

**Lemma** *Let $p \in (1, \infty)$, $1 \le D(\phi) < 2^n$ and $\gamma = \log 2^n / \log D$. If $\Omega \in L^q(\Sigma_{n-1})$ and $q \ge p'$ or $q > \min\{p, \gamma'\}$, then for any $k > 0$,*

$$\int_{B_r(x_0)} |M_\Omega(f_k)(x)|^p dx \le C \left(\frac{D}{2^{n\theta}}\right)^k \|f\|^p_{L^{p,\phi}} \phi(r),$$

*where $\theta \in (1/\gamma, 1)$ for the case $q \ge p'$ or $q > p$; $\theta \in (1/\gamma, 1/q')$ for the case $q > \gamma'$; and $C$ is a constant independent of $k$ and $f$.*

*Proof.* By the properties of $A_p$ weights (see [4]), we have

$$(M\chi_{B_r(x_0)})^\theta \quad \in A_{p/q'} \quad \text{for} \quad q \ge p';$$

$$\left[(M\chi_{B_r(x_0)})^\theta\right]^{-1/(p-1)} \in A_{p'/q'} \quad \text{for} \quad q > p;$$

and

$$\left[(M\chi_{B_r(x_0)})^\theta\right]^{q'} \in A_p \quad \text{for} \quad q > \gamma'.$$

Then, by [10] we see that

$$\begin{aligned}\int_{B_r(x_0)} |M_\Omega(f_k)(x)|^p dx &\le \int_{\mathbf{R}^n} |M_\Omega(f_k)(x)|^p (M\chi_{B_r(x_0)})^\theta(x) dx \\ &\le C \int_{\mathbf{R}^n} |f_k(x)|^p (M\chi_{B_r(x_0)})^\theta(x) dx \\ &\le C(2^{1-k})^{n\theta} \int_{B_{2^{k+1}r}(x_0)} |f_k(x)|^p dx\end{aligned}$$

$$\leq C\left(\frac{D}{2^{n\theta}}\right)^k \|f\|^p_{L^{p,\phi}}\phi(r).$$

This finishes the proof of the lemma. □

Now, for the boundedness of sublinear operators on these spaces, we have the following general theorem.

**Theorem 1** *Let $p \in (1,\infty)$, $1 \leq D(\phi) < 2^n$ and $\gamma = \log 2^n/\log D$. If a sublinear operator $T$ is bounded on $L^p(\mathbf{R}^n)$ and for any $f \in L^1(\mathbf{R}^n)$ with compact support and $x \notin C_0 \operatorname{supp} f$,*

$$|Tf(x)| \leq C\int_{\mathbf{R^n}} \frac{|\Omega(x-y)|}{|x-y|^n}|f(y)|dy, \tag{4}$$

*where $C_0 \geq 1$, $C > 0$ are absolute constants, $\Omega$ is homogeneous of degree zero and $\Omega \in L^q(\Sigma_{n-1})$ for some $q \geq p'$ or some $q > \min\{p,\gamma'\}$, then $T$ is also bounded on $L^{p,\phi}(\mathbf{R^n})$.*

*Proof.* Without loss of generality, we may assume $C_0 = 1$. For any $x_0 \in \mathbf{R}^n$ and $r > 0$, we write $f$ as in (3). For $f_0$, by the $L^p(\mathbf{R}^n)$-boundedness of $T$ and (1)–(2), we have

$$\begin{aligned}\left(\int_{B_r(x_0)} |Tf_0(x)|^p dx\right)^{1/p} &\leq C\left(\int_{\mathbf{R}^n} |f_0(x)|^p dx\right)^{1/p}\\ &\leq C\|f\|_{L^{p,\phi}}\phi^{1/p}(2r) \leq C\|f\|_{L^{p,\phi}}\phi^{1/p}(r).\end{aligned}$$

For $k > 0$, we choose $\theta$ as in the lemma. Then

$$\begin{aligned}&\int_{B_r(x_0)} |Tf_k(x)|^p dx\\ &\quad\leq C\int_{B_r(x_0)}\left(\int_{B_{2^{k+1}r}(x_0)\backslash B_{2^k r}(x_0)} \frac{|\Omega(x-y)|}{|x-y|^n}|f(y)|dy\right)^p dx\\ &\quad\leq C\int_{B_r(x_0)}\left(\frac{1}{(2^k r)^n}\int_{B_{2^{k+1}r}(x_0)} |\Omega(x-y)f_k(y)|dy\right)^p dx\\ &\quad\leq C\int_{B_r(x_0)} |M_\Omega(f_k)(x)|^p dx.\end{aligned}$$

Since $D < 2^{n\theta}$, we have

$$\begin{aligned}\left(\int_{B_r(x_0)} |Tf(x)|^p dx\right)^{1/p} &\\ &\le \sum_{k=0}^{\infty}\left(\int_{B_r(x_0)} |Tf_k(x)|^p dx\right)^{1/p}\\ &\le C\left[1+\sum_{k=1}^{\infty}\left(\frac{D}{2^{n\theta}}\right)^{k/p}\right]\|f\|_{L^{p,\phi}}\phi^{1/p}(r) \le C\|f\|_{L^{p,\phi}}\phi^{1/p}(r).\end{aligned}$$

This finishes the proof of Theorem 1. □

We remark that (4) is first proposed by Soria and Weiss in [8]. Obviously, (4) is satisfied by many operators in harmonic analysis, such as the Calderón-Zygmund operators, Carleson's maximal operator, Hardy-Littlewood maximal operator, C. Fefferman's singular multiplier operators, R. Fefferman's singular integral operators, Ricci-Stein's oscillatory singular integral and the Bochner-Riesz means at the critical index and so on. In particular, Theorem 1 contains Theorem 2.2 (i) and Theorem 3.2 (i) in [5] as special cases.

Recently, the linear commutators are revealed to be very useful in studying the regularity of solutions to nondivergence elliptic equations with $VMO$ coefficients; see [1, 2, 3]. For the boundedness of these commutators on Morrey spaces, we have

**Theorem 2** *Let $p \in (1,\infty)$, $a \in \mathrm{BMO}(\mathbf{R}^n)$, $1 \le D(\phi) < 2^n$ and $\gamma = \log 2^n/\log D$. If a linear operator $T$ satisfies (4) with $\Omega \in L^q(\Sigma_{n-1})$ for some $q \ge p'$ or some $q > \min\{p, \gamma'\}$ and $[a,T]$ is bounded from $L^p(\mathbf{R}^n) \times \mathrm{BMO}(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$, then $[a,T]$ is also bounded from $L^{p,\phi}(\mathbf{R}^n)\times\mathrm{BMO}(\mathbf{R}^n)$ to $L^{p,\phi}(\mathbf{R}^n)$.*

*Proof.* Without loss of generality, we may assume $C_0 = 1$. For any $x_0 \in \mathbf{R}^n$ and $r > 0$, we write $f$ as in (3). By the $L^p(\mathbf{R}^n)$-boundedness of $[a,T]$, we obtain

$$\begin{aligned}\left(\int_{B_r(x_0)} |[a,T]f_0(x)|^p\, dx\right)^{1/p} &\le C\|a\|_*\|f_0\|_{L^p(\mathbf{R}^n)}\\ &\le C\phi^{1/p}(r)\|f\|_{L^{p,\phi}}\|a\|_*.\end{aligned}$$

For $k > 0$ and $x \in B_r(x_0)$, we write

$$\begin{aligned}
&|[a,T]f_k(x)| \\
&\quad \le \frac{C}{(2^k r)^n} \int_{B_{2^{k+1}r}(x_0)} |a(x) - a_r||\Omega(x-y) f_k(y)| dy \\
&\qquad + \frac{C}{(2^k r)^n} \int_{B_{2^{k+1}r}(x_0)} |a_r - a_{2^{k+1}r}||\Omega(x-y) f_k(y)| dy \\
&\qquad + \frac{C}{(2^k r)^n} \int_{B_{2^{k+1}r}(x_0)} |a(y) - a_{2^{k+1}r}||\Omega(x-y) f_k(y)| dy \\
&\quad \equiv I_1 + I_2 + I_3,
\end{aligned}$$

where $a_\delta$ $(\delta > 0)$ is defined by

$$a_\delta = \frac{1}{|B_\delta(x_0)|} \int_{B_\delta(x_0)} a(y) dy.$$

By the fact that for any $r > 0$ and $k \in \mathbf{N}$

$$|a_{2^{k+1}r} - a_r| \le C(n)(k+1)\|a\|_* \qquad \text{(see [9])},$$

we obtain

$$I_2 \le C(n)(k+1)\|a\|_* M_\Omega(f_k)(x).$$

Then for $\theta_1 \in (1/\gamma, 1)$ as in Lemma, we have

$$\begin{aligned}
\int_{B_r(x_0)} I_2^p dx \;&\le\; C(k+1)^p \|a\|_*^p \int_{B_r(x_0)} |M_\Omega(f_k)(x)|^p dx \\
&\le\; C(k+1)^p \|a\|_*^p \left(\frac{D}{2^{n\theta_1}}\right)^k \|f\|_{L^{p,\phi}}^p \phi(r),
\end{aligned}$$

where we have used the Lemma.

For $I_3$, if we choose $1 < u < \min\{p, q\}$ such that $u \le qp/(q+p)$ for the case $q \ge p'$ (then $q/u \ge (p/u)'$) and $q/u > \gamma'$ for the case $q > \gamma'$, then

$$\begin{aligned}
I_3 \;&\le\; C\left[\frac{1}{(2^k r)^n} \int_{B_{2^{k+1}r}(x_0)} |a(y) - a_{2^{k+1}r}|^{u'} dy\right]^{1/u'} \\
&\quad \times \left[\frac{1}{(2^k r)^n} \int_{B_{2^{k+1}r}(x_0)} |\Omega(x-y)|^u |f_k(y)|^u dy\right]^{1/u} \\
&\le\; C\|a\|_* (M_{|\Omega|^u}(|f_k|^u)(x))^{1/u},
\end{aligned}$$

where we have used the John-Nirenberg Lemma on BMO functions (see [9]). Therefore, by the lemma, we obtain

$$\begin{aligned}\int_{B_r(x_0)} I_3^p\,dx &\le C\|a\|_*^p \int_{B_r(x_0)} (M_{|\Omega|^u}(|f_k|^u)(x))^{p/u}dx\\ &\le C\|a\|_*^p \left(\frac{D}{2^{n\theta_2}}\right)^k \|f\|_{L^{p,\phi}}^p \phi(r),\end{aligned}$$

where $\theta_2 \in (1/\gamma, 1)$.

For $I_1$, let $s \in (0,1)$ and $u \in (1,\infty)$ such that $sq > 1$ and $1/(pu') + 1/(sq) = 1$. Set $A = pu'$, $B = sqv$ and $E = sqv'$ with $v \in (1,\infty)$. Then $1/A + 1/B + 1/E = 1$. From Hölder's inequality, it follows that

$$\begin{aligned}I_1 &\le \frac{C|a(x)-a_r|}{(2^kr)^n}\int_{B_{2^{k+1}r}(x_0)} |f_k(y)|^{1/u'}|\Omega(x-y)|^{1/v}\\ &\qquad \times |f_k(y)|^{1/u}|\Omega(x-y)|^{1/v'}dy\\ &\le C|a(x)-a_r)|\left[\frac{1}{(2^kr)^n}\int_{B_{2^{k+1}r}(x_0)} |f_k(y)|^p dy\right]^{1/(pu')}\\ &\quad \times \left[\frac{1}{(2^kr)^n}\int_{B_{2^{k+1}r}(x_0)} |\Omega(x-y)|^{sq}|f_k(y)|^{sqv/u}dy\right]^{1/(sqv)}\\ &\quad \times \left[\frac{1}{(2^kr)^n}\int_{B_{2^{k+1}r}(x_0)} |\Omega(x-y)|^{sq}dy\right]^{1/(sqv')}\\ &\le \frac{C|a(x)-a_r|}{(2^kr)^{n/(pu')}}\phi^{1/(pu')}(2^{k+1}r)\|f\|_{L^{p,\phi}}^{1/u'}(M_{|\Omega|^{sq}}(|f_k|^{sqv/u})(x))^{1/(sqv)}.\end{aligned}$$

By Hölder's inequality again, we have

$$\begin{aligned}&\int_{B_r(x_0)} I_1^p dx\\ &\le \frac{C}{(2^kr)^{n/u'}}\phi^{1/u'}(2^{k+1}r)\|f\|_{L^{p,\phi}}^{p/u'}\int_{B_r(x_0)} |a(x)-a_r|^p\\ &\qquad \times (M_{|\Omega|^{sq}}(|f_k|^{sqv/u})(x))^{p/(sqv)}dx\\ &\le \frac{C}{(2^kr)^{n/u'}}\phi^{1/u'}(2^{k+1}r)\|f\|_{L^{p,\phi}}^{p/u'}\left[\int_{B_r(x_0)} |a(x)-a_r|^{pu'}dx\right]^{1/u'}\\ &\quad \times \left[\int_{B_r(x_0)} (M_{|\Omega|^{sq}}(|f_k|^{sqv/u})(x))^{pu/(sqv)}dx\right]^{1/u}\\ &\le \frac{C}{2^{kn/u'}}\|a\|_*^p\phi^{1/u'}(2^{k+1}r)\|f\|_{L^{p,\phi}}^{p/u'}\end{aligned}$$

$$\times \Big[\int_{B_r(x_0)} (M_{|\Omega|^{sq}}(|f_k|^{sqv/u})(x))^{pu/(sqv)} dx\Big]^{1/u}.$$

Now we choose $s$, $u$ and $v$ such that (i) $1 < pu/(sqv)$; (ii) $u > v$ and $1/q < s \le pu/(qv + pu)$ for the case $q \ge p'$ (hence $1/s \ge (pu/sqv)'$); $u = v$ for the case $q > p$ (hence $1/s > pu/(sqv)$); and $u = v$, $s < 1/\gamma'$ for the case $q > \gamma'$ (hence $1/s > \gamma'$). Since $\Omega \in L^{1/s}(\Sigma_{n-1})$, from Lemma, it follows that

$$\begin{aligned}\int_{B_r(x_0)} I_1^p dx &\le C\|a\|_*^p \phi^{1/u'}(2^{k+1}r)2^{-kn/u'}\|f\|_{L^{p,\phi}}^{p/u'} \\ &\quad \times \left(\frac{D}{2^{n\theta}}\right)^{k/u} \||f|^{sqv/u}\|_{L^{pu/(sqv),\phi}}^{p/(sqv)} \phi^{1/u}(r) \\ &\le C\|a\|_*^p \left(\frac{D}{2^{n\theta_3}}\right)^k \|f\|_{L^{p,\phi}}^p \phi(r),\end{aligned}$$

where $\theta_3 \in (1/\gamma, 1)$. Let $\theta_0 = \min\{\theta_1, \theta_2, \theta_3\}$. Then $\theta_0 \in (1/\gamma, 1)$. Summing up the above estimates, we obtain

$$\begin{aligned}&\left(\int_{B_r(x_0)} |[a,T]f(x)|^p dx\right)^{1/p} \\ &\le \sum_{k=0}^{\infty}\left(\int_{B_r(x_0)} |[a,T]f_k(x)|^p dx\right)^{1/p} \\ &\le C\|a\|_*\|f\|_{L^{p,\phi}}\phi^{1/p}(r) + \sum_{k=1}^{\infty}\left(\int_{B_r(x_0)} I_1^p + I_2^p + I_3^p dx\right)^{1/p} \\ &\le C\|a\|_*\|f\|_{L^{p,\phi}}\phi^{1/p}(r) \\ &\qquad + C\sum_{k=1}^{\infty}\left(\frac{D}{2^{n\theta_0}}\right)^{k/p}(k+2)\|a\|_*\|f\|_{L^{p,\phi}}\phi^{1/p}(r) \\ &\le C\|a\|_*\|f\|_{L^{p,\phi}}\phi^{1/p}(r).\end{aligned}$$

This finishes the proof of Theorem 2. □

*Remark.* When $\phi(r) = r^\lambda$, $\lambda \in (1, n)$, we re-obtain Theorems 2.1–2.2 in [2] again.

It is well known that the Riesz potentials $I_\alpha$ are bounded from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$, where $1/q = 1/p - \alpha/n$ and $\alpha$ is the index of $I_\alpha$. Nakai [7] has obtained the boundedness of $I_\alpha$ on the more generalized Morrey spaces. We partially generalize the results in [7] to the case of sublinear operators on the generalized Morrey spaces in [5] as follows.

**Theorem 3** *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $1 \le D(\phi) < 2^{n-\alpha p}$ and $\gamma = \log 2^{n-\alpha p}/\log D$. Assume sublinear $T$ satisfies that for all $f \in L^1(\mathbf{R}^n)$ with compact support and $x \notin C_0 \operatorname{supp} f$,*

$$Tf(x) \le C \int_{\mathbf{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |f(y)|\, dy, \tag{5}$$

*where $C_0 \ge 1$ is a constant, $\Omega$ is homogeneous of degree zero and $\Omega \in L^\beta(\Sigma_{n-1})$ for some $\beta > n/(n-\alpha)$ and $\beta \ge p'$ or $\beta > \min\{p, n\gamma'/(n-\alpha)\}$. If $T$ is bounded from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$, then $T$ is also bounded from $L^{p,\phi}(\mathbf{R}^n)$ to $L^{q,\phi^{q/p}}(\mathbf{R}^n)$.*

*Proof.* We first point out that if we replace $q$ and $\gamma$ in Lemma, respectively, by $\beta$ and $\gamma$ in this theorem then the conclusion of Lemma still holds.

Without loss of generality, we may also assume $C_0 = 1$. For any $x_0 \in \mathbf{R}^n$ and $r > 0$, we write $f$ as in (3).

For $f_0$, since $T$ is bounded from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$, we have

$$\begin{aligned}\int_{B_r(x_0)} |Tf_0(x)|^q\, dx &\le C\|f_0\|^q_{L^p(\mathbf{R}^n)} \\ &\le C\Big\{\int_{B_{2r}(x_0)} |f(x)|^p dx\Big\}^{q/p} \\ &\le C\phi^{q/p}(r)\|f\|^q_{L^{p,\phi}}.\end{aligned}$$

For $k > 0$, let $u = q/p$ and $1/(pu') + 1/(\theta\beta) = 1$. Then we have $0 < \theta < 1$. For $y \in B_{2^{k+1}r}(x_0)\backslash B_{2^k r}(x_0)$, it is easy to verify that

$$M\chi_{B_r(x_0)}(y) \le \frac{C}{2^{nk}} \tag{6}$$

with $C$ independent of $k$, $r$ and $x_0$. For any $x \in B_r(x_0)$ and $y \in (B_{2r}(x_0))^c$, we easily verify that $(1/|x-y|)^{n-\alpha}$ is comparable to $(M\chi_{B_r(x_0)}(y)/r^n)^{1-\alpha/n}$. Now let $\delta = (n-\alpha p)/n$. Then by Hölder's inequality and (6), we have

$$\begin{aligned}|Tf_k(x)| &\le \frac{C}{r^{n-\alpha}} \int_{B_{2^{k+1}r}(x_0)\backslash B_{2^k r}(x_0)} |\Omega(x-y)| \\ &\qquad \times |f_k(y)|(M_{B_r(x_0)}(y))^{1-\alpha/n} dy \\ &\le \frac{C}{r^{n-\alpha}} \Big[\int_{B_{2^{k+1}r}(x_0)\backslash B_{2^k r}(x_0)} |f_k(y)|^p\end{aligned}$$

$$\begin{aligned}
&\quad \times (M\chi_{B_r(x_0)}(y))^{\delta} dy\Big]^{1/(pu')} \\
&\quad \times \Big[\int_{B_{2^{k+1}r}(x_0)\setminus B_{2^k r}(x_0)} |\Omega(x-y)|^{\theta\beta} \\
&\quad \times (M\chi_{B_r(x_0)}(y))^{(1-\alpha/n-\delta/(pu')-1/(\theta\beta u))\theta\beta u'} dy\Big]^{1/(\theta\beta u')} \\
&\quad \times \Big[\int_{B_{2^{k+1}r}(x_0)\setminus B_{2^k r}(x_0)} |\Omega(x-y)|^{\theta\beta} \\
&\quad \times |f_k(y)|^{\theta\beta} M\chi_{B_r(x_0)}(y) dy\Big]^{1/(\theta\beta u)} \\
&\le C r^{\alpha-n} \left(\frac{D}{2^{n\delta}}\right)^{k/(pu')} \phi^{1/(pu')}(r) \|f\|_{L^{p,\phi}}^{1/u'} (2^k r)^{n/(\theta\beta u')} \\
&\quad \times (2^k)^{-n(1-\alpha/n-\delta/(pu')-1/(\theta\beta u))} \\
&\quad \times r^{n/(\theta\beta u)} (M_{|\Omega|^{\theta\beta}} |f_k|^{\theta\beta}(x))^{1/(\theta\beta u)} \\
&\le C (2^k)^{\alpha p/q} \left(\frac{D}{2^{n\delta}}\right)^{k/(pu')} \phi^{1/(pu')}(r) \|f\|_{L^{p,\phi}}^{1/u'} \\
&\quad \times (M_{|\Omega|^{\theta\beta}} |f_k|^{\theta\beta}(x))^{1/(\theta\beta u)}.
\end{aligned}$$

By Lemma, we obtain

$$\begin{aligned}
\int_{B_r(x_0)} |Tf_k(x)|^q dx &\le C \left(\frac{D}{2^{n\delta}}\right)^{kq/(pu')} \phi^{q/(pu')}(r) \|f\|_{L^{p,\phi}}^{q/u'} \\
&\quad \times \left(\frac{D}{2^{n\epsilon-\alpha p}}\right)^{k} \||f|^{\theta\beta}\|_{L^{q/(\theta\beta u),\phi}}^{q/(\theta\beta u)} \phi(r) \\
&\le C \left(\frac{D}{2^{n\delta}}\right)^{kq/(pu')} \phi^{q/p}(r) \|f\|_{L^{p,\phi}}^{q},
\end{aligned}$$

where $\epsilon \in (1/\gamma, 1)$ such that $n\epsilon - \alpha p > \log_2 D$. Theorefore,

$$\begin{aligned}
&\left(\int_{B_r(x_0)} |Tf(x)|^q dx\right)^{1/q} \\
&\quad \le \sum_{k=0}^{\infty} \left(\int_{B_r(x_0)} |Tf_k(x)|^q dx\right)^{1/q} \\
&\quad \le C\|f\|_{L^{p,\phi}} \phi^{1/p}(r) + C \sum_{k=1}^{\infty} \left(\frac{D}{2^{n\delta}}\right)^{k/(pu')} \|f\|_{L^{p,\phi}} \phi^{1/p}(r) \\
&\quad \le C\|f\|_{L^{p,\phi}} \phi^{1/p}(r).
\end{aligned}$$

Then the proof is finished. □

For the commutators, we also have

**Theorem 4** *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $1 \le D(\phi) < 2^{n-\alpha p}$, $\gamma = \log 2^{n-\alpha p}/\log D$ and $a \in \mathrm{BMO}(\mathbf{R}^n)$. Assume that linear operator $T$ satisfies (5) for some $\beta > n/(n-\alpha)$ and $\beta \ge p'$ or $\beta > \min\{p, n\gamma'/(n-\alpha)\}$. If $[a,T]$ is bounded from $L^p(\mathbf{R}^n) \times \mathrm{BMO}(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$, then $[a,T]$ is also bounded from $L^{p,\phi}(\mathbf{R}^n) \times \mathrm{BMO}(\mathbf{R}^n)$ to $L^{q,\phi^{q/p}}(\mathbf{R}^n)$.*

*Proof.* The proof of this theorem is similar to that of Theorem 2 and Theorem 3, and we only give a sketch for $C_0 = 1$.

For $x_0 \in \mathbf{R}^n$ and $r > 0$, write $f$ as (3). If $k = 0$, by the boundedness of $[a,T]$ from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$, we can obtain the desired inequality. To estimate $Tf_k$ for $k > 0$, we first point out that in the proof of Theorem 3 we have proved

$$\begin{aligned}\Bigg\{\int_{B_r(x_0)}\Bigg[\frac{1}{r^{n-\alpha}}\int_{B_{2^{k+1}r(x_0)}\setminus B_{2^k r}(x_0)}&|\Omega(x-y)f_k(y)|\\ &\times (M\chi_{B_r(x_0)}(y))^{1-\alpha/n}dy\Bigg]^q dx\Bigg\}^{1/q}\\ \le C\left(\frac{D}{2^{n\delta}}\right)^{k/(pu')}\phi^{1/p}\|f\|_{L^{p,\phi}},&\end{aligned}\tag{7}$$

where $\alpha$, $p$, $q$, $u$ and $\delta$ are as in the proof of Theorem 3. Now we set

$$\begin{aligned}I_1 &= \Bigg\{\int_{B_r(x_0)}\Bigg[|a_r - a(x)|\\ &\quad\times\int_{B_{2^{k+1}r}(x_0)\setminus B_{2^k r}(x_0)}\frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}|f(y)|dy\Bigg]^q dx\Bigg\}^{1/q},\\ I_2 &= \Bigg\{\int_{B_r(x_0)}\Bigg[\int_{B_{2^{k+1}r}(x_0)\setminus B_{2^k r}(x_0)}|a_r - a_{2^{k+1}r}|\\ &\quad\times\frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}|f(y)|dy\Bigg]^q dx\Bigg\}^{1/q},\end{aligned}$$

and

$$I_3 = \Bigg\{\int_{B_r(x_0)}\Bigg[\int_{B_{2^{k+1}r}(x_0)\setminus B_{2^k r}(x_0)}|a(y) - a_{2^{k+1}r}|$$

$$\times \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}|f(y)|dy\Big]^q dx\Big\}^{1/q}.$$

For $I_1$, if $\beta \geq p'$, by Hölder's inequality and (6), we have

$$\begin{aligned}
I_1 &\leq Cr^{\alpha-n}\Big\{\int_{B_r(x_0)}|a(x)-a_r|^q\int_{B_{2^{k+1}r}(x_0)\backslash B_{2^k r}(x_0)}\Big[|\Omega(x-y)f_k(y)| \\
&\quad \times (M\chi_{B_r(x_0)}(y))^{1-\alpha/n}dy\Big]^q dx\Big\}^{1/q} \\
&\leq Cr^{\alpha-n}\Big\{\int_{B_r(x_0)}|a(x)-a_r)|^q\Big[\int_{B_{2^{k+1}r}(x_0)\backslash B_{2^k r}(x_0)}|\Omega(x-y)|^{p'} \\
&\quad \times (M\chi_{B_r(x_0)}(y))^{(1-\alpha/n-\delta/p)p'}dy\Big]^{q/p'} \\
&\quad \times \Big[\int_{B_{2^{k+1}r}(x_0)\backslash B_{2^k r}(x_0)}|f_k(y)|^p(M\chi_{B_r(x_0)}(y))^\delta dy\Big]^{q/p}dx\Big\}^{1/q} \\
&\leq Cr^{\alpha-n}(2^k r)^{n/p'}(2^{-k})^{n(1-\alpha/n-\delta/p)}\left(\frac{D}{2^{n\delta}}\right)^{k/p} \\
&\quad \times \|f\|_{L^{p,\phi}}\phi_{1/p}(r)\Big(\int_{B_r(x_0)}|a(x)-a_r|^q dx\Big)^{1/q} \\
&\leq C\left(\frac{D}{2^{n\delta}}\right)^{k/p}\|a\|_*\|f\|_{L^{p,\phi}}\phi^{1/p}(r).
\end{aligned}$$

If $\beta > \min\{p, n\gamma'/(n-\alpha)\}$, we let $\epsilon \in (0,1)$ to be determined late. Let $u = q/(p\epsilon)$, $s = pu/q$, and $\theta = \frac{1}{\beta(1-1/(pu'))}$. Then $s > 1$, $qs = pu$, $1/(pu') + 1/(\theta\beta) = 1$ and $pu/(\theta\beta) > 1$. For the case $\beta > \max\{n/(n-\alpha), p\}$ we choose $\epsilon > 1 - q((n-\alpha)/n - 1/\beta)$ and $v > 1$ such that $v \in (pu/\beta, pu/(\theta\beta))$; and for the case $\beta > n\gamma'/(n-\alpha)$ we choose $\epsilon > 1 - q((n-\alpha) - \gamma'/\beta)$ and $v \in (1, pu/(\theta\beta))$. Then we have $1/\theta > pu/(\theta\beta v) > 1$ for the case $\beta > p$; and $1/\theta > \gamma'$ for the case $\beta > n\gamma'/(n-\alpha)$. Therefore, by Hölder's inequality, Lemma and (6), we have

$$\begin{aligned}
I_1 &\leq Cr^{\alpha-n}\Big\{\int_{B_r(x_0)}|a(x)-a_r|^q \\
&\quad \times \Big[\int_{B_{2^{k+1}r}(x_0)\backslash B_{2^k r}(x_0)}|f_k(y)|^p(M\chi_{B_r(x_0)}(y))^\delta dy\Big]^{q/(pu')} \\
&\quad \times \Big[\int_{B_{2^{k+1}r}(x_0)\backslash B_{2^k r}(x_0)}|\Omega(x-y)|^{\theta\beta}
\end{aligned}$$

$$\times\, (M\chi_{B_r(x_0)}(y))^{(1-\alpha/n-\delta/(pu')-1/(\theta\beta v))\theta\beta v'}dy\Big]^{\frac{q}{\theta\beta v'}}$$

$$\times \Big[\int_{B_{2^{k+1}r}(x_0)\setminus B_{2^k r}(x_0)} |\Omega(x-y)|^{\theta\beta}|$$

$$\times\, f_k(y)|^{\theta\beta v/u} M\chi_{B_r(x_0)}(y)dy\Big]^{q/(\theta\beta v)}dx\Big\}^{/q}$$

$$\leq Cr^{\alpha-n+n/(\theta\beta)}\left(\frac{D}{2^{n\delta}}\right)^{k/(pu')}\phi^{1/(pu')}(r)$$

$$\times\, \|f\|_{L^{p,\phi}}^{1/u'}(2^k)^{-n(1-\alpha/n-\delta/(pu')-1/(\theta\beta))}$$

$$\times\left(\int_{B_r(x_0)}|a_r-a(x)|^{qs'}dx\right)^{1/(qs')}$$

$$\times\left\{\int_{B_r(x_0)}\left[M_{|\Omega|^{\theta\beta}}(|f_k|^{\theta\beta v/u})(x)\right]^{qs/(\theta\beta v)}dx\right\}^{1/(qs)}$$

$$\leq C(2^k)^{\epsilon\alpha p/q}\left(\frac{D}{2^{n\delta}}\right)^{k/(pu')}\|a\|_*\left(\frac{D}{2^{n\epsilon_1}}\right)^{kq/(pu)}\|f\|_{L^{p,\phi}}\phi^{1/p}(r)$$

$$\leq C\left(\frac{D}{2^{n\delta}}\right)^{k/(pu')}\left(\frac{D}{2^{n\epsilon_1-\alpha p}}\right)^{k/(pu)}\|f\|_{L^{p,\phi}}\|a\|_*\phi^{1/p}(r)$$

$$\leq C\left(\frac{D}{2^{n\delta}}\right)^{k/(pu')}\|a\|_*\|f\|_{L^{p,\phi}}\phi^{1/p}(r),$$

where $\epsilon_1 \in (1/\gamma, 1)$ for the case $\beta > p$ and $\epsilon_1 \in (1/\gamma, 1-\theta)$ for the case $\beta > n\gamma'/(n-\alpha)$.

For $I_2$, from (7) and the properties of BMO functions, it follows that

$$I_2 \leq C|a_r - a_{2^{k+1}r}|\left\{\int_{B_r(x_0)}\left[\frac{1}{r^{n-\alpha}}\right.\right.$$

$$\times\int_{B_{2^{k+1}r}(x_0)\setminus B_{2^k r}(x_0)}|\Omega(x-y)f_k(y)|$$

$$\left.\left.\times\,(M\chi_{B_r(x_0)}(y))^{1-\alpha/n}dy\right]^q dx\right\}^{\frac{1}{q}}$$

$$\leq C(k+1)\left(\frac{D}{2^{n\delta}}\right)^{k/(pu')}\|a\|_*\|f\|_{L^{p,\phi}}\phi^{1/p}(r),$$

where $u = q/p$.

For $I_3$, let $1 < s < \min\{p, q\}$. By Hölder's inequality and (7), we have

$$\begin{aligned} I_3 &\leq Cr^{\alpha-n}\Big\{\int_{B_r(x_0)}\Big[\int_{B_{2^{k+1}r}(x_0)\setminus B_{2^k r}(x_0)}|a(y)-a_{2^{k+1}r}|^{s'}dy\Big]^{q/s'} \\ &\quad\times\Big[\int_{B_{2^{k+1}r}(x_0)\setminus B_{2^k r}(x_0)}|\Omega(x-y)|^s|f_k(y)|^s \\ &\qquad\times(M\chi_{B_r(x_0)}(y))^{s(1-\alpha)}dy\Big]^{q/s}dx\Big\}^{1/q} \\ &\leq C\|a\|_*(2^k r)^{n/s'}r^{\alpha-n+(n-s\alpha)/s}(2^{-k})^{(s-1)n/s} \\ &\quad\times\Big\{\int_{B_r(x_0)}\Big[\frac{1}{r^{n-s\alpha}}\int_{B_{2^{k+1}r}(x_0)\setminus B_{2^k r}(x_0)}|\Omega(x-y)|^s|f_k(y)|^s \\ &\qquad\times(M\chi_{B_r(x_0)}(y))^{1-s\alpha/n}dy\Big]dx\Big\}^{s/q\cdot 1/s} \\ &\leq C\|a\|_*\left(\frac{D}{2^{n\delta}}\right)^{k/pu'}\|f\|_{L^{p,\phi}}\phi^{1/p}(r), \end{aligned}$$

where $\delta = (n-\alpha p)/n$ and $u = q/p$.

Summing up the above estimates, we finish the proof of Theorem 4. □

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