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Existence and nonexistence of global solutions to quasilinear parabolic equations with convection

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Abstract. We consider nonnegative solutions to the Cauchy problem for the quasilinear parabolic equations $u_t = \triangle u^m + a \cdot \nabla u^q + u^p$ where $m \geq 1$, $p, q > 1$, $a \in \mathbf{R}^{\mathrm{N}}$ and $a \neq 0$. In this paper we show: (a) if $q > m-1$ and $\max\{m, q\} \leq p < \min\{m+2/N, m+2(q-m+1)/(N+1)\}$ or $m+2/N \geq q \geq m+1/N$ and $p = m+2/N$, then all nontrivial solutions do not exist globally in time; (b) if $p > m+2/N$, then there are nontrivial global solutions. Further, in case (b) we study the asymptotic behavior of the global solutions. We also study the asymptotic behavior of the global solutions of $u_t = \triangle u^m + a \cdot \nabla u^q$.

Key words: asymptotic behavior, blow-up, Cauchy problem, convection, critical exponent, global solution, $L^\infty - L^\ell$ estimate, quasilinear parabolic equation.

1. Introduction

In this paper we shall consider the Cauchy problem

$$\partial_t u = \triangle u^m + a \cdot \nabla u^q + u^p \quad (x,t) \in \mathbf{R}^{\mathrm{N}} \times (0,T), \tag{1.1}$$

$$u(x,0) = u_0(x) \quad x \in \mathbf{R}^{\mathrm{N}}, \tag{1.2}$$

where $m \geq 1$, $p, q > 1$, $a \in \mathbf{R}^{\mathrm{N}}$, $a \neq 0, u_0(x) \geq 0$ and $u_0(x) \in BC(\mathbf{R}^{\mathrm{N}})$ (bounded continuous functions). It is well known that if $T > 0$ is small enough then a nonnegative continuous weak solution of (1.1) (1.2) exists (see [19], [26], [4]). The definition of a weak solution of (1.1) (1.2) is given in Section 2.

We use the following notation: $L^p\,(1 \leq p \leq \infty)$ is the usual space of all L^p-functions in $\mathbf{R}^{\mathrm{N}}$ with norm $\|f\|_p \equiv \|f\|_{L^p(\mathbf{R}^{\mathrm{N}})}$.

When $a = 0$, the following results are known to hold:

(I) If $1 < p \leq m+2/N$ then all nontrivial nonnegative weak solutions of (1.1) (1.2) blow up in finite time. Namely, $\lim_{t\uparrow T} \|u(t)\|_\infty = \infty$ for some $T \in (0,\infty)$.

(II) If $p > m+2/N$, then global solutions of (1.1) (1.2) exist when the initial data are sufficiently small.

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We note that in case (II) solutions of (1.1) (1.2) also blow up in finite time when the initial data are large enough (see [15] etc.).

Case (I) is called the blow-up case; (II) is called the global existence case. The cut off number

$$p_m^* = m + 2/N \tag{1.3}$$

is called the critical exponent.

In case $p \neq p_m^*$, these results are due to Fujita [9] for $m = 1$ and Galaktionov et al. [11] for $m > 1$. In case $p = p_m^*$, these results are due to Hayakawa [13] and Weissler [30] for $m = 1$ and Galaktionov [10], Kawanago [17] and Mochizuki-Suzuki [24] independently for $m > 1$. We note that similar results were obtained in the exterior domain case. Namely, when $N \geq 2$ Mochizuki-Suzuki [24] showed that p_m^* is the critical exponent and when $N \geq 3$ Suzuki [29] showed that $p = p_m^*$ is in the blow-up case.

Especially, when $p > m+2/N$ $(= p_m^*)$, Kawanago [17] obtained a precise L^∞-decay estimates of global solutions $u(t) = u(x,t)$ of (1.1) (1.2) with $a = 0$ as follows: If $\|u_0\|_{p_0}$ is sufficiently small, then

$$\|u(t)\|_\infty \leq Kt^{-1/(p-1)} = Kt^{-N/[N(m-1)+2p_0]} \quad \text{for } t \geq 0$$

where K is some constant and

$$p_0 = \frac{N(p-m)}{2}. \tag{1.4}$$

Furthermore, he obtained that if $u_0(x) \in L^1(\mathbf{R^N})$ then the solution $u(t)$ converges to the heat kernel (when $m = 1$) and the Barenblatt solution (when $m > 1$) with the convergence rate $t^{-N/\{N(m-1)+2\}}$.

Our aim in this paper is to extend these results to case $a \neq 0$.

In the blow-up case, we get the following theorem. Put

$$p_{m,q}^* = \min\left\{m + \frac{2}{N}, m + \frac{2(q-m+1)}{N+1}\right\}. \tag{1.5}$$

Theorem 1 *Let $q > m-1$. If*

$$\max\{m, q\} \leq p < p_{m,q}^*, \tag{1.6}$$

then all nonnegative nontrivial weak solutions $u(x,t)$ of (1.1) (1.2) *do not exist globally in time. Furthermore, if $2q \geq m+1$ and $u_0(x) \in L^1(\mathbf{R^N})$,* *then*

$$\lim_{t \uparrow T} \|u(t)\|_\infty = \infty \tag{1.7}$$

for some $T \in (0, \infty)$.

The methods of the proof of Theorem 1 are the same as those of Aguirre-Escobedo [1].

In the global existence case, we obtain the following L^∞-estimates for the solution of (1.1) (1.2).

Theorem 2 *Let* $p > m + 2/N$. *Assume that* $u_0 \in L^{p_0}(\mathbf{R^N})$ *with* $p_0 \equiv N(p-m)/2\,(>1)$. *Then there exists some constant* $\delta_0 = \delta_0(N, m, p) > 0$ *such that if* $\|u_0\|_{p_0} < \delta_0$ *then* (1.1) (1.2) *has a weak solution* $u(x,t)$ *with* $T = \infty$ *satisfying*

$$\|u(t)\|_\infty \le K_1 t^{-1/(p-1)} = K_1 t^{-N/\{N(m-1)+2p_0\}} \quad \text{for} \quad t > 0, \tag{1.8}$$

where $K_1 = K_1(N, m, p, \delta_0)$.

Further, we assume that $u_0 \in L^1(\mathbf{R^N})$.

Theorem 3 *Let* $p > m + 2/N$. *Assume that* $u_0 \in L^1(\mathbf{R^N}) \cap L^{p_0}(\mathbf{R^N})$. *Then there exists some constant* $\delta_1 = \delta_1(N, m, p)$ *such that if* $\|u_0\|_{p_0} < \delta_1$ *then* (1.1) (1.2) *has a weak solution* $u(t) = u(x,t)$ *with* $T = \infty$ *satisfying*

$$\|u(t)\|_\infty \le K_2 \min\left\{t^{-N/\{N(m-1)+2p_0\}}, t^{-N/\{N(m-1)+2\}}\right\} \quad \text{for} \quad t > 0 \tag{1.9}$$

and

$$\sup_{t>0} \|u(t)\|_1 \le K_3 \tag{1.10}$$

where $K_2 = K_2(N, m, p, \|u_0\|_1, \delta_1)$ *and* $K_3 = K_3(N, p, m, \|u_0\|_1, \delta_1) < \infty$. *Moreover, if* $q > m + 1/N$, *then the weak solution* $u(t)$ *of* (1.1) (1.2) *is unique and satisfies that*

$$t^{N/\{N(m-1)+2\}} \left|u(x,t) - V_m(x, t, M_\infty)\right| \to 0 \quad \text{as} \quad t \to \infty \tag{1.11}$$

uniformly on the set $\{x \in \mathbf{R^N}; |x| \le bt^{1/\{N(m-1)+2\}}\}$ $(b > 0)$ *where*

$V_m(x,t,L)$ is a unique weak solution of

$$\begin{cases} v_t = \triangle v^m & (x,t) \in \mathbf{R^N} \times (0,\infty) \\ v(x,0) = L\delta(x) & x \in \mathbf{R^N} \end{cases} \tag{1.12}$$

with Dirac's δ-function $\delta(x)$ and a positive constant L, and

$$M_\infty = \int_{\mathbf{R^N}} u_0(x)\,dx + \int_0^\infty \int_{\mathbf{R^N}} u^p\,dxdt \quad (< \infty). \tag{1.13}$$

We note that (1.8), (1.9) and (1.10) also hold for the equation with convection term $a \cdot \nabla u^q$ replaced by more general $K(x) \cdot \nabla u^q$ where $K(x) = (k_1(x), \dots, k_N(x))$ with $k_i(x) \in C^1(\mathbf{R^N}) \cap L^\infty(\mathbf{R^N})$ $(0 \le i \le N)$ and $\nabla \cdot K(x) = 0$ in $\mathbf{R^N}$.

Our proof of Theorem 2 and 3 is based on the energy estimates due to Kawanago [17] which treats the case $a = 0$. He showed these theorems with $a = 0$ using $L^\infty - L^\ell$ estimates for solutions which are obtained by virtue of $L^\infty - L^\ell$ estimates for solutions of a semilinear equation. But, it seems that his methods can not be directly applied to equation (1.1) with $a \neq 0$. Therefore, we need the other $L^\infty - L^\ell$ estimates for solutions to prove Theorem 2 and 3. And in order to get these estimates we directly apply the Moser's iteration method to equation (1.1).

Theorem 1 and 2 show that when $m + 2/N > q > m + 1/N$ (which is called the weak convection case), the number $p^*_{m,q}$ $(= m + 2/N)$ is a critical exponent. In case $p = p^*_{m,q}$, we get the following theorem.

Theorem 4 *If $p \ge q \ge m + 1/N$ and $p = p^*_{m,q} (= m + 2/N)$, then all nonnegative nontrivial weak solutions $u(x,t)$ of (1.1) (1.2) do not exist globally in time. Furthermore, if $u_0(x) \in L^1(\mathbf{R^N})$, then*

$$\lim_{t \uparrow T} \|u(t)\|_\infty = \infty. \tag{1.14}$$

for some $T \in (0,\infty)$.

The methods of the proof are the same as those of the proof of R. Suzuki [29] and Mochizuki-Mukai [23] in the critical case. Namely, we use the L^1-estimate and some transformation for the solutions. See also J. Aguirre-M. Escobedo [1].

We note that in case $p > p^*_{m,q}$ solutions $u(x,t)$ of (1.1) (1.2) do not exist globally in time either when the initial data are large enough (see [21] and [16]).

Here we must mention the interesting work of J. Aguirre-M. Escobedo [1] concerning with the blow-up or global existence of solutions of (1.1) (1.2) when $m = 1$. Roughly speaking, they showed that if $p \geq q$, then $p^*_{1,q}$ is the critical exponent and $p = p^*_{1,q}$ belongs to the blow-up case. Therefore it seems that our results are not complete in the strong convection case $q < m + 1/N$. Because, we can not see whether p belongs to the blow-up case or global existence case when $m - 1 < q < m + 1/N$ and $p^*_{m,q}$ $(= m + 2(q - m + 1)/(N + 1)) \leq p \leq m + 2/N$. However, in case $m = 1$ and $q > 1 + 1/N$, our results refine their results. Because, their sufficient condition on the existence of global solution need the smallness of $\|u_0\|_\infty$ and $\|u_0\|_{L^1}$ and we further obtain a precise L^∞-decay estimate of global solutions.

Finally, we note that the methods of the proof of Theorem 3 can be applied to the following problem:

$$v_t = \triangle v^m + a \cdot \nabla v^q, \tag{1.15}$$

$$v(x, 0) = v_0(x), \tag{1.16}$$

where $m \geq 1$, $q > 1$, $v_0(x) \geq 0$ and $v_0(x) \in BC(\mathbf{R^N})$. We obtain the following theorem which is used in the proof of Theorem 4.

Theorem 5 *Let $v_0(x) \in L^1(\mathbf{R^N})$. Then there exists a weak solution of* (1.15) (1.16) *satisfying*

$$M = \int_{\mathbf{R^N}} v(x,t)\,dx = \int_{\mathbf{R^N}} v_0(x)\,dx \quad t \geq 0 \tag{1.17}$$

and

$$\|v(t)\|_\infty \leq K_4 t^{-N/[N(m-1)+2]} \quad \textit{for} \ \ t \geq 0 \tag{1.18}$$

where $K_4 = K_4(m, N, M)$. Moreover, if $q > m + 1/N$, then the weak solution $v(t)$ of (1.15) (1.16) *is unique and satisfies that*

$$t^{N/\{N(m-1)+2\}} |v(x,t) - V_m(x,t,M)| \to 0 \quad \textit{as} \ \ t \to \infty \tag{1.19}$$

uniformly on set $\{x \in \mathbf{R^N} \mid |x| \leq bt^{1/\{N(m-1)+2\}}\}$ $(b > 0)$ where $V_m(x,t,L)$ is as in Theorem 3.

When $m = 1$, these results were obtained by Escobedo-Zuazua [6]. In their results, the convergence in (1.19) is uniform convergence in $\mathbf{R^N}$.

Furthermore, they showed that if $q = 1 + 1/N$, then the L^1-valued solution of (1.15) (1.16) with $m = 1$ converges to the self-similar solution of (1.15) with convergence rate $t^{-N/\{N(m-1)+2\}}$. But, we can not extend this result to case $m > 1$. We note that in case $m = 1$ and $1 < q < 1+1/N$, Escobedo-Vázquez-Zuazua [5] obtained the interesting results which show that the solution $u(t)$ of (1.15) (1.16) converges to some self-similar solution of the reduced equation $u_t = \triangle' u + a \cdot \nabla u^q$ as $t \to \infty$ where $\triangle'$ is the $(N-1)$-dimensional Laplacian in the hyperplane orthogonal to a.

Remark 1.1. In case $q = 1$, noting the suitable linear change of variables we can transform equation (1.1) or (1.15) into (1.1) or (1.15) with $a = 0$ respectively and so we obtain the similar results to those with $a = 0$.

Now, we state some related papers to those problems. There are several papers in one dimension case. When $m = 1$, Friedman-Lacey [8] studied the blow-up conditions and the asymptotic behavior of blow-up solutions for (1.1) in bounded intervals. Levine et al. [21] also studied the stability and instability for the solutions of (1.1) in bounded intervals. For quasilinear equation, R. Suzuki [28] and Imai-Mochizuki-Suzuki [16] studied the blow-up condition and existence of single point blow-up solutions in bounded intervals or **R**. In larger dimension case $N > 1$, we do not know the paper which treat (1.1) with $a \neq 0$ but Aguirre-Escobedo [1]. For equation (1.15) Hui [14] studied the uniqueness and existence of solutions and discussed the asymptotic behaviour of solutions as $q \to \infty$.

We refer to the review article [20] for a lot of literature on blow-up theorems for problems related to (1.1).

The rest of the paper is organized as follows. In the next Section 2, we define a weak solution of (1.1) and prepare the fundamental propositions and several preliminary lemmas. In Section 3, we consider the blow-up cases and prove Theorem 2. In section 4, we give the $L^\infty - L^\ell$ estimates for the solutions of (1.1) (1.2) in order to show Theorem 3 which is proved in Section 5. Also, in Section 5, we prepare the several lemmas in order to obtain the $L^\infty - L^\ell$ estimates for the solution when $u_0(x) \in L^1(\mathbf{R^N})$ and in section 6 we show these estimates. In Section 7, using them we prove Theorem 3 and Theorem 5. In Section 8 we consider the critical case and prove Theorem 4. Finally, in Appendix we show the comparison theorem for the solutions.

2. Definition and preliminary

We begin with the definition of weak solutions of (1.1) (1.2).

Definition 2.1 By a *weak solution* of equation (1.1) in $\mathbf{R}^\mathbf{N} \times [0, T)$, we mean a function $u(x,t)$ in $\mathbf{R}^\mathbf{N} \times [0, T)$ such that

(i) $u(x,t) \geq 0$ in $\mathbf{R}^\mathbf{N} \times [0,T)$ and $\in BC(\mathbf{R}^\mathbf{N} \times [0,\tau])$ (bounded continuous) for each $0 < \tau < T$.

(ii) For any $0 < \tau < T$ and nonnegative $\varphi(x,t) \in C_0^\infty(\mathbf{R}^\mathbf{N} \times [0,T))$,

$$\int_{\mathbf{R}^\mathbf{N}} u(x,\tau)\varphi(x,\tau)\,dx - \int_{\mathbf{R}^\mathbf{N}} u(x,0)\varphi(x,0)\,dx$$
$$= \int_0^\tau \int_{\mathbf{R}^\mathbf{N}} \{u\partial_t\varphi + u^m\triangle\varphi - u^q a\cdot\nabla\varphi + u^p\varphi\}\,dx\,dt. \tag{2.1}$$

A *supersolution* [or *subsolution*] is similarly defined with equality of (2.1) replaced by $\geq$ [or $\leq$].

The following comparison theorem holds.

Proposition 2.2 (comparison theorem) *Assume $2q \geq m+1$. Let v and u be weak solutions of* (1.1) *in $\mathbf{R}^\mathbf{N} \times [0,T)$, and suppose that $v(x,0)$ and $u(x,0)$ belong to $L^1(\mathbf{R}^\mathbf{N})$. If $v(x,0) \geq u(x,0)$ in $\mathbf{R}^\mathbf{N}$, then we have $v \geq u$ in the whole $\mathbf{R}^\mathbf{N} \times [0,T)$.*

Proof. This proposition immediately follows from Corollary 9.2, since in equation (1.1) condition $2q \geq m+1$ is equivalent to condition (A1) of this corollary. □

Next we construct a weak solution of (1.1) (1.2) as follows: First, we assume

$$u_0(x) \leq Ce^{-|x|/m} \quad \text{for some} \quad C > 0. \tag{2.2}$$

Let

$$u_{0,n}(x) = \begin{cases} \max\{u_0(x), e^{-n/m}\} & \text{in} \quad x \in B_n \\ e^{-n/m} & \text{in} \quad x \notin B_n \end{cases} \tag{2.3}$$

and $u_n(x,t)$ be the classical solution of the initial boundary value problem

$$u_t = \triangle u^m + a\cdot\nabla u^q + u^p \quad (x,t) \in B_n \times (0,T) \tag{2.4}$$

$$u(x,0) = u_{0,n}(x) \quad x \in B_n \tag{2.5}$$

$$u(x,t) = e^{-n/m} \quad |x| = n,\ t > 0, \tag{2.6}$$

where

$$B_R \equiv \{x \in \mathbf{R^N} \mid |x| < R\}.$$

Let

$$y(t;M) = (M^{-(p-1)} - (p-1)t)^{-1/(p-1)} \tag{2.7}$$

and

$$T(M) = \frac{1}{p-1} M^{-(p-1)}. \tag{2.8}$$

Then, we obtain the local existence theorem for solutions by the following two propositions.

Proposition 2.3 *If* $T = T(\|u_{0,n}\|_\infty)$, *then there exists a unique solution* $u_n(x,t)$ *of* (2.4) (2.5) (2.6) *satisfying*

$$\|u_n(x,t)\|_\infty \le y(t; \|u_{0,n}\|_\infty) \quad \textit{for} \ \ t \in [0,T). \tag{2.9}$$

Proof. The proof is obvious. See Kawanago [17]. □

Proposition 2.4 (existence I) (Ref. [26], [14]) *Assume* (2.2) *and assume that for some subsequence* $\{n'\} \subset \{n\}$,

$$\sup_{[0,T]} \|u_{n'}(t)\|_\infty \le C' \quad \textit{for large} \ \ n'. \tag{2.10}$$

Then, there exists a weak solution $u(x,t)$ *of* (1.1) (1.2) *such that for some subsequence* $\{n''\} \subset \{n'\}$

$$u_{n''}(x,t) \to u(x,t) \quad \textit{as} \ \ n'' \to \infty \tag{2.11}$$

uniformly in $\mathbf{R^N} \times [0,T]$,

$$\nabla u_{n''}^{(\ell+m-1)/2} \to \nabla u^{(\ell+m-1)/2} \quad \textit{as} \ \ n'' \to \infty \tag{2.12}$$

weakly in $L^2_{loc}(\mathbf{R^N} \times [0,T])$ *for each* $\ell > 1$ *and*

$$\int_{B_{n''}} u_{n''}(x,t)^\ell \, dx \to \int_{\mathbf{R^N}} u(x,t)^\ell \, dx \quad \textit{as}\ n'' \to \infty \tag{2.13}$$

uniformly in $[0,T]$ *for each* $\ell \ge 1$. *Here we extended* u_n *to* 0 *in* $B_n^c \times [0,T]$.

Furthermore, $u(x,t)$ *satisfies that*

$$u(x,t) \leq C'' e^{-|x|/m} \quad \text{in } \mathbf{R^N} \times [0,T] \tag{2.14}$$

for some constant $C'' > 0$,

$$\begin{array}{l} u(x,t) \text{ is uniformly continuous in } \mathbf{R^N} \times [0,T] \text{ with the} \\ \text{modulo of continuity depending only on } \sup_{[0,T]} \|u(t)\|_\infty, \end{array} \tag{2.15}$$

$$\int_{\mathbf{R^N}} u(x,t)^\ell \, dx \text{ is continuous in } t \in [0,T] \tag{2.16}$$

for each $\ell \geq 1$ *and*

$$\int_0^T \int_{\mathbf{R^N}} |\nabla u^{(\ell+m-1)/2}|^2 \, dxdt \leq \liminf_{n' \to \infty} \int_0^T \int_{B_{n'}} |\nabla u_{n'}^{(\ell+m-1)/2}|^2 \, dxdt \tag{2.17}$$

for each $\ell > 1$.

Proof. It is enough to show that

$$u_n(x,t) \leq C'' e^{-|x|/m} \quad \text{in } B_n \times [0,T] \tag{2.18}$$

where $C'' > 0$ is a constant depending only on C and C'. Because, if (2.18) holds, then by the methods of [26], the equicontinuity of the solution of (2.4) (see DiBenedetto [4]) and the next lemma we can prove this proposition easily.

Put $v = u_n^m$. Then v is a solution of the problem

$$\begin{cases} v_t = mv^{(m-1)/m} \triangle v + qv^{(q-1)/m} a \cdot \nabla v + mv^{(p-1)/m} v \\ \qquad\qquad\qquad\qquad (x,t) \in B_n \times [0,T] \\ v(x,0) = u_{0,n}^m \quad x \in B_n \\ v(x,t) = e^{-n} \quad |x| = n. \end{cases} \tag{2.19}$$

We note that

$$v(x,0) \leq L e^{-|x|} \quad \text{for } x \in B_n, \tag{2.20}$$

where $L = C^m$.

Put

$$\tilde{v}(x,t) = L e^{\tilde{C} t} e^{-|x|} \tag{2.21}$$

where

$$\tilde{C} = m\left(M^{(m-1)/m} + |a|qM^{(q-1)/m}/m + M^{(p-1)/m}\right) \tag{2.22}$$

with $M = \{C'\}^m$. Then, by the simple calculation we get

$$\begin{aligned} & mv^{(m-1)/m}\triangle\tilde{v} + qv^{(q-1)/m}a\cdot\nabla\tilde{v} + mv^{(p-1)/m}\tilde{v} \\ &= m\Bigg\{v^{(m-1)/m}\left(\frac{\partial^2}{\partial r^2} + \frac{N-1}{r}\frac{\partial}{\partial r}\right)e^{-r} \\ &\qquad + v^{(q-1)/m}\frac{q}{m}\frac{a\cdot x}{|x|}\frac{\partial}{\partial r}e^{-r} + v^{(p-1)/m}e^{-r}\Bigg\} \times Le^{\tilde{C}t} \\ &\le \tilde{C}Le^{\tilde{C}t}e^{-r} = \tilde{v}_t \end{aligned}$$

with $r = |x|$ and

$$\tilde{v}(x,t) = Le^{\tilde{C}t}e^{-n} \quad \text{on} \quad |x| = n. \tag{2.23}$$

Hence, using the comparison theorem for the classical solution of a semilinear equation we have

$$v(x,t) \le \tilde{v}(x,t) \quad \text{in} \quad B_n \times [0,T],$$

and so we obtain (2.18). The proof is complete. □

This solution $u_n(x,t)$ of (2.4) (2.5) (2.6) satisfies the following usual energy inequality which is used in the proof of the previous proposition and after this section.

Lemma 2.5 *For each $\ell \ge 1$,*

$$\begin{aligned} & \frac{\partial}{\partial t}\int_{B_n} u_n^\ell\,dx + \frac{4m\ell(\ell-1)}{(m+\ell-1)^2}\int_{B_n}|\nabla u_n^{(m+\ell-1)/2}|^2\,dx \\ &\quad \le \ell\int_{B_n} u_n^{p+\ell-1}\,dx \end{aligned} \tag{2.24}$$

Proof. By the comparison theorem we see that $u_n \ge e^{-n/m}$ and so $\frac{\partial u}{\partial n} \le 0$ on $|x| = n$ where n denotes the outer unit normal to the boundary. Hence, multiplying the both sides by u_n^ℓ and integrating by parts over B_n we get (2.24). □

Next, we consider the Cauchy problem (1.1) (1.2) in the case

$$u_0(x) \in L^\infty(\mathbf{R^N}). \tag{2.25}$$

Let $\{u_{0,n}(x)\}$ be a sequence of continuous functions in $\mathbf{R^N}$ such that

$$0 \le u_{0,n}(x) \le C(n)e^{-|x|/m} \quad \text{in} \quad \mathbf{R^N} \tag{2.26}$$

for some $C(n) > 0$ and

$$u_{0,n}(x) \to u_0(x) \quad \text{as} \quad n \to \infty \tag{2.27}$$

locally uniformly in $\mathbf{R^N}$.

Proposition 2.6 (existence II) *Assume* (2.25). *Let* $u_n(x,t)$ *be a classical solution of* (1.1) (1.2) *with* $u_0(x)$ *replaced by* $u_{0,n}(x)$ *where* $u_{0,n}(x)$ *satisfies* (2.26) *and* (2.27). *If* (2.10) *holds for some subsequence* $\{n'\} \subset \{n\}$, *then there exists a weak solution* $u(x,t)$ *of* (1.1) (1.2) *such that for some subsequence* $\{n''\} \subset \{n'\}$,

$$u_{n''}(x,t) \to u(x,t) \quad \text{as } n'' \to \infty \tag{2.28}$$

locally uniformly in $\mathbf{R^N} \times [0,T]$.

Proof. The proof is the same as that of Proposition 2.4. □

Finally, we state some versions of the Gagliardo-Nirenberg inequality. They are the essential inequality in the proof of Theorem 2 and 3. First, we recall the Gagliardo-Nirenberg inequality (c.f. Ladyzenskaja et al. [19], Ohara [25]):

Lemma 2.7 *For any* $f \in C_0^\infty(\mathbf{R^N})$,

$$\|f\|_{\tilde r} \le C\tilde r \|f\|_r^{1-\theta}\|\nabla f\|_2^\theta \tag{2.29}$$

where

$$\theta = \frac{r^{-1} - \tilde r^{-1}}{N^{-1} - 2^{-1} + r^{-1}},$$

C is a constant independent of r, $\tilde r$ *and* θ, *and*:

(1) *for* $N > 2$, $0 < r \le \max\{1, r\} < \tilde r < 2N/(N-2)$;
(2) *for* $N = 2$, $0 < r \le \max\{1, r\} < \tilde r < \infty$;
(3) *for* $N = 1$, $0 < r \le \max\{1, r\} < \tilde r \le \infty$.

Proof. When $r \ge 1$, the inequality is well known. When $0 < r < 1$, using the Hölder inequality we can get it easily (see [25] and [19]). □

Inequality (2.29) is reduced to the following two inequalities:

Lemma 2.8 (see (4.12) and (4.13) in Kawanago [17]) *For any* $u \in C_0^\infty(\mathbf{R^N})$, *the following two inequalities hold:*

$$\int_{\mathbf{R^N}} u^{p+\ell-1}\,dx \le A_\ell \|u\|_{p_0}^{p-m} \|\nabla u^{(m+\ell-1)/2}\|_2^2 \tag{2.30}$$

where $A_\ell > 0$ *is a constant and* $\ell > \max\{0, p_0 - p + 1\}$;

$$\begin{aligned}\|u(t)\|_\ell \le\ & C\|u(t)\|_\beta^{\beta[N(m-1)+2\ell]/\ell[\beta(2-N)+N(m+\ell-1)]} \\ & \times \|\nabla u^{(m+\ell-1)/2}\|_2^{2N(\ell-\beta)/\ell[\beta(2-N)+N(m+\ell-1)]}\end{aligned} \tag{2.31}$$

where $C > 0$ *is a constant and* $0 < \beta \le \ell$.

Proof. If we put $f = u^{(m+\ell-1)/2}$, $r = N(p-m)/(m+\ell-1), \tilde{r} = 2(p+\ell-1)/(m+\ell-1)$ or $f = u^{(m+\ell-1)/2}$, $r = 2\beta/(m+\ell-1), \tilde{r} = 2\ell/(m+\ell-1)$ in (2.29), then (2.30) or (2.31) follows respectively. □

Lemma 2.9 *For any* $f \in C_0^\infty(\mathbf{R^N} \times [0,T])$ *and* $\ell > p-1$,

$$\left[\int_0^T \int_{\mathbf{R^N}} f^{\tilde{r}}\,dxdt\right]^{1/k} \le C\left[\sup_{t\in[0,T]} \int_{\mathbf{R^N}} f^r\,dx + \int_0^T \int_{\mathbf{R^N}} |\nabla f|_2^2\,dxdt\right] \tag{2.32}$$

where $r = 2(\ell-p+1)/(\ell+m-p)$, $\tilde{r} = 2(r/N+1)$, $k = 1+2/N$ *and* C *is some constant independent of* T.

Proof. For any $f \in C_0^\infty(\mathbf{R^N} \times [0,T])$, inequality (2.29) leads to

$$\begin{aligned}\int_0^T \|f\|_{\tilde{r}}^{\tilde{r}}\,dt \le\ & (\tilde{r}C)^{\tilde{r}} \int_0^T \|f\|_r^{(1-\theta)\tilde{r}} \|\nabla f\|_2^{\theta\tilde{r}}\,dt \\ \le\ & (\tilde{r}C)^{\tilde{r}} \sup_{[0,T]} \|f\|_r^{(1-\theta)\tilde{r}} \int_0^T \|\nabla f\|_2^{\theta\tilde{r}}\,dt.\end{aligned} \tag{2.33}$$

Let $s > 1$. By the Hölder inequality we get

$$\begin{aligned}&\left[\int_0^T \|f\|_{\tilde{r}}^{\tilde{r}}\,dt\right]^{1/s} \\ &\quad\le (\tilde{r}C)^{\tilde{r}/s}\left\{\frac{1}{s'}\sup_{[0,T]}\|f\|_r^{(1-\theta)\tilde{r}s'/s} + \frac{1}{s}\int_0^T \|\nabla f\|_2^{\theta\tilde{r}}\,dt\right\},\end{aligned} \tag{2.34}$$

where s' is the conjugate number of s.

Now, we choose numbers $r = 2(\ell-p+1)/(\ell-p+m) \in (0,2)$, $\tilde{r} =$

$2(1+r/N)$ and $s = k = 1 + 2/N$ in (2.34). Then, since r, $\tilde{r}$ and s satisfy the above conditions (1) (2) (3), and $\theta = 2/\tilde{r}$, we have

$$\left[\int_0^T \|f\|_{\tilde{r}}^{\tilde{r}}\, dt\right]^{1/k} \le (\tilde{r}C)^{\tilde{r}/k} \left\{\frac{1}{k'} \sup_{[0,T]} \|f\|_r^r + \frac{1}{k}\int_0^T \|\nabla f\|_2^2\, dt\right\}.$$

Thus, noting $\tilde{r} \le 2(1+2/N)$ we get (2.32). The proof is complete. □

Remark 2.10 When $r = 1$, similar result was obtained by D. Lortz-R. Meyer-Spasche- E. W. Stredunlinsky [22] in a bounded domain.

3. Blow-up cases I

In this section we prove Theorem 1. The methods of the proof are similar to those of Aguirre-Escobedo [1] and Mochizuki-Suzuki [24].

First of all, there is no loss of generality if we make a linear change of variables to transform equation (1.1) into

$$\partial_t u = \triangle u^m + a\frac{\partial}{\partial x_1}u^q + u^p \tag{3.1}$$

with $a \in \mathbf{R}/\{0\}$. As usual we let $x = (x_1, x')$ with $x' \in \mathbf{R}^{\mathbf{N-1}}$.

We fix a positive function $s(x) \in C^2(\mathbf{R}^{\mathbf{N}})$ with s, ∇s and $\triangle s \in L^1(\mathbf{R}^{\mathbf{N}})$ such that

$$s(0) = 1, \quad \triangle s(x) \ge -s(x) \quad \text{and} \quad \left|\frac{\partial}{\partial x_1}s(x)\right| \le Ks(x) \tag{3.2}$$

for some constant $K > 0$. Explicit examples were given in [1].

Put

$$s_\varepsilon(x) \equiv s(\varepsilon^{1+\gamma}x_1, \varepsilon x') \tag{3.3}$$

where $\gamma = 0$ if $q \ge m + 1/N$ and $\gamma = \{1-(q-m)N\}/(q-m+1)$ if $m - 1 < q < m + 1/N$. Let $u(x,t)$ be a weak solution of (3.1) (1.2) and set

$$J(t) = \int_{\mathbf{R^N}} u(x,t)s_\varepsilon(x)\, dx \Big/ \int_{\mathbf{R^N}} s_\varepsilon(x)\, dx \tag{3.4}$$

for each $t \ge 0$. Then we establish the following blow-up theorem.

Proposition 3.1 *Assume* $p \ge \max\{m, q\}$ (> 1). *If* u_0 *is large enough to satisfy*

$$J(0) > c_0\varepsilon^{k_0} \tag{3.5}$$

for small $\varepsilon \in (0,1)$ *where*

$$k_0 = \min\left\{\frac{2}{p-m}, \frac{1+\gamma}{p-q}\right\} \tag{3.6}$$

and

$$c_0 = \max\left\{2^{1/(p-m)}, (2|a|K)^{1/(p-q)}\right\},$$

then the corresponding weak solution u *of* (3.1) (1.2) *is not global in time. Here, we defined* $2/(p-m) = \infty$ *and* $2^{1/(p-m)} = 0$ *when* $p = m$*, and* $(1+\gamma)/(p-q) = \infty$ *and* $(2|a|K)^{1/(p-q)} = 0$ *when* $p = q$.

Proof. The methods of the proof are similar to those of the proof in Mochizuki-Suzuki [24] and Imai-Mochizuki [15].

Let $u(x,t)$ be a weak solution of (3.1) (1.2) and let $\varphi(x) \in C^2(\mathbf{R^N})$ satisfy

$$\int_{\mathbf{R^N}} \{|\varphi| + |\nabla\varphi| + |\triangle\varphi|\}\, dx < \infty.$$

Then, by a limit procedure we have from (2.1) with $a \cdot \nabla u^q$ replaced by $a\frac{\partial}{\partial x_1}u^q$,

$$\int_{\mathbf{R^N}} u(x,\tau)\varphi(x)\, dx - \int_{\mathbf{R^N}} u(x,0)\varphi(x)\, dx$$
$$= \int_0^\tau \int_{\mathbf{R^N}} \left\{u^m\triangle\varphi - u^q a\frac{\partial}{\partial x_1}\varphi + u^p\varphi\right\} dxdt.$$

We choose $\varphi(x) = s_\varepsilon(x)$. Then, since

$$\triangle s_\varepsilon(x) \geq -\varepsilon^2 s_\varepsilon(x) \quad \text{and} \quad \left|a\frac{\partial}{\partial x_1}s_\varepsilon(x)\right| \leq |a|K\varepsilon^{1+\gamma}s_\varepsilon(x)$$

for $0 < \varepsilon < 1$, we have

$$\int_{\mathbf{R^N}} u(x,\tau)s_\varepsilon(x)\, dx - \int_{\mathbf{R^N}} u(x,0)s_\varepsilon(x)\, dx$$
$$\geq \int_0^\tau \int_{\mathbf{R^N}} \left\{-\varepsilon^2 u^m - \varepsilon^{1+\gamma}|a|Ku^q + u^p\right\} s_\varepsilon(x)\, dxdt. \tag{3.7}$$

We define function $\Gamma_1(\xi)$ and $\Gamma_2(\xi)$ as follows: When $p > m$,

$$\Gamma_1(\xi) = \begin{cases} -\varepsilon^2\xi^m + \dfrac{1}{2}\xi^p & \text{for } \ \xi \geq \left(\dfrac{2m\varepsilon^2}{p}\right)^{1/(p-m)} \\ -\dfrac{p-m}{2m}\left(\dfrac{2m\varepsilon^2}{p}\right)^{p/(p-m)} & \text{for } \ 0 \leq \xi < \left(\dfrac{2m\varepsilon^2}{p}\right)^{1/(p-m)} \end{cases}$$

and when $p = m > 1$

$$\Gamma_1(\xi) = (1/2 - \varepsilon^2)\xi^p \quad \text{for } \ \xi \geq 0.$$

When $p > q$,

$$\Gamma_2(\xi) = \begin{cases} -\varepsilon^{1+\gamma}|a|K\xi^q + \dfrac{1}{2}\xi^p \ \text{ for } \ \xi \geq \left(\dfrac{2q\varepsilon^{1+\gamma}|a|K}{p}\right)^{1/(p-q)} \\ -\dfrac{p-q}{2q}\left(\dfrac{2q\varepsilon^{1+\gamma}|a|K}{p}\right)^{q/(p-q)} \\ \qquad\qquad\qquad\qquad \text{for } \ 0 \leq \xi < \left(\dfrac{2q\varepsilon^{1+\gamma}|a|K}{p}\right)^{1/(p-q)} \end{cases}$$

and when $p = q > 1$,

$$\Gamma_2(\xi) = (1/2 - \varepsilon^{1+\gamma}|a|K)\xi^p \quad \text{for } \xi \geq 0.$$

Then, if we set $\Gamma = \Gamma_1 + \Gamma_2$, we get

$$\begin{aligned} &\int_{\mathbf{R^N}} u(x,\tau)s_\varepsilon(x)\,dx - \int_{\mathbf{R^N}} u(x,0)s_\varepsilon(x)\,dx \\ &\qquad \geq \int_0^\tau \int_{\mathbf{R^N}} \Gamma(u)s_\varepsilon(x)\,dxdt. \end{aligned} \tag{3.8}$$

Since Γ_1 and Γ_2 are convex functions, Γ is also a convex function when $\varepsilon > 0$ is small enough. Since Γ_1 or Γ_2 is a positive increasing function in $\xi > (2\varepsilon^2)^{1/(p-m)}$ or $\xi > (2\varepsilon^{1+\gamma}|a|K)^{1/(p-q)}$ respectively, $\Gamma = \Gamma_1 + \Gamma_2$ is a positive increasing function in $\xi > m(\varepsilon) = \max\{(2\varepsilon^2)^{1/(p-m)}, (2\varepsilon^{1+\gamma}|a|K)^{1/(p-q)}\}$ for small $\varepsilon > 0$. We note that $c_0\varepsilon^{k_0} \geq m(\varepsilon)$ for small $\varepsilon > 0$. Hence, as in the proof of Theorem 1.1 of Imai-Mochizuki [15] (see also the proof of Lemma 5.2), if $J(0) > c_0\varepsilon^{k_0}$ then it follows from (3.8), (3.4) and Jensen's inequality that

$$J(\tau) \geq J(0) + \int_0^\tau \Gamma(J(t))\,dt,$$

from which we have

$$t \le \int_{J(0)}^{J(t)} \frac{d\xi}{\Gamma(\xi)} \le \int_{J(0)}^{\infty} \frac{d\xi}{\xi^p - \varepsilon^2\xi^m - \varepsilon^{1+\gamma}|a|K\xi^q} < \infty,$$

as long as $u(x,t)$ exists. This leads to contradiction if the solution is global. □

The next lemma follows from the above proposition immediately.

Lemma 3.2 *Assume $p \ge \max\{m,q\}$ (>1). Let $u(x,t)$ be a global weak solution of* (3.1) (1.2) *in time. Then, we have*

$$\int_{\mathbf{R}^N} u(x,t)s(\varepsilon^{1+\gamma}x_1, \varepsilon x')\,dx \le C(N)\varepsilon^{k_0-N-\gamma} \tag{3.9}$$

for any $t \ge 0$ and $1 \gg \varepsilon > 0$, where $C(N)$ is a positive constant depending only on N, and k_0 is as in Proposition 3.1.

Proof. Since

$$\int_{\mathbf{R}^N} s_\varepsilon\,dx = \int_{\mathbf{R}^N} s(\varepsilon^{1+\gamma}x_1, \varepsilon x')\,dx = \int_{\mathbf{R}^N} s(y)\,dy \times \varepsilon^{-N-\gamma},$$

the blow-up condition (3.5) is reduced to

$$\int_{\mathbf{R}^N} u_0(x)s(\varepsilon^{1+\gamma}x, \varepsilon x')\,dx > c_0\varepsilon^{k_0-N-\gamma}\int_{\mathbf{R}^N} s(y)\,dy.$$

Thus, if $C(N) = c_0\int_{\mathbf{R}^N} s(y)\,dy$, every global weak solution u of (1.1) (1.2) must satisfy the inverse inequality (3.9). □

Proof of Theorem 1. Let $u(x,t)$ be a global solution of (1.1) (1.2). As is mentioned-above, without of loss of generality we can assume that $u(x,t)$ is a global solution of (3.1) (1.2).

We note that if $m-1 < q \le p$ and $m \le p$ then

$$k_0 = \min\left\{\frac{2}{p-m}, \frac{1+\gamma}{p-q}\right\} > N+\gamma$$

is equivalent to

$$p < p^*_{m,q} = \min\left\{m + \frac{2}{N}, m + \frac{2(q-m+1)}{N+1}\right\}.$$

Therefore, when $\max\{m,q\} \le p < p^*_{m,q}$ namely $k_0 > N+\gamma$, letting $\varepsilon \downarrow 0$ in

(3.9), by Fatou's lemma we get

$$\int_{\mathbf{R^N}} u(x,t)\,dx = 0 \quad \text{for any} \quad t \geq 0, \tag{3.10}$$

which leads to $u(x,t) \equiv 0$. If $2q \geq m+1$ and $u_0(x) \in L^1(\mathbf{R^N})$ then solutions $u(x,t)$ of (3.1) (1.2) are unique by Proposition 2.2, and (1.7) holds for the maximum existence time T of u, since otherwise $u(x,t)$ exists beyond the time T by the local existence theorem for the solutions. The proof is complete. □

4. $L^\infty - L^\ell$ estimates for solutions I.

In this section we show the L^∞-estimates for the solutions of (1.1) (1.2) which play a very important role in the proof of Theorem 2 and Theorem 3. We shall show the following $L^\infty - L^\ell$ estimates:

Proposition 4.1 *Assume $p > m + 2/N$ and assume (2.2) for the initial data u_0. Let $u(x,t)$ be a weak solution of (1.1) (1.2) which is constructed in Proposition 2.4. Suppose that*

$$\sup_{[0,T]} \int_{\mathbf{R^N}} u^{p_0}(x,t)\,dx = h < \infty \tag{4.1}$$

and $\ell > \alpha \equiv p_0+p-1$ $(> p)$ where p_0 (> 1) is as in (1.4). Let $0 < \rho < \tau < T$ and $\varepsilon > 0$ satisfy

$$\rho\varepsilon^{p-1} \leq 1. \tag{4.2}$$

Then

$$\|u(\tau)\|_\infty \tag{4.3}$$
$$\leq C_1\Big[\rho^{-N/2}\varepsilon^{-(p-1)(N/2+1)}\Big\{\rho^{-1}\int_{\tau-\rho}^{T}\int_{\mathbf{R^N}} u^\ell\,dxdt + \varepsilon^{\ell-p_0}\Big\}\Big]^{1/(\ell-\alpha)}$$

where $C_1 = C_1(N, m, p, h, \ell)$.

Similar results were obtained by D. Lortz, R. Meyer-Spasche and E.W. Stredulinskly [22] for a linear equation in a bounded domain in virtue of Moser's iteration methods. Our methods of the proof are similar to their ones. However, their methods can not be applied to our quasilinear equation directly. We must develop them.

In the proof of Theorem 2 and Theorem 3, we use this proposition in the following versions.

Proposition 4.2 *Let $u(x,t)$ be as in Proposition 4.1. Suppose for some $\ell > \alpha$,*

$$\|u(t)\|_\ell \text{ is nonincreasing in } t \in [0,T]. \tag{4.4}$$

Then

$$\|u(\tau)\|_\infty \le C_2 \left\{ \frac{\tau}{2} \left(\int_{\mathbf{R^N}} u^\ell(\tau/2)\, dx + (\tau/2)^{-(\ell-p_0)/(p-1)} \right) \right\}^{1/(\ell-\alpha)} \tag{4.5}$$

for $0 < \tau \le T$, where $C_2 = C_2(N,m,p,h,\ell)$.

Proof. Put $\rho = \varepsilon^{-(p-1)} = \tau/2$ in Proposition 4.1. Then, since

$$\int_{\tau-\rho}^{\tau} \int_{\mathbf{R^N}} u^\ell \, dxdt = \int_{\tau/2}^{\tau} \int_{\mathbf{R^N}} u^\ell \, dxdt \le \frac{\tau}{2} \int_{\mathbf{R^N}} u^\ell(\tau/2)\, dx$$

by (4.4), (4.3) is reduced to (4.5). □

Proposition 4.3 *Let $u(x,t)$ be as in Proposition 4.2. Then*

$$\|u(\tau)\|_\infty \le C_3 \left\{ \int_{\mathbf{R^N}} u^\ell(\tau-1)\, dx \right\}^{\left\{1-\frac{p-1}{\ell-p_0}\left(\frac{N}{2}+1\right)\right\}\frac{1}{\ell-\alpha}} \tag{4.6}$$

for any $\tau \ge 2$, where $C_3 = C_3(N,m,p,h,\|u(1)\|_\ell,\ell)$.

Proof. Put $\rho = 1$ and $\varepsilon = \{\int_{\mathbf{R^N}} u^\ell(\tau-1)\, dx\}^{1/(\ell-p_0)} / \|u(1)\|_\ell^{\ell/(\ell-p_0)}$ (≤ 1) in Proposition 4.1. Since

$$\int_{\tau-1}^{\tau} \int_{\mathbf{R^N}} u^\ell \, dxdt \le \int_{\mathbf{R^N}} u^\ell(\tau-1)\, dx,$$

we obtain (4.6). □

We will prove Proposition 4.1 in a series of lemmas. We first show the following lemma.

Lemma 4.4 *Assume $p > m + 2/N$ and assume (2.2) for the initial data u_0. Let $u(x,t)$ be a weak solution of (1.1) (1.2) which is constructed in Proposition 2.4 and satisfies (4.1). Let $\ell > \alpha \equiv p_0 + p - 1$. Put $I = [\tau, \tau+s]$ and $I' = [\tau-\sigma, \tau+s]$ with $T > \tau > \sigma > 0$ and $T - \tau > s > 0$. Then, for* *any* $\varepsilon > 0$ *satisfying*

$$\sigma\varepsilon^{p-1} \le 1, \tag{4.7}$$

$$\left\{\int_I \int_{\mathbf{R^N}} u^{(\ell-p+1)k+m-1}\,dxdt + \frac{s+\sigma}{\varepsilon^{p_0}}\varepsilon^{(\ell-p+1)k+m-1}\right\}^{1/k} \le C_4(\ell+1)\frac{\varepsilon^{-p+1}}{\sigma}\left\{\int_{I'}\int_{\mathbf{R^N}} u^\ell\,dxdt + \frac{s+\sigma}{\varepsilon^{p_0}}\varepsilon^\ell\right\} \tag{4.8}$$

where $k = 1 + 2/N$ *and* $C_4 = C_4(N, m, p, h)$.

We need the following lemma to prove the above lemma.

Lemma 4.5 *Let* $u_n(x,t)$ *be a classical solution of* (2.4) (2.5) (2.6) *and let* $\varphi(t) \ge 0$ *in* $[t_1, t_2]$ *be a* C^1*-function with* $\varphi(t_1) = 0$*. Then, for* $\ell \ge \ell_0 > p$,

$$\varphi(t_2)\int_{B_n} u_n^{\ell-p+1}(t_2)\,dx + \nu_0\int_{t_1}^{t_2}\varphi(t)\int_{B_n}|\nabla u_n^{(\ell+m-p)/2}|^2\,dxdt \le \int_{t_1}^{t_2}\varphi'\int_{B_n}u_n^{\ell-p+1}\,dxdt + (\ell-p+1)\int_{t_1}^{t_2}\varphi(t)\int_{B_n}u_n^\ell\,dxdt \tag{4.9}$$

where

$$\nu_0 = \min\left\{\inf_{\ell\ge\ell_0}\frac{4m(\ell-p+1)(\ell-p)}{(\ell+m-p)^2}, 1\right\}.$$

Proof. Multiplying (2.24) with ℓ replaced by $\ell - p + 1$ by $\varphi(t)$ and integrating by part over (t_1, t_2), we obtain (4.9). □

Proof of Lemma 4.4 Let $u_n(x,t)$ be a classical solution of (2.4) (2.5) (2.6). Choose $\hat{t} \in I = [\tau, \tau + s]$ such that

$$\max_I \int_{B_n} u_n^{\ell-p+1}(t)\,dx = \int_{B_n} u_n^{\ell-p+1}(\hat{t})\,dx.$$

In the following, we shall choose suitable $\varphi(t)$, t_1 and t_2 in (4.9). First, put $t_1 = \tau - \sigma$, $t_2 = \hat{t}$ and

$$\varphi(t) = \frac{t-\tau+\sigma}{\hat{t}-\tau+\sigma}.$$

Then, since $0 \le \varphi'(t) = 1/(\hat{t} - \tau + \sigma) \le 1/\sigma$, $\varphi \le 1$ and $[t_1, t_2] \subset I'$ we have

$$\max_I \int_{B_n} u_n^{\ell-p+1}(t)\,dx$$
$$\le \frac{1}{\sigma}\int_{I'}\int_{B_n} u_n^{\ell-p+1}\,dxdt + (\ell - p + 1)\int_{I'}\int_{B_n} u_n^{\ell}\,dxdt. \tag{4.10}$$

Next, we put $t_1 = \tau - \sigma$, $t_2 = \tau + s$ and

$$\varphi = \begin{cases} 1 & t \in I = [\tau, \tau + s] \\ -\sigma^{-2}(t-\tau)^2 + 1 & \tau - \sigma \le t \le \tau. \end{cases}$$

Since $0 \le \varphi' \le 2/\sigma$, $\varphi \le 1$ and $[t_1, t_2] = I'$ we have

$$\nu_0 \int_I \int_{B_n} |\nabla u_n^{(\ell+m-p)/2}|^2\,dxdt$$
$$\le \frac{2}{\sigma}\int_{I'}\int_{B_n} u_n^{\ell-p+1}\,dxdt + (\ell - p + 1)\int_{I'}\int_{B_n} u_n^{\ell}\,dxdt. \tag{4.11}$$

Therefore, combining (4.10) and (4.11) we get

$$\max_I \int_{B_n} u_n^{\ell-p+1}(t)\,dx + \nu_0 \int_I \int_{B_n} |\nabla u_n^{(\ell+m-p)/2}|^2\,dxdt$$
$$\le \frac{3}{\sigma}\int_{I'}\int_{B_n} u_n^{\ell-p+1}\,dxdt + 2(\ell - p + 1)\int_{I'}\int_{B_n} u_n^{\ell}\,dxdt. \tag{4.12}$$

Hence, letting $n \to \infty$ in (4.12), by Proposition 2.4 we have

$$\max_I \int_{\mathbf{R^N}} u^{\ell-p+1}(t)\,dx + \nu_0 \int_I \int_{\mathbf{R^N}} |\nabla u^{(\ell+m-p)/2}|^2\,dxdt$$
$$\le \frac{3}{\sigma}\int_{I'}\int_{\mathbf{R^N}} u^{\ell-p+1}\,dxdt + 2(\ell - p + 1)\int_{I'}\int_{\mathbf{R^N}} u^{\ell}\,dxdt. \tag{4.13}$$

We estimate the first term of the right side of (4.13) from above in the following way: Set

$$G_\varepsilon(t) = \{x \in \mathbf{R^N} \mid u(x,t) \ge \varepsilon\}$$

for each $t > 0$ and $\varepsilon > 0$. It follows from (4.1) and $|I'| = s + \sigma$ that for each $\ell \ge p_0 + p - 1$

$$\int_{I'}\int_{\mathbf{R^N}} u^{\ell-p+1}\,dxdt = \left(\int_{I'}\int_{G_\varepsilon(t)} + \int_{I'}\int_{\mathbf{R^N}/G_\varepsilon(t)}\right) u^{\ell-p+1}$$
$$\le \varepsilon^{-p+1}\int_{I'}\int_{G_\varepsilon(t)} u^{\ell}\,dxdt + \varepsilon^{\ell-p_0-p+1}\int_{I'}\int_{\mathbf{R^N}/G_\varepsilon(t)} u^{p_0}\,dxdt.$$

$$\leq \varepsilon^{-p+1}\int_{I'}\int_{\mathbf{R^N}} u^\ell\,dxdt + (s+\sigma)h\varepsilon^{\ell-p_0-p+1}. \tag{4.14}$$

Further, estimating the left side of (4.13) from below by inequality (2.32) with $f = u^{(\ell+m-p)/2}$ we see that (4.13) is reduced to

$$\begin{aligned}
&\nu_0\Big\{\int_I\int_{\mathbf{R^N}} u^{(\ell+m-p)\tilde{r}/2}\,dxdt\Big\}^{1/k}\\
&\leq C\left\{\max_I \int_{\mathbf{R^N}} u^{\ell-p+1}(t)\,dx + \nu_0\int_I\int_{\mathbf{R^N}} |\nabla u^{(\ell+m-p)/2}|^2\,dxdt\right\}\\
&\leq C\Big\{(3\varepsilon^{-p+1}\sigma^{-1} + 2(\ell-p+1))\int_{I'}\int_{\mathbf{R^N}} u^\ell\,dxdt\\
&\qquad\qquad + \frac{3(\sigma+s)}{\sigma}h\varepsilon^{\ell-p_0-p+1}\Big\},
\end{aligned} \tag{4.15}$$

since $\nu_0 < 1$. Note

$$\frac{\ell+m-p}{2}\tilde{r} = k(\ell-p+1) + m - 1 \quad (\text{since } k = 1 + 2/N)$$

and add $C\frac{\sigma+s}{\sigma}h\varepsilon^{\ell-p_0-p+1}$ to the both sides of (4.15). Then we obtain

$$\begin{aligned}
&\nu_0\Big\{\int_I\int_{\mathbf{R^N}} u^{k(\ell-p+1)+m-1}\,dxdt\Big\}^{1/k} + C\frac{\sigma+s}{\sigma}h\varepsilon^{\ell-p_0-p+1}\\
&\leq C\Big\{(3\varepsilon^{-p+1}\sigma^{-1} + 2(\ell-p+1))\int_{I'}\int_{\mathbf{R^N}} u^\ell\,dxdt\\
&\qquad\qquad + \frac{4(\sigma+s)}{\sigma}h\varepsilon^{\ell-p_0-p+1}\Big\}\\
&= C\Big\{(3\varepsilon^{-p+1}\sigma^{-1} + 2(\ell-p+1))\int_{I'}\int_{\mathbf{R^N}} u^\ell\,dxdt\\
&\qquad\qquad + 4\varepsilon^{-p+1}\sigma^{-1}\frac{h(\sigma+s)\varepsilon^\ell}{\varepsilon^{p_0}}\Big\}\\
&\leq 4C(3\varepsilon^{-p+1}\sigma^{-1} + 2(\ell-p+1))\Big\{\int_{I'}\int_{\mathbf{R^N}} u^\ell\,dxdt + \frac{h(\sigma+s)\varepsilon^\ell}{\varepsilon^{p_0}}\Big\}.
\end{aligned} \tag{4.16}$$

Furthermore, we estimate the left side of this inequality from below by using inequality

$$\begin{aligned}
&\frac{\sigma+s}{\sigma}h\varepsilon^{\ell-p_0-p+1} \\
&= h^{1-1/k}\Big(\sigma\varepsilon^{p-1}\Big)^{-1/k}(\frac{\sigma+s}{\sigma})^{(k-1)/k}\left\{\frac{h(\sigma+s)}{\varepsilon^{p_0}}\varepsilon^{(\ell-p+1)k+m-1}\right\}^{1/k} \\
&\qquad\qquad \text{(Here, we used relation } (\ell-p+1)k+m-1 \\
&\qquad\qquad\qquad\qquad = (\ell-p_0-p+1)k+p_0+p-1) \\
&\geq h^{1-1/k}\left\{\frac{h(\sigma+s)}{\varepsilon^{p_0}}\varepsilon^{(\ell-p+1)k+m-1}\right\}^{1/k} \\
&\qquad\qquad \text{(for any } \varepsilon>0 \text{ satisfying } \sigma\varepsilon^{p-1}\leq 1);
\end{aligned}$$

and we estimate the right side of inequality (4.16) from above by using inequality

$$\begin{aligned}
3\varepsilon^{-p+1}\sigma^{-1}+2(\ell-p+1) &\leq \varepsilon^{-p+1}\sigma^{-1}(3+2\sigma\varepsilon^{p-1}(\ell-p+1)) \\
&\leq 5(\ell+1)\varepsilon^{-p+1}\sigma^{-1} \\
&\qquad \text{(for any } \varepsilon>0 \text{ satisfying } \sigma\varepsilon^{p-1}\leq 1).
\end{aligned}$$

Then, we have

$$\begin{aligned}
&\nu_0\left\{\int_I\int_{\mathbf{R^N}} u^{k(\ell-p+1)+m-1}\,dxdt\right\}^{1/k} \\
&\quad + Ch^{1-1/k}\left\{\frac{h(\sigma+s)}{\varepsilon^{p_0}}\varepsilon^{(\ell-p+1)k+m-1}\right\}^{1/k} \\
&\quad\quad \leq 20C(\ell+1)\varepsilon^{-p+1}\sigma^{-1}\left\{\int_{I'}\int_{\mathbf{R^N}} u^\ell\,dxdt+\frac{h(\sigma+s)\varepsilon^\ell}{\varepsilon^{p_0}}\right\}.
\end{aligned} \tag{4.17}$$

Thus, if we put

$$C_4=\frac{20C}{\min\{\nu_0, Ch^{1-1/k}\}}$$

and use the inequality $(a+b)^{1/k}\leq a^{1/k}+b^{1/k}$, we obtain (4.8). □

We can now use Moser's iteration methods.

Lemma 4.6 *Let u be as in Lemma 4.4. Suppose $\ell>\alpha=p_0+p-1$. Let*

$0 < \rho < \tau < T$, $0 < s < T - \tau$ and $\varepsilon > 0$ satisfy (4.2). *Then*

$$\sup_{[\tau,\tau+s]} \|u(t)\|_\infty \le C_5 \Big\{ \rho^{-N/2} \varepsilon^{-(p-1)(N/2+1)} \Big\{ \rho^{-1} \int_{\tau-\rho}^{\tau+s} \int_{\mathbf{R^N}} u^\ell \, dxdt + \frac{s + 2^{-1}\rho}{\rho} \varepsilon^{\ell - p_0} \Big\} \Big\}^{1/(\ell-\alpha)} \tag{4.18}$$

where $C_5 = C_5(N, m, p, h, \ell)$.

Proof. Let $\ell > \alpha = p_0 + p - 1$ and $\{\lambda_n\}$ be the sequence of real numbers satisfying the following inductive formula:

$$\begin{cases} \lambda_n = (\lambda_{n-1} - p + 1)k + m - 1 \\ \lambda_1 = \ell. \end{cases} \tag{4.19}$$

Then

$$\lambda_n = \alpha + (\ell - \alpha)k^{n-1} \tag{4.20}$$

and

$$\lambda_n > \ell \quad \text{and} \quad \lim_{n\to\infty} \lambda_n = \infty. \tag{4.21}$$

We put $\ell = \lambda_n$ in (4.8). Then, since $\lambda_n \le \ell k^{n-1}$, from (4.19) we have

$$\Big\{ \int_I \int_{\mathbf{R^N}} u^{\lambda_{n+1}} \, dxdt + \frac{s+\sigma}{\varepsilon^{p_0}} \varepsilon^{\lambda_{n+1}} \Big\}^{1/k} \le C_6 \frac{k^{n-1}\varepsilon^{-p+1}}{\sigma} \Big\{ \int_{I'} \int_{\mathbf{R^N}} u^{\lambda_n} \, dxdt + \frac{s+\sigma}{\varepsilon^{p_0}} \varepsilon^{\lambda_n} \Big\} \tag{4.22}$$

where $C_6 = 2\ell C_4$. Let $0 < \rho < \tau < T$ and $\varepsilon > 0$ satisfy (4.2). We further put in (4.22)

$$I = I_{n+1} \equiv [\tau - 2^{-n}\rho, \tau + s]$$

and

$$I' = I_n \equiv [\tau - 2^{-n+1}\rho, \tau + s].$$

Namely, we note

$$\sigma = (\tau - 2^{-n}\rho) - (\tau - 2^{-n+1}\rho) = 2^{-n}\rho \tag{4.23}$$

and so

$$\sigma\varepsilon^{p-1} \le \rho\varepsilon^{p-1} \le 1 \quad \text{for} \quad n \ge 1.$$

If we set

$$J_n = \int_{I_n} \int_{\mathbf{R^N}} u^{\lambda_n} \, dxdt + \frac{s + 2^{-n}\rho}{\varepsilon^{p_0}} \varepsilon^{\lambda_n}, \tag{4.24}$$

then from (4.22) and (4.23) we get

$$\{J_{n+1}\}^{1/k} \leq C_{7,n} J_n \tag{4.25}$$

where

$$C_{7,n} = C_6 \varepsilon^{-p+1} \rho^{-1} k^{n-1} 2^n.$$

We now use Moser's iteration methods: Iterating (4.25), we have

$$\{J_{n+1}\}^{1/k^n} \leq C_{8,n} J_1 \tag{4.26}$$

where

$$\begin{aligned} C_{8,n} &= \prod_{i=1}^{n} C_{7,i}^{k^{-(i-1)}} \\ &= \{C_6 \varepsilon^{-p+1} \rho^{-1} 2\}^{\Sigma_{i=1}^n k^{-(i-1)}} \times (2k)^{\Sigma_{i=1}^n (i-1) k^{-(i-1)}} \end{aligned}$$

and

$$\lim_{n \to \infty} C_{8,n} = \{2 C_6 \varepsilon^{-p+1} \rho^{-1}\}^{N/2+1} (2k)^{k/(k-1)^2}.$$

Since

$$\begin{aligned} \sup_{[\tau, \tau+s]} \|u(t)\|_\infty^{\ell-\alpha} &= \lim_{n \to \infty} \left\{ \int_\tau^{\tau+s} \int_{\mathbf{R^N}} u^{\lambda_{n+1}} \, dxdt \right\}^{1/k^n} \\ &\leq \lim_{n \to \infty} \inf J_{n+1}^{1/k^n}, \end{aligned}$$

if $n \to \infty$ in (4.26) then we have

$$\sup_{[\tau, \tau+s]} \|u(t)\|_\infty^{\ell-\alpha} \leq \{2 C_6 \varepsilon^{-p+1} \rho^{-1}\}^{N/2+1} (2k)^{k/(k-1)^2} J_1$$

and so we get (4.18). □

Proof of Proposition 4.1 If $s \downarrow 0$ in (4.18), then by Proposition 2.4 we get (4.3). The proof is complete. □

5. Proof of Theorem 2

In this section, by using Proposition 4.2, we prove Theorem 2. For this aim, we need the following key lemma which was established by Kawanago [17] for a classical solution. But, we must prove it for a weak solution directly.

Lemma 5.1 *Assume* $p > m + 2/N$ *and assume* (2.2) *for the initial data* u_0. *Let* $u(x,t)$ *be a weak solution of* (1.1) (1.2) *which is constructed in Proposition* 2.4. *Put*

$$B_\ell = \left[4m(\ell-1)/A_\ell(m+\ell-1)^2\right]^{1/(p-m)} \tag{5.1}$$

where A_ℓ *is in Lemma* 2.8. *Then, for any* $\infty > \ell > \max\{1, p_0 - p + 1\}$ *with* $p_0 = N(p-m)/2$, *if* $\|u_0\|_{p_0} < \min\{B_{p_0}, B_\ell\}$ *then*

$$\|u(t)\|_\ell \,(\le \|u_0\|_\ell) \text{ is nonincreasing in } t \ge 0. \tag{5.2}$$

Furthermore, if

$$A \equiv \sup_{t\in[0,T]} \|u(t)\|_\beta < \infty \quad \text{for some constant } \beta \in [1,\ell), \tag{5.3}$$

then

$$\|u(t)\|_\ell^\ell \le C_9 t^{-N(\ell-\beta)/\{N(m-1)+2\beta\}} \quad \text{for } t \in [0,T] \tag{5.4}$$

where $C_9 = C_9(N, m, p, A, \beta, \ell)$.

The methods of the proof are similar to those of the proof in Kawanago [17] and so we only state the outline of the proof. But, in order to show the above lemma for a weak solution of (1.1) (1.2) directly, we need the next lemma.

Lemma 5.2 *Let* $q > 1$ *and let* $h(t) \ge 0$ *in* $[0,T]$ *be a continuous function satisfying*

$$h(s) + C\int_\tau^s h(t)^q\,dt \le h(\tau) \quad \text{for } 0 \le \tau \le s \le T \tag{5.5}$$

where $C > 0$ *is a constant. Then,*

$$h(s) \le \{C(q-1)s\}^{-1/(q-1)} \quad \text{for } 0 \le s \le T. \tag{5.6}$$

Proof. Let $0 < s \leq T$ be fixed. For $0 < \tau < s$, we define

$$\alpha(\tau) \equiv h(s) + C \int_\tau^s h(t)^q \, dt. \tag{5.7}$$

Suppose $h(s) > 0$. Then, since $0 < \alpha(\tau) \leq h(\tau)$ by (5.5), we get

$$1 \leq \frac{h(\tau)^q}{\alpha(\tau)^q}. \tag{5.8}$$

Integrating the both sides over $(0, s)$ and noting $d\alpha(\tau) = -Ch(\tau)^q d\tau = d\xi$, we have

$$\begin{aligned} s &= \int_0^s \frac{h(\tau)^q}{\alpha(\tau)^q} \, d\tau \leq \int_{\alpha(0)}^{h(s)} \frac{1}{\xi^q} \times \left(-\frac{1}{C}\right) d\xi \\ &\leq \frac{1}{C} \int_{h(s)}^\infty \frac{1}{\xi^q} \, d\xi = \frac{1}{C(q-1)} h(s)^{-q+1}, \end{aligned}$$

that is,

$$h(s) \leq [C(q-1)s]^{-1/(q-1)}.$$

When $h(s) = 0$, (5.6) is obvious. The proof is complete. □

Proof of Lemma 5.1 Let $u(x, t)$ be a weak solution of (1.1) (1.2) which is constructed in Proposition 2.4. We use Lemma 2.5. Integrating (2.24) over $[\tau, s]$ and letting $n \to \infty$, we have for $\ell \geq 1$,

$$\begin{aligned} &\int_{\mathbf{R^N}} u^\ell \, dx \Big|_\tau^s + \frac{4m\ell(\ell-1)}{(m+\ell-1)^2} \int_\tau^s \|\nabla u^{(m+\ell-1)/2}\|_2^2 \, dt \\ &\qquad \leq \ell \int_\tau^s \int_{\mathbf{R^N}} u^{p+\ell-1} \, dx ds. \end{aligned} \tag{5.9}$$

By (2.30), we obtain

$$\begin{aligned} &\int_{\mathbf{R^N}} u^\ell \, dx \Big|_\tau^s \\ &\quad + \int_\tau^s \left(\frac{4m\ell(\ell-1)}{(m+\ell-1)^2} - \ell A_\ell \|u\|_{p_0}^{p-m} \right) \|\nabla u^{(m+\ell-1)/2}\|_2^2 \, dt \leq 0. \end{aligned} \tag{5.10}$$

Put $\ell = p_0 \, (> 1)$ in the above inequality. Since $\|u(t)\|_{p_0}$ is continuous in $[0, T]$, if $\|u_0\|_{p_0} < B_{p_0}$ then $\|u(t)\|_{p_0}$ is nonincreasing in $t \geq 0$. Therefore, if $\|u_0\|_{p_0} < \min\{B_{p_0}, B_\ell\}$ and $\ell > \max\{1, p_0 - p + 1\}$, then

$$\int_{\mathbf{R^N}} u^\ell \, dx \Big|_\tau^s + C \int_\tau^s \|\nabla u^{(m+\ell-1)/2}\|_2^2 \, dt \leq 0 \tag{5.11}$$

for some $C > 0$, and so we see that $\|u(t)\|_\ell$ is nonincreasing in $t \geq 0$.

Assume (5.3). Then, it follows from (2.31) and (5.11) that

$$\|u(s)\|_\ell^\ell + C\int_\tau^s \left\{\|u(t)\|_\ell^\ell\right\}^{1+\frac{N(m-1)+2\beta}{N(\ell-\beta)}} dt \\ \leq \|u(\tau)\|_\ell^\ell \quad \text{for } 0 \leq \tau \leq s \leq T. \tag{5.12}$$

Applying Lemma 5.2 to (5.12) we get

$$\|u(s)\|_\ell^\ell \leq \left\{C \times \frac{N(m-1)+2\beta}{N(\ell-\beta)} s\right\}^{-N(\ell-\beta)/\{N(m-1)+2\beta\}}.$$

The proof is complete. □

Proposition 5.3 *Let $u(x,t)$ be as in Lemma 5.1. Then, if $\|u_0\|_{p_0} < \min\{B_{p_0}, B_\ell\} \equiv \delta_0$ for some $\ell > \max\{1, p_0+p-1\}$,*

$$\|u(t)\|_\infty \leq K_1 t^{-1/(p-1)} \quad \textit{for } t \in [0,T] \tag{5.13}$$

where $K_1 = K_1(N, m, p, \delta_0)$.

Proof. Let $u(x,t)$ be as in Lemma 5.1. Assume $\|u_0\|_{p_0} < B_{p_0}$. Then, by Lemma 5.1 we have

$$\|u(t)\|_{p_0} \leq \|u_0\|_{p_0} \quad \text{for } t \in [0,T]. \tag{5.14}$$

It follows from Proposition 4.2 with $h = \|u_0\|_{p_0}^{p_0}$ and Lemma 5.1 with $\beta = p_0$ that if $\|u_0\|_{p_0} < \delta_0 = \min\{B_{p_0}, B_\ell\}$ for some $\ell > p_0+p-1$ then

$$\begin{aligned}\|u(t)\|_\infty &\leq C_2\left\{\left(\frac{t}{2}\int_{\mathbf{R^N}} u^\ell(t/2)\,dx + (t/2)^{-(\ell-p_0)/(p-1)}\right)\right\}^{1/(\ell-\alpha)} \\ &\leq C_2\left\{\frac{t}{2}\left(C_7\left(\frac{t}{2}\right)^{\frac{-N(\ell-p_0)}{N(m-1)+2p_0}} + \left(\frac{t}{2}\right)^{-\frac{\ell-p_0}{p-1}}\right)\right\}^{1/(\ell-\alpha)} \\ &\leq K_1 t^{-1/(p-1)} \quad \text{for } t \in [0,T].\end{aligned} \tag{5.15}$$

Here we used the relation

$$\frac{N}{N(m-1)+2p_0} = \frac{1}{p-1} \quad \text{and} \quad \alpha = p_0+p-1.$$

□

Proof of Theorem 2. We show Theorem 2 in the case that u_0 satisfies

(2.2). In the general case, using Proposition 2.6 with adding the assumption that $u_{0,n} \to u_0$ in L^{p_0} we can also show Theorem 2 (See the proof of Theorem 1.2 in Kawanago [17]).

Assume (2.2) and $\|u_0\|_{p_0} < \delta_0$ where δ_0 is as in Proposition 5.3. Let $u_n(x,t)$ be a classical solution of (2.4) (2.5) (2.6). Put $T_1 = T(\|u_0\|_\infty)/2$ and

$$M_1 = \max\{y(T_1; \|u_0\|_\infty),\, K_1 T_1^{-1/(p-1)} + 1\} \tag{5.16}$$

where $y(t;M)$ and $T(M)$ are defined by (2.7) and (2.8) respectively.

We shall show that for any subsequence $\{n'\} \subset \{n\}$ and any $T > 0$, there exists a subsequence $\{n''\} \subset \{n'\}$ such that

$$\sup_{[0,T]} \|u_{n''}(t)\| \le M_1 \quad \text{for all } \ n''. \tag{5.17}$$

Let $\{n'\} \subset \{n\}$ be fixed arbitrarily and set

$$\begin{aligned} T' = \sup\{T\,;\ & \text{there exists a subsequence } \{n''\} \subset \{n'\} \text{ such that} \\ & \sup_{[0,T]} \|u_{n''}(t)\|_\infty \le M_1 \quad \text{for all } \ n''\}. \end{aligned} \tag{5.18}$$

Then, by Proposition 2.3 and the fact $\|u_0\|_\infty = \|u_{0,n}\|_\infty$ for large n, we have

$$T' \ge T_1.$$

Suppose $T' < \infty$. Then for any $T < T'$ satisfying $T' - T < T(M_1)/2$, there exists a subsequence $\{n''\} \subset \{n'\}$ such that

$$\sup_{[0,T]} \|u_{n''}(t)\|_\infty \le M_1 \quad \text{for all } \ n''. \tag{5.19}$$

Hence, from Proposition 2.3, we see that $u_{n''}(t)$ exists in $[0, T_2]$ with $T_2 = T + T(M_1)/2$ beyond T' and satisfies

$$\sup_{[0,T_2]} \|u_{n''}(t)\|_\infty \le y(T(M_1)/2; M_1) \quad \text{for all } \ n''. \tag{5.20}$$

It follows from Proposition 2.4 and 5.3 that for some subsequence $\{\tilde{n}\} \subset \{n''\}$,

$$u_{\tilde{n}}(x,t) \to u(x,t) \quad \text{as } \ \tilde{n} \to \infty \tag{5.21}$$

uniformly in $\mathbf{R^N} \times [0, T_2]$ and $u(x,t)$ is a weak solution of (1.1) (1.2) satisfying (5.13). Therefore, if $\tilde{n}$ is large enough, then

$$\|u_{\tilde{n}}(t)\|_\infty \le K_1 t^{-1/(p-1)} + 1 \le K_1 T_1^{-1/(p-1)} + 1 \le M_1 \quad \text{for } t \in [T_1, T_2]. \tag{5.22}$$

This is a contradiction to the definition of T' and thus we get $T = \infty$ namely (5.17) for any $T > 0$.

Hence, using the diagonal methods, we can choose a subsequence $\{n''\} \subset \{n'\}$ satisfying that for any $T > 0$,

$$\sup_{[0,T]} \|u_{n''}(t)\|_\infty \le M_1 \quad \text{for large } n''.$$

Therefore, by Proposition 2.4 and 5.3 we see that there exists a weak solution $u(x,t)$ of (1.1) (1.2) in $\mathbf{R}^\mathbf{N} \times [0,\infty)$ satisfying (1.8). The proof is complete. □

The rest of this section, we prepare for the proof of Theorem 3 which treats the case $u_0(x) \in L^1$. The next lemma is proved by Kawanago [17] for a classical solution of (1.1) (1.2) with $a = 0$.

Lemma 5.4 *Assume $p > m + 2/N$ and (2.2). Let $u(x,t)$ be a global weak solution of (1.1) (1.2) which is constructed in Proposition 2.4 and the proof of Theorem 2. Then, there exists a constant $\delta_2 = \delta_2(N,p,m)$ such that if $\|u_0\|_{p_0} < \delta_2$,*

$$\int_2^\infty \|u(t)\|_\infty^{p-1}\, dt \le C_{10} < \infty \tag{5.23}$$

and

$$\sup_{t \ge 0} \|u(t)\|_1 \le K_3 \tag{5.24}$$

where $C_{10} = C_{10}(N,p,m,\|u_0\|_1,\delta_2)$ and $K_3 = K_3(N,p,m,\|u_0\|_1,\delta_2)$.

The methods of the proof are similar to those of the proof in Kawanago [17] if we use Proposition 4.3. However, we must treat the weak solutions directly and so we need the next lemma:

Lemma 5.5 *Let $g(t) \ge 0$ in $[0,T]$ and $h(t) \ge 0$ in $[0,T]$ be continuous functions. Then, if u satisfies that for some $0 < q < 1$*

$$h(\tau) \le h(0) + \int_0^T g(t) h^q(t)\, dt \quad \text{for any } \ 0 \le \tau \le T, \tag{5.25}$$

$$\frac{1}{1-q}h(\tau)^{1-q} \le \int_0^\tau g(t)\,dt + \frac{1}{1-q}h(0)^{1-q} \quad \text{for } 0 \le \tau \le T. \tag{5.26}$$

Proof. Put

$$\alpha(t) = h(0) + \int_0^t g(t)h(t)^q\,dt.$$

Suppose that there exists $t_0 \in [0, T]$ such that

$$\alpha(t) > 0 \quad \text{for} \quad t \in (t_0, T] \quad \text{and} \quad \alpha(t_0) = 0.$$

By (5.25) we get

$$h(t) \le \alpha(t)$$

and hence we have

$$\frac{1}{\alpha^q(t)}g(t)h^q(t) \le g(t).$$

Integrating the both sides over (t_0, τ) and noting $d\alpha(t) = g(t)h(t)^q\,dt = d\xi$, we have

$$\int_{\alpha(t_0)}^{\alpha(\tau)} \frac{1}{\xi^q}\,d\xi = \int_{t_0}^\tau \frac{1}{\alpha^q(t)}g(t)h^q(t)\,dt \le \int_{t_0}^\tau g(t)\,dt.$$

Therefore, noting $\alpha(t_0) = 0$ we obtain for $\tau \in (t_0, T]$

$$\begin{aligned}\frac{1}{1-q}h(\tau)^{1-q} &\le \frac{1}{1-q}\alpha(\tau)^{1-q} \le \int_{t_0}^\tau g(t)\,dt \\ &\le \int_0^\tau g(t)\,dt + \frac{1}{1-q}h(0)^{1-q}\end{aligned}$$

and

$$h(\tau) \le \alpha(\tau) \le \alpha(t_0) = 0 \quad \text{for} \quad \tau \in [0, t_0].$$

Thus we obtain (5.26).

Similarly, we also get (5.26) in case that $\alpha(0) = h(0) > 0$. The proof is complete. □

Proof of Lemma 5.4 (see Kawanago [17]) Assume (2.2) and let $u(x,t)$ be a global weak solution of (1.1) (1.2) which is constructed in Proposition 2.4 and the proof of Theorem 2.

First, we prove (5.23). We choose a small $\eta > 0$ such that $p_0 - \eta > \max\{1, p_0 - p + 1\}$. We use (5.2) with $\ell = p_0 - \eta$. Then, if $\|u_0\|_{p_0} < \min\{B_{p_0}, B_{p_0-\eta}\}$,

$$\sup_{t\geq 0} \|u(t)\|_{p_0-\eta} \leq \|u_0\|_{p_0-\eta} \leq \max\{\|u_0\|_1, \|u_0\|_{p_0}\}. \tag{5.27}$$

Further, it follows from Proposition 4.3 and (5.4) with $\beta = p_0 - \eta$ that if $\|u_0\|_{p_0} < B_\ell$ for some $\ell > p_0 - \eta$ then

$$\begin{aligned}\|u(t)\|_\infty &\leq C_3 \left\{\int_{\mathbf{R^N}} u^\ell(t-1)\,dx\right\}^{\{1-\frac{p-1}{\ell-p_0}(\frac{N}{2}+1)\}\frac{1}{\ell-\alpha}} \\ &= C(t-1)^{-\ell'} \quad \text{for } \; t \geq 2,\end{aligned} \tag{5.28}$$

where

$$\ell' = \frac{N(1-(p_0-\eta)/\ell)}{N(m-1)+2(p_0-\eta)} \left\{1 - \frac{p-1}{\ell-p_0}(N/2+1)\right\} \frac{\ell}{\ell-\alpha}. \tag{5.29}$$

Let $\eta_1 \in (0, \eta)$. Then, we can choose large ℓ such that

$$\ell' > \frac{N}{N(m-1)+2(p_0-\eta_1)}.$$

Hence, we have from (5.28),

$$\|u(t)\|_\infty \leq C(t-1)^{-N/\{N(m-1)+2(p_0-\eta_1)\}} \quad \text{for } \; t \geq 2, \tag{5.30}$$

in order to obtain

$$\int_2^\infty \|u(t)\|_\infty^{p-1}\,dt \leq C^{p-1}\frac{N(m-1)+2(p_0-\eta_1)}{2\eta_1} < \infty. \tag{5.31}$$

Next, we show (5.24). It follows from (5.9) with $\ell = 1$ that

$$\begin{aligned}\int_{\mathbf{R^N}} u(t)\,dx &= \int_{\mathbf{R^N}} u(2)\,dx + \int_2^t \int_{\mathbf{R^N}} u^p\,dxdt \\ &\leq \int_{\mathbf{R^N}} u(2)\,dx + \int_2^t \|u(t)\|_\infty^{p-1} \int_{\mathbf{R^N}} u\,dxdt \quad \text{for } \; t \geq 2\end{aligned}$$

which leads to

$$\int_{\mathbf{R^N}} u(t)\,dx \leq \int_{\mathbf{R^N}} u(2)\,dx \exp\left(\int_2^\infty \|u(t)\|_\infty^{p-1}\,dt\right) < \infty \quad \text{for } \; t \geq 2. \tag{5.32}$$

On the other hand, we use (5.4) with $\ell = 2Np$ and $\beta = p_0$. Then, if $\|u_0\|_{p_0} < \min\{B_{p_0}, B_{2Np}\}$, we have by the Hölder inequality,

$$\begin{aligned}\int_{\mathbf{R^N}} u\,dx\Big|_0^t &= \int_0^t \int_{\mathbf{R^N}} u^p\,dxdt \\ &\le \int_0^t \|u\|_{2Np}^{2Np(p-1)/(2Np-1)} \|u\|_1^{p(2N-1)/(2Np-1)}\,dt \\ &\le C\int_0^t t^{-\frac{(3p+m)N}{2(2Np-1)}} \|u\|_1^{p(2N-1)/(2Np-1)}\,dt. \end{aligned} \tag{5.33}$$

It follows from Lemma 5.5 with $h(t) = \|u(t)\|_1$ and $q = p(2N-1)/(2Np-1)\,(<1)$ that

$$\begin{aligned}&\frac{2Np-1}{p-1}\|u(t)\|_1^{(p-1)/(2Np-1)} \\ &\le C\int_0^t t^{\frac{-(3p+m)N}{2(2Np-1)}}\,dt + \frac{2Np-1}{p-1}\|u_0\|_1^{(p-1)/(2Np-1)} \\ &= C\frac{2Np-1}{p_0-1}t^{(p_0-1)/(2Np-1)} + \frac{2Np-1}{p-1}\|u_0\|_1^{(p-1)/(2Np-1)}. \end{aligned} \tag{5.34}$$

Combining this and (5.32) we get (5.24). Thus, if set $\delta_2 = \min\{B_{p_0}, B_{p_0-\eta}, B_\ell, B_{2Np}\}$, then we obtain Lemma 5.4. □

Let $u_n(x,t)$ be a approximate solution of (2.4) (2.5) (2.6) for above $u(x,t)$ in Lemma 5.4. Then, if we put

$$v_n(x,t) = \exp\left(-\int_2^t \|u_n(t)\|_\infty^{p-1}\,dt\right) \times u_n, \tag{5.35}$$

$v_n(x,t)$ satisfies the following differential inequality:

$$\begin{aligned} v_{n.t} \le &\exp\left((m-1)\int_2^t \|u_n\|_\infty^{p-1}\,dt\right) \times \triangle v_n^m \\ &+ \exp\left((q-1)\int_2^t \|u_n\|_\infty^{p-1}\,dt\right) \times a\cdot\nabla v_n^q. \end{aligned} \tag{5.36}$$

As prove (2.24) and (4.9), we have for any $\ell \ge m$,

$$\begin{aligned}&\int_{B_n} v_n^{\ell-m+1}(s)\,dx + \nu_\ell \int_\tau^s \int_{B_n} |\nabla v_n^{\ell/2}|^2\,dxdt \\ &\le \int_{B_n} v_n^{\ell-m+1}(\tau)\,dx \quad \text{for } 2 \le \tau \le s \end{aligned} \tag{5.37}$$

where

$$\nu_\ell = \frac{4m(\ell - m + 1)(\ell - m)}{\ell^2}.$$

If $n \to \infty$, we get the following lemma.

Lemma 5.6 *Let $u(x,t)$ be as in Lemma 5.4 with $\|u_0\|_{p_0} < \delta_2$. Put*

$$v = \exp\left(-\int_2^t \|u(t)\|_\infty^{p-1}\, dt\right) \times u. \tag{5.38}$$

Then $v(x,t)$ satisfies that

$$\sup_{t \ge 2} \|v(t)\|_1 \le K_3 \tag{5.39}$$

where K_3 is as in Lemma 5.4, and for any $\ell \ge \ell_0 \ge m$

$$\begin{aligned} \int_{\mathbf{R^N}} v^{\ell-m+1}(s)\, dx + \nu_0 \int_\tau^s \int_{\mathbf{R^N}} |\nabla v^{\ell/2}|^2\, dxdt \\ \le \int_{\mathbf{R^N}} v^{\ell-m+1}(\tau)\, dx \quad \text{for } \ 2 \le \tau \le s \end{aligned} \tag{5.40}$$

where

$$\nu_0 = \inf_{\ell \ge \ell_0} \nu_\ell.$$

Hence, we see that

$$\|v(t)\|_\ell \quad (1 \le \ell \le \infty) \ \ \text{is nonincreasing in} \ \ t \ge 2. \tag{5.41}$$

Proof. (5.39) and (5.40) are obvious by Lemma 5.4 and (5.37). (5.40) is reduced to (5.41) when $1 \le \ell < \infty$. Noting (2.14) we can also show (5.41) in case $\ell = \infty$. □

In the following section, we shall estimate this $v(x,t)$.

6. $L^\infty - L^\ell$ Estimates for solutions II

In this section, we shall show the following L^∞-estimates for $v(x,t)$ which is very important to study the asymptotic behavior of the solution $u(x,t)$ in case $u_0(x) \in L^1(\mathbf{R^N})$.

Proposition 6.1 *Let $v(x,t)$ in $\mathbf{R^N} \times [0,\infty)$ be a nonnegative continuous function satisfying* (5.39), (5.40), (5.41) *and* (2.14) *with u replaced by v.*

Let $\ell > m$. Then, for any $t \geq 4$ and $\varepsilon > 0$ satisfying

$$t^{1/2}\varepsilon^{1/N+(m-1)/2} \leq 1, \tag{6.1}$$

$$\|v(t)\|_\infty^\ell \leq C_{11}(t\varepsilon^{m-1})^{-N/2}\left(\int_{\mathbf{R^N}} v^\ell(t/2)\,dx + K_3\varepsilon^{\ell-1}\right) \tag{6.2}$$

where $C_{11} = C_{11}(N, m, K_3, \ell)$.

The methods of the proof are similar to those of the proof of Proposition 4.1. That is, we use Moser's iteration methods (Ref. Kawanago [18]). First we show the next lemma.

Lemma 6.2 *Let $v(x,t)$ in $\mathbf{R^N} \times [0,\infty)$ be a nonnegative continuous function satisfying* (5.39), (5.40), (5.41) *and* (2.14) *with u replaced by v. Let $\ell \geq \ell_0$ for some $\ell_0 > m$. Then, when $N = 1$,*

$$C(\ell_0)(s-\tau)^{1/2}\varepsilon^{(m-1)/2}\|v(s)\|_\infty^\ell \leq \int_{\mathbf{R^N}} v^\ell(\tau)\,dx + \frac{K_3}{2}\varepsilon^{\ell-1} \tag{6.3}$$

and, when $N > 1$, for any $2 \leq \tau \leq s \leq t$ and $\varepsilon > 0$ satisfying (6.1),

$$\begin{aligned} C(\ell_0)(s-\tau)^{1/2}\varepsilon^{(m-1)/2}&\left\{\int_{\mathbf{R^N}} v^{\tilde{k}\ell}(s)\,dx + \frac{K_3}{\varepsilon}\varepsilon^{\tilde{k}\ell}\right\}^{1/\tilde{k}} \\ &\leq \int_{\mathbf{R^N}} v^\ell(\tau)\,dx + \frac{K_3}{\varepsilon}\varepsilon^\ell \end{aligned} \tag{6.4}$$

where $\tilde{k} = \frac{N}{N-1}$ and $C(\ell_0) = C(N, m, K_3, \ell_0)$.

Proof. We use the Gagliardo-Nirenberg inequality (see (2.29))

$$\|f\|_{2N/(N-1)} \leq C\|\nabla f\|_2^{1/2}\|f\|_2^{1/2} \quad \text{for } f \in C_0^\infty(\mathbf{R^N}) \tag{6.5}$$

where $2N/(N-1) = \infty$ when $N = 1$. Let $\ell \geq \ell_0 > m$ and $t \geq s \geq \tau \geq 2$. Combining this inequality with $f = v^{\ell/2}$ and (5.40) and noting (5.41), we have

$$\begin{aligned} &\frac{\nu_0}{C^4}(s-\tau)\frac{\|v^{\ell/2}(s)\|_{2N/(N-1)}^4}{\|v^{\ell/2}(\tau)\|_2^2} \\ &\quad\leq \frac{\nu_0}{C^4}\int_\tau^s \frac{\|v^{\ell/2}\|_{2N/(N-1)}^4}{\|v^{\ell/2}\|_2^2}\,dt \leq \nu_0\int_\tau^s\int_{\mathbf{R^N}} |\nabla v^{\ell/2}|^2\,dxdt \\ &\qquad\leq \int_{\mathbf{R^N}} v^{\ell-m+1}(\tau)\,dx. \end{aligned} \tag{6.6}$$

Hence, we obtain

$$(s-\tau)\|v^{\ell/2}(s)\|_{2N/(N-1)}^4 \le \frac{C^4}{\nu_0}\|v^{\ell/2}(\tau)\|_2^2 \int_{\mathbf{R^N}} v^{\ell-m+1}(\tau)\,dx,$$

that is,

$$\begin{aligned} &C(s-\tau)^{1/2}\|v^{\ell/2}(s)\|_{2N/(N-1)}^2 \\ &\quad\le \left\{\int_{\mathbf{R^N}} v^{\ell}(\tau)\,dx\right\}^{1/2}\left\{\int_{\mathbf{R^N}} v^{\ell-m+1}(\tau)\,dx\right\}^{1/2} \end{aligned} \tag{6.7}$$

for some constant $C>0$.

Similarly to the proof of (4.14), we get by (5.39),

$$\begin{aligned} \int_{\mathbf{R^N}} v^{\ell-m+1}(\tau)\,dx &\le \varepsilon^{\ell-m}\int_{\mathbf{R^N}} v(\tau)\,dx + \varepsilon^{-(m-1)}\int_{\mathbf{R^N}} v^{\ell}(\tau)\,dx \\ &\le \varepsilon^{-(m-1)}\left\{\varepsilon^{\ell-1}K_3 + \int_{\mathbf{R^N}} v^{\ell}(\tau)\,dx\right\}. \end{aligned} \tag{6.8}$$

Therefore, it follows from (6.7) and the Schwarz inequality that

$$\begin{aligned} &C(s-\tau)^{1/2}\varepsilon^{(m-1)/2}\|v^{\ell/2}(s)\|_{2N/(N-1)}^2 \\ &\quad\le \left\{\int_{\mathbf{R^N}} v^{\ell}(\tau)\,dx\right\}^{1/2}\left\{\varepsilon^{\ell-1}K_3 + \int_{\mathbf{R^N}} v^{\ell}(\tau)\,dx\right\}^{1/2} \\ &\quad\le \int_{\mathbf{R^N}} v^{\ell}(\tau)\,dx + \frac{K_3\varepsilon^{\ell-1}}{2} \end{aligned} \tag{6.9}$$

which is equal to (6.3) when $N=1$. When $N>1$, in order to obtain (6.4) we must add $K_3\varepsilon^{\ell-1}/2$ to the both sides of (6.9). Then, putting $\tilde{k}=\frac{N}{N-1}$ we have

$$\begin{aligned} &C(s-\tau)^{1/2}\varepsilon^{\frac{(m-1)}{2}}\left\{\left(\int_{\mathbf{R^N}} v^{\tilde{k}\ell}(s)\,dx\right)^{1/\tilde{k}} \right.\\ &\qquad\left. + \frac{K_3^{1/N}}{2C\varepsilon^{\frac{1}{N}+\frac{m-1}{2}}(s-\tau)^{\frac{1}{2}}}\left(\frac{K_3}{\varepsilon}\varepsilon^{\tilde{k}\ell}\right)^{1/\tilde{k}}\right\} \\ &\quad= C(s-\tau)^{1/2}\varepsilon^{(m-1)/2}\|v^{\ell/2}(s)\|_{2N/(N-1)}^2 + \frac{K_3\varepsilon^{\ell-1}}{2} \\ &\quad\le \int_{\mathbf{R^N}} v^{\ell}(\tau)\,dx + \frac{K_3}{\varepsilon}\varepsilon^{\ell}. \end{aligned} \tag{6.10}$$

Let $\varepsilon>0$ satisfy (6.1). Then, since $\varepsilon^{\frac{1}{N}+\frac{m-1}{2}}(s-\tau)^{\frac{1}{2}} \le \varepsilon^{\frac{1}{N}+\frac{m-1}{2}}t^{\frac{1}{2}} \le$

1, putting $C(\ell_0) = C\min\{1, 2^{-1}C^{-1}K_3^{1/N}\}$ and using the inequality $(a+b)^{1/\tilde{k}} \le a^{1/\tilde{k}} + b^{1/\tilde{k}}$ we obtain (6.4). The proof is complete. □

Proof of Proposition 6.1 In the case $N = 1$, if we put $s = t$ and $\tau = t/2$ in (6.3) we get (6.2).

Next, we consider the case $N > 1$. Let $\ell > m$ and $t^{\frac{1}{2}}\varepsilon^{\frac{1}{N}+\frac{m-1}{2}} \le 1$. For any $t \ge 4$, let $\{t_n\}$ be the sequence of numbers satisfying the recurrence formula

$$\begin{cases} t_0 = t/2 \\ t_n = t/2^{n+1} + t_{n-1} \quad (n \ge 1). \end{cases} \tag{6.11}$$

Then,

$$t_n = \{1 - (1/2)^{n+1}\}t \quad (\le t) \tag{6.12}$$

and

$$t_n \uparrow t \quad \text{as} \quad n \to \infty. \tag{6.13}$$

Put $s = t_n$ and $\tau = t_{n-1}$ in (6.4) with ℓ replaced by $\tilde{k}^{n-1}\ell$ $(\ge \ell)$. We note that $2 \le \tau \le s \le t$ and $(s-\tau)^{1/2} = (t_n - t_{n-1})^{1/2} = t^{1/2}(\sqrt{2})^{-(n+1)}$. Then, if we set

$$J_n = \int_{\mathbf{R^N}} v^{\tilde{k}^n\ell}(t_n)\,dx + \frac{K_3}{\varepsilon}\varepsilon^{\tilde{k}^n\ell}, \tag{6.14}$$

then we have

$$C(\ell)t^{1/2}\varepsilon^{(m-1)/2}(\sqrt{2})^{-(n+1)}\{J_n\}^{1/\tilde{k}} \le J_{n-1}. \tag{6.15}$$

We now use Moser's iteration methods. Iterating (6.15) we have

$$c_n\{J_n\}^{1/\tilde{k}^n} \le J_0 \tag{6.16}$$

where

$$\begin{aligned} c_n &= \prod_{i=1}^{n}\left\{C(\ell)t^{1/2}\varepsilon^{(m-1)/2}(\sqrt{2})^{-(i+1)}\right\}^{\tilde{k}^{-(i-1)}} \\ &= \{C(\ell)t^{1/2}\varepsilon^{(m-1)/2}2^{-1}\}^{\Sigma_{i=1}^n\tilde{k}^{-(i-1)}}(\sqrt{2})^{-\Sigma_{i=1}^n(i-1)\tilde{k}^{-(i-1)}} \end{aligned}$$

and

$$\lim_{n\to\infty} c_n = \{C(\ell)t^{1/2}\varepsilon^{(m-1)/2}2^{-1}\}^{\tilde{k}/(\tilde{k}-1)}(\sqrt{2})^{-\tilde{k}/(\tilde{k}-1)^2}.$$

Since by (2.14)

$$\|v(t)\|_\infty^\ell = \lim_{n\to\infty}\left\{\int_{\mathbf{R}^N} v^{\tilde{k}^n\ell}(t)\,dx\right\}^{1/\tilde{k}^n} \le \lim_{n\to\infty}\inf J_n^{1/\tilde{k}^n},$$

if $n \to \infty$ in (6.16), then we have by $\tilde{k} = N/(N-1)$,

$$\{C(\ell)t^{1/2}\varepsilon^{(m-1)/2}2^{-1}\}^N \sqrt{2}^{-N(N-1)}\|v(t)\|_\infty^\ell \le J_0,$$

and so we get (6.2). The proof is complete. □

7. Proof of Theorem 3 and 5

In this section we prove Theorem 3 and 5. We begin with the following proposition:

Proposition 7.1 *Let* $u(x,t)$ *be a global weak solution of* (1.1) (1.2) *in Lemma* 5.4 *satisfying* $\|u_0\|_{p_0} < \delta_2$. *Then,*

$$\|u(t)\|_\infty \le C_{12}t^{-N/[N(m-1)+2]} \quad \textit{for} \quad t \ge 4 \tag{7.1}$$

where $C_{12} = C_{12}(N, m, p, \|u_0\|_1, \delta_2)$.

Proof. Let $u(x,t)$ be a global weak solution of (1.1) (1.2) in Lemma 5.4 satisfying $\|u_0\|_{p_0} < \delta_2$. If we put $v(x,t) = \exp(-\int_2^t \|u\|_\infty^{p-1}\,dt)u$ then $v(x,t)$ satisfies (6.2) by Lemma 5.6 and Proposition 6.1. By (5.31) we note that for some $C > 0$ $\|u(t)\|_\infty \le C\|v(t)\|_\infty$ for $t \ge 0$. Hence, we see that $u(x,t)$ satisfies (6.2) also. Putting $\varepsilon = t^{-N/[N(m-1)+2]}$ in (6.2) and using (5.4) with $\beta = 1$ (see (5.24)), we have (7.1). The proof is complete. □

Proof of Theorem 3. (Ref. Kawanago [17], R. Suzuki [29] and Friedman-Kamin [7]) First, we show (1.9) and (1.10) of Theorem 3 in the case that $u_0(x)$ satisfies (2.2). In the general case, using Proposition 2.6 with adding the assumption that $u_{0,n} \to u_0$ in $L^1 \cap L^{p_0}$, we can show (1.9) and (1.10) of Theorem 3 also.

Suppose (2.2) for the initial data $u_0(x)$ and let $u(x,t)$ be a global weak solution of (1.1) (1.2) which is constructed in Lemma 5.4 with $\|u_0\|_{p_0} < \delta_1 \equiv \min\{\delta_0, \delta_2\}$ where δ_0 is in Theorem 2. Then, (1.9) and (1.10) follow from Theorem 2, Lemma 5.4 and Proposition 7.1 that (1.7) and (1.8) hold.

Next, we shall show (1.11). Assume $q > m + 1/N$ and set

$$u_k(x,t) = k^N u(kx, k^{N(m-1)+2}t), \quad k > 0. \tag{7.2}$$

Then u_k is a unique weak solution of

$$\begin{cases} u_t = \triangle u^m + k^{-\eta} a \cdot \nabla u^q + k^{-\mu} u^p & \text{in } \mathbf{R^N} \times (0,\infty), \\ u(x,0) = k^N u_0(kx) & \text{in } \mathbf{R^N} \times (0,\infty). \end{cases} \tag{7.3}$$

where $\eta = N(q-m) - 1 > 0$ and $\mu = N(p-m) - 2 > 0$. Furthermore, by (1.9) we have

$$\begin{aligned} \|u_k(t)\|_\infty &= k^N \|u(k^{N(m-1)+2} t)\|_\infty \\ &\le C t^{-N/[N(m-1)+2]} \quad \text{for } t > 0. \end{aligned} \tag{7.4}$$

Thus, since $u_k(x,t)$ is uniformly bounded on $\mathbf{R^N} \times [\tau,\infty)$, $\tau > 0$, by Di Benedetto [4], we see that $u_k(x,t)$ is equicontinuous on every compact subset in $\mathbf{R^N} \times (0,\infty)$. Therefore, there exist a subsequence $\{k'\} \subset \{k\}$ and a continuous function $v(x,t)$ in $\mathbf{R^N} \times (0,\infty)$ such that

$$u_{k'}(x,t) \to v(x,t) \tag{7.5}$$

uniformly on every compact subset of $\mathbf{R^N} \times (0,\infty)$. In the following, by using the uniqueness of the solution of (1.12) due to Pierre [27] we shall show

$$v(x,t) = V_m(x,t,M_\infty) \tag{7.6}$$

where $V_m(x,t,M_\infty)$ is as in Theorem 3.

Since $u_k(x,t)$ is a weak solution of (7.3), it satisfies the integral identity

$$\begin{aligned} &\int_{\mathbf{R^N}} u_k(x,T)\varphi(x,T)\,dx \\ &= \int_0^T \int_{\mathbf{R^N}} \{u_k \partial_t \varphi + u_k^m \triangle \varphi\}\,dx\,dt - k^{-\eta} \int_0^T \int_{\mathbf{R^N}} u_k^q a \cdot \nabla \varphi\,dx\,dt \\ &\quad + k^{-\mu} \int_0^T \int_{\mathbf{R^N}} u_k^p \varphi\,dx\,dt + k^N \int_{\mathbf{R^N}} u_0(kx)\varphi(x,0)\,dx \\ &\equiv S_1 + S_2 + S_3 + S_4 \end{aligned} \tag{7.7}$$

for all $\varphi \in C_0^\infty(\mathbf{R^N} \times [0,\infty))$.

We shall estimate each S_i $(i = 1,2,3,4)$. First, we consider S_1. We note by (5.24),

$$\int_{\mathbf{R^N}} u_k(x,t)\,dx = \int_{\mathbf{R^N}} u(x, k^{N(m-1)+2} t)\,dx \le K_3 \quad \text{for } t \ge 0. \tag{7.8}$$

Hence, since by (7.4) we have

$$\begin{aligned}
&\left|\int_0^\delta \int_{\mathbf{R^N}} \{u_k\varphi_t + u_k^m \triangle\varphi\}\, dx\, dt\right| \\
&\le C\int_0^\delta \int_{\mathbf{R^N}} \{u_k + u_k^m\}\, dxdt \\
&\le C\int_0^\delta \left\{\int_{\mathbf{R^N}} u_k\, dx + \sup_x u_k^{m-1} \int_{\mathbf{R^N}} u_k\, dx\right\} dt \\
&\le CK_3\left\{\delta + C\int_0^\delta t^{-N(m-1)/[N(m-1)+2]}\, dt\right\} \\
&= CK_3\left(\delta + \frac{N(m-1)+2}{2}\delta^{2/[N(m-1)+2]}\right) \to 0 \quad (\text{as } \delta \downarrow 0),
\end{aligned} \tag{7.9}$$

we get

$$S_1 \to \int_0^T \int_{\mathbf{R^N}} \{v\partial_t\varphi + v^m \triangle\varphi\}\, dx\, dt \quad \text{as} \quad k = k' \to \infty. \tag{7.10}$$

Next, we consider S_2. Similarly as above we get

$$\begin{aligned}
&|S_2| \\
&\le Ck^{-1} \int_0^{k^{N(m-1)+2}T} \int_{\mathbf{R^N}} u^q dxdt \\
&\le CK_3k^{-1}\left\{C\int_1^{k^{N(m-1)+2}T} t^{-N(q-1)/[N(m-1)+2]}dt + \int_0^1 \|u\|_\infty^{q-1} dt\right\} \\
&\begin{cases}
\le CK_3k^{-1}\Big\{C\dfrac{N(m-1)+2}{N(m-q)+2}[T^{\frac{N(m-q)+2}{N(m-1)+2}}k^{N(m-q)+2} - 1] \\
\qquad\qquad + \displaystyle\int_0^1 \|u\|_\infty^{q-1}\, dt\Big\} \quad (\text{if } q \ne m + 2/N) \\
\le CK_3k^{-1}\Big\{C\log(k^{N(m-1)+2}T) + \displaystyle\int_0^1 \|u\|_\infty^{q-1}\, dt\Big\} \\
\qquad\qquad\qquad\qquad (\text{if } q = m + 2/N)
\end{cases} \\
&\to 0 \quad (\text{as } k \to \infty)
\end{aligned} \tag{7.11}$$

when $q > m + 1/N$.

Next, we consider S_3. Since $u(x,t)$ satisfies (1.10) and (2.1), if we choose the suitable test function $\varphi(x,t)$ and use the limit procedure (see [29]) then we get

$$\int_0^\infty \int_{\mathbf{R^N}} u^p \, dxdt + \int_{\mathbf{R^N}} u_0(x) \, dx = \lim_{T\to\infty} \int_{\mathbf{R^N}} u(x,T) \, dx \le K_3. \tag{7.12}$$

Hence, it follows from the Lebesgue dominated theorem that

$$\begin{aligned} S_3 &= \int_0^{k^{N(m-1)+2}T} \int_{\mathbf{R^N}} u(x,t)^p \varphi(x/k, t/k^{N(m-1)+2}) \, dxdt \\ &\to \varphi(0,0) \int_0^\infty \int_{\mathbf{R^N}} u^p \, dxdt \quad \text{as } k \to \infty. \end{aligned} \tag{7.13}$$

Similarly, we get

$$S_4 \to \varphi(0,0) \int_{\mathbf{R^N}} u_0(x) \, dx \quad \text{as } k \to \infty. \tag{7.14}$$

Thus, if $k = k_i \to \infty$ in (7.7), we have

$$\begin{aligned} &\int_{\mathbf{R^N}} v(x,T)\varphi(x,T) \, dx \\ &\quad = \int_0^T \int_{\mathbf{R^N}} \{v\varphi_t + v^m \triangle \varphi\} \, dxdt \\ &\quad + \varphi(0,0) \left\{ \int_{\mathbf{R^N}} u_0 \, dx + \int_0^\infty \int_{\mathbf{R^N}} u^p \, dxdt \right\} \end{aligned} \tag{7.15}$$

which shows that $v(x,t)$ is a weak solution of (1.12) with $L = M_\infty$ which is defined by (1.13). Therefore, since

$$\begin{aligned} \int_0^T \int_{\mathbf{R^N}} \{v + v^m\} \, dxdt &\le \liminf_{k\to\infty} \int_0^T \int_{\mathbf{R^N}} \{u_k + u_k^m\} \, dxdt \\ &\le CK_3 \Big(T + \frac{N(m-1)+2}{2} T^{2/[N(m-1)+2]} \Big) \end{aligned} \tag{7.16}$$

by (7.9), the uniqueness theorem for solutions of (1.12) due to Pierre [27] (see also Lemma 2.2 of R. Suzuki [29]) implies (7.6) and so (1.11) (see Friedman-Kamin [7] and Kawanago [17]). The proof is complete. □

Proof of Theorem 5. The proof is the same as that of Theorem 3. Therefore, we only state the outline of the proof. We assume that the initial data $v_0(x)$ satisfies (2.2) and we construct a weak solution $v(x,t)$ of (1.15) (1.16) similarly in Proposition 2.4 (In this case $v(x,t)$ exists globally in time).

Then, we have for any $\ell \geq 1$

$$\int_{\mathbf{R^N}} v^\ell \, dx \Big|_\tau^s + \frac{4m\ell(\ell-1)}{(m+\ell-1)^2} \int_\tau^s \|\nabla v^{(m+\ell-1)/2}\|_2^2 \, dt \leq 0 \tag{7.17}$$

for $0 \leq \tau \leq s$. Hence, according to the proof of Lemma 5.1, we get the assertions of this lemma for $v(x,t)$ without the assumption $\|v_0\|_{p_0} < \min\{B_{p_0}, B_\ell\}$. Furthermore we see that $v(x,t)$ satisfies the assertions of Proposition 6.1, since $v(x,t)$ satisfies the assumptions of this proposition. Therefore, combining these lemma and proposition, we have

$$\|v(t)\|_\infty \leq Ct^{-N/[N(m-1)+2]} \quad \text{for } \ t > 0 \tag{7.18}$$

where $C = C(m, N, \|v_0\|_1)$ (see the proof of Proposition 7.1). The rest of the assertions of Theorem 5 are also showed by using the similar methods to those of Theorem 3. The proof is complete. □

8. Blow up cases II

In this section, we prove Theorem 4 in a series of lemmas. The methods of the proof are the same as those of R. Suzuki [29].

Lemma 8.1 *Assume $p \geq q \geq m + 1/N$ and $p = p^*_{m,q}$ $(= m + 2/N)$. Let $u(x,t)$ be a nonnegative global weak solution of* (1.1) (1.2). *Then, $u(\cdot,t) \in L^1(\mathbf{R^N})$ and*

$$\int_{\mathbf{R^N}} u(x,t) \, dx \leq C(N) \quad \textit{for all } \ t \geq 0 \tag{8.1}$$

where $C(N)$ is as in Lemma 3.2.

Proof. Let γ and k_0 be as in Lemma 3.2. We note $k_0 = 2/(p-m) = N$ and $\gamma = 0$, when $p \geq q \geq m + 1/N$ and $p = p^*_{m,q}$ $(= m + 2/N)$. Then since $k_0 - N - \gamma = 0$, it follows from Lemma 3.2 that

$$\int_{\mathbf{R^N}} u(x,t) s_\varepsilon(x) \, dx \leq C(N) \quad \text{for all } \ t \geq 0. \tag{8.2}$$

Therefore if $\varepsilon \downarrow 0$, by Fatou's lemma we get (8.1). □

Lemma 8.2 *Let $u(x,t)$ be as in Lemma* 8.1. *Then we have for any $T > 0$,*

$$\int_0^T \int_{\mathbf{R^N}} u(x,t)^p \, dx dt \leq C(N). \tag{8.3}$$

Proof. By Lemma 8.1 we see that u^m, u^p, u^q, $\in L^1(\mathbf{R^N} \times (0,T))$ for any $T > 0$. Therefore, from (2.1) of Definition 2.1 we have

$$\int_0^T \int_{\mathbf{R^N}} u^p \, dxdt \leq \int_{\mathbf{R^N}} u(x,T) \, dx \leq C(N)$$

(see (7.12)). The proof is complete. □

Proof of Theorem 4. Assume that the Cauchy problem (1.1) (1.2) has a global solution u. Suppose $u_0(x) \not\equiv 0$. Then, by Lemma 8.1 we get $u_0(x) \in L^1(\mathbf{R^N})$. Putting $u_k(x,t) = k^N u(kx, k^{N(m-1)+2}t)$ we see that it is a global weak solution of (7.3) with $\nu = N(p-m) - 2 = 0$, when $p = m + 2/N$. Furthermore, when $q \geq m + 1/N$ we see that $|k^{-\eta}a| \leq |a|$, since $\eta = N(q-m) - 1 \geq 0$. Therefore, since (8.3) holds with $u = u_k$ for all $k \geq 1$, we have

$$\int_0^T \int_{\mathbf{R^N}} u_k(x,t)^p \, dxdt \leq C(N) \quad \text{for } T > 0. \tag{8.4}$$

Let $v(x,t)$ be a weak solution of (1.15) (1.16) with $v_0(x) = u_0(x)$. Then, if we define v_k similarly as u_k, we get by the comparison theorem (Proposition 2.2),

$$v_k(x,t) \leq u_k(x,t) \quad \text{in } \mathbf{R^N} \times (0,\infty). \tag{8.5}$$

Here, we note by the proof of (1.19) (see also (7.5) and (7.6)),

$$v_k(x,t) \to V_m(x,t,M) \quad \text{as } k \to \infty \tag{8.6}$$

locally uniformly in $\mathbf{R^N} \times (0,\infty)$ where $M = \int_{\mathbf{R^N}} u_0(x) \, dx$ (> 0). Therefore, it follows from (8.4), (8.5) and Fatou's lemma that

$$\int_0^T \int_{\mathbf{R^N}} V_m(x,t,M)^p \, dxdt \leq \liminf_{k\to\infty} \int_0^T \int_{\mathbf{R^N}} v_k^p \, dxdt \leq C(N).$$

On the other hand, since $V_m(x,t,M)$ is the given concrete form (see Friedman-Kamin [7] and Lemma 2.1 in R. Suzuki [29]), we see that if $p = p^*_{m,q}$ $(= m + 2/N)$ and $M > 0$ then

$$\int_0^T \int_{\mathbf{R^N}} V_m(x,t,M)^p \, dxdt = \infty.$$

This is a contradiction and so $u_0(x) \equiv 0$. Therefore, by the uniqueness of solutions with L^1-valued initial data (see Proposition 2.2), we obtain

$u(x,t) \equiv 0$ in $(x,t) \in \mathbf{R^N} \times (0,\infty)$. As in the proof of Theorem 1, (1.14) is obvious by the comparison and existence theorems for solutions. The proof is complete. □

9. Appendix

In this appendix, for the convenience of readers we state the comparison theorem for the weak solution of the Cauchy problem of equation

$$u_t = \triangle\eta(u) + a \cdot \nabla h(u) + k(u) \quad (x,t) \in \mathbf{R^N} \times (0,T) \tag{9.1}$$

where $a \in \mathbf{R^N}$, $\eta(\xi), h(\xi), k(\xi) \in C^1([0,\infty))$, $\eta'(\xi), k'(\xi) \geq 0$ for $\xi \geq 0$ and $\eta(0) = h(0) = k(0) = 0$. We assume

(A1) $\quad |h'(\xi)| \leq C(M)\sqrt{\eta'(\xi)}$ for $0 \leq \xi \leq M$

for some positive constant $C(M) > 0$. We define a supersolution [or subsolution] similarly as in Definition 2.1. We shall prove the following proposition:

Proposition 9.1 *Assume* (A1). *Let* v [*or* u] *be a supersolution* [*or subsolution*] *of* (9.1) *in* $\mathbf{R^N} \times [0,T)$ *and suppose*

$$\sup_{[0,\tau]} \|v(\cdot,t) - u(\cdot,t)\|_1 < \infty \quad \textit{for any} \;\; \tau \in (0,T). \tag{9.2}$$

If $v(x,0) \geq u(x,0)$ *in* $\mathbf{R^N}$, *then we have* $v \geq u$ *in the whole* $\mathbf{R^N} \times (0,T)$.

Corollary 9.2 *Assume* (A1). *Let* v *and* u *be weak solutions of* (9.1) *in* $\mathbf{R^N} \times [0,T)$ *and suppose that* $v(x,0)$ *and* $u(x,0)$ *belong to* $L^1(\mathbf{R^N})$. *If* $v(x,0) \geq u(x,0)$ *in* $\mathbf{R^N}$, *then we have* $v \geq u$ *in the whole* $\mathbf{R^N} \times (0,T)$.

Proof. By the above proposition, it is enough to show that if $u(x,0) \in L^1(\mathbf{R^N})$ then there exists a nondecreasing function $C(t)$ ($< \infty$) such that

$$\|u(\cdot,t)\|_1 \leq C(t)\|u(x,0)\|_1 \quad \text{for} \;\; t \in (0,T). \tag{9.3}$$

Let $s(x)$ be a positive bounded C^2-function with s, ∇s and $\triangle s \in L^1(\mathbf{R^N})$ satisfying

$$s(0) = 1, \quad |\triangle s(x)| \leq s(x) \quad \text{and} \quad |a \cdot \nabla s(x)| \leq K s(x)$$

for some constant $K > 0$ (Explicit examples were given in [1]). If we put

$$s_\varepsilon(x) = s(\varepsilon x)$$

for each $\varepsilon \in (0,1)$, then

$$|\triangle s_\varepsilon(x)| \leq \varepsilon^2 s_\varepsilon(x) \quad \text{and} \quad |a \cdot \nabla s_\varepsilon(x)| \leq \varepsilon K s_\varepsilon(x). \tag{9.4}$$

Now, we consider $s_\varepsilon(x)$ as a test function $\varphi(x,t)$ in (2.1) (see Section 3). Then, we have from (9.4),

$$\begin{aligned} &\int_{\mathbf{R^N}} u(x,\tau) s_\varepsilon(x)\,dx \\ &\leq \int_{\mathbf{R^N}} u(x,0) s_\varepsilon(x)\,dx + \int_0^\tau \int_{\mathbf{R^N}} (\varepsilon^2 \eta(u) \\ &\qquad + \varepsilon K h(u) + k(u)) s_\varepsilon(x)\,dxdt \quad \text{for} \quad \tau > 0. \end{aligned} \tag{9.5}$$

Hence, if we set

$$g(t) = \sup_{\mathbf{R^N} \times [0,t]} \{(\eta(u) + K h(u) + k(u))/u\} \quad (< \infty) \tag{9.6}$$

for each $t \in (0,T)$, then for any $\varepsilon \in (0.1)$ and $\tau \in [0,t]$ we obtain

$$\begin{aligned} &\int_{\mathbf{R^N}} u(x,\tau) s_\varepsilon(x)\,dx \\ &\leq \int_{\mathbf{R^N}} u(x,0) s_\varepsilon(x)\,dx + g(t) \int_0^\tau \int_{\mathbf{R^N}} s_\varepsilon(x) u(x,t)\,dxdt \\ &\qquad \text{for} \quad \tau \in [0,t] \end{aligned} \tag{9.7}$$

which leads to

$$\int_{\mathbf{R^N}} u(x,\tau) s_\varepsilon(x)\,dx \leq e^{g(t)\tau} \int_{\mathbf{R^N}} u(x,0) s_\varepsilon(x)\,dx \qquad \text{for} \quad \tau \in [0,t]. \tag{9.8}$$

Put $\tau = t$ and

$$C(t) = e^{g(t)t}, \tag{9.9}$$

and let $\varepsilon \downarrow 0$ in (9.8). Then, noting $u(x,0) \in L^1(\mathbf{R^N})$ we get (9.3). The proof is complete. □

Remark 9.3 When $N = 1$, Proposition 9.1 was proved by Gillding [12] under weaker conditions. They do not need condition (9.2). Our methods of the proof are different from ones of [12] and similar to ones of [2] and [3].

Proof of Proposition 9.1 Since v [or u] is a supersolution [or subsolution] of (9.1), for any test function $\varphi \geq 0, \in C_0^\infty(\mathbf{R^N} \times [0,T))$ we have

$$\int_{\mathbf{R^N}} (u(\tau) - v(\tau))\varphi(\tau)\,dx - \iint_{Q_\tau} (u-v)(\varphi_t + \tilde{\eta}\triangle\varphi - \tilde{h}a\cdot\nabla\varphi)\,dxdt \\ \leq \int_{\mathbf{R^N}} (u(x,0) - v(x,0))\varphi(0)\,dx + \iint_{Q_\tau} (u-v)\tilde{k}\,dxdt \qquad (9.10)$$

where $Q_\tau = \mathbf{R^N} \times [0,\tau)$ and

$$\tilde{f}(x,t) = \begin{cases} \dfrac{f(u)-f(v)}{u-v} & \text{if } u \neq v \\ 0 & \text{otherwise.} \end{cases} \qquad (9.11)$$

Here we note that

$$\tilde{\eta},\ \tilde{h},\ \tilde{k} \in L^\infty(Q_\tau) \quad \text{for each } \ 0 < \tau < T, \qquad (9.12)$$

$$\tilde{\eta},\ \tilde{k} \geq 0 \quad \text{in } \ Q_T \qquad (9.13)$$

and

$$|\tilde{h}| \leq C(\tau)\sqrt{\tilde{\eta}} \quad \text{for } \ (x,t) \in Q_\tau \qquad (9.14)$$

for some constant $C(\tau) > 0$. Let $\chi \in C_0^\infty(\mathbf{R^N})$ and $0 \leq \chi \leq 1$. Let $R > 0$ be so large that $\operatorname{supp}\chi \subset B_{R/2}$. Furthermore, define sequences of smooth positive functions $\{\eta_n\}$ and $\{h_n\}$ as follows (see Aronson-Crandall-Peletier [2]) :

$$\frac{1}{n} \leq \eta_n \leq \|\tilde{\eta}\|_{L^\infty(Q_{\tau,R})} + \frac{1}{n}, \qquad (9.15)$$

$$\frac{\eta_n - \tilde{\eta}}{\sqrt{\eta_n}} \to 0 \quad \text{in } \ L^2(Q_{\tau,R}) \quad \text{as } \ n \to \infty, \qquad (9.16)$$

$$h_n \to \tilde{h} \quad \text{in } \ L^2(Q_{\tau,R}) \quad \text{as } \ n \to \infty \qquad (9.17)$$

and

$$|h_n| \leq C(\tau)\sqrt{\eta_n} \quad \text{in } \ Q_{\tau,R}, \qquad (9.18)$$

where $Q_{\tau,R} = B_R \times [0,\tau)$.

Finally, we define a sequence of smooth functions $\{\varphi_n\}$ by a smooth solution of

$$\begin{cases} \varphi_{n,t} + \eta_n \triangle \varphi_n - h_n a \cdot \nabla \varphi_n = \lambda \varphi_n & \text{in } Q_{\tau,R} \\ \varphi_n = 0 \quad \text{on } \partial B_R \times [0,\tau) \\ \varphi_n(x,\tau) = \chi \quad \text{on } B_R. \end{cases} \tag{9.19}$$

We need the following lemma. □

Lemma 9.1 *If λ is large enough, then*
(i) $0 \le \varphi_n \le e^{\lambda(t-\tau)}$ *in* $Q_{\tau,R}$;
(ii) $\int\int_{Q_{\tau,R}} \eta_n (\triangle\varphi_n)^2\, dxdt < C$;
(iii) $\sup_{0\le t\le\tau} \int_{B_R} |\nabla\varphi_n|^2(t)\, dx < C$,
where C is a constant depending only on χ.

Proof. (i) is obvious by the comparison theorem.

Next we prove (ii) and (iii). Multiply the both sides of equation (9.19) by $\triangle\varphi_n$ and integrate by parts over $B_R \times (t,\tau)$. Then

$$\frac{1}{2}\int_{B_R} |\nabla\varphi_n|^2(t)\, dx + \int_t^\tau \int_{B_R} \eta_n (\triangle\varphi_n)^2\, dxdt + \lambda \int_t^\tau \int_{B_R} |\nabla\varphi_n|^2\, dxdt$$
$$\le \int_t^\tau \int_{B_R} h_n a \cdot \nabla\varphi_n \triangle\varphi\, dxdt + \frac{1}{2}\int_{B_R} |\nabla\chi|^2\, dx. \tag{9.20}$$

Since we have

$$|h_n a \cdot \nabla\varphi_n \triangle\varphi| \le \frac{1}{2}\eta_n(\triangle\varphi_n)^2 + \frac{1}{2}C(\tau)^2|a|^2|\nabla\varphi_n|^2$$

by (9.18), we get

$$\frac{1}{2}\int_{B_R} |\nabla\varphi_n|^2(t)\, dx + \frac{1}{2}\int_t^\tau \int_{B_R} \eta_n (\triangle\varphi_n)^2\, dxdt$$
$$+ (\lambda - C(\tau)^2|a|^2/2) \int_t^\tau \int_{B_R} |\nabla\varphi_n|^2\, dxdt$$
$$\le \frac{1}{2}\int_{B_R} |\nabla\chi|^2\, dx$$

which leads to (ii) and (iii). □

Proof of Proposition 9.1 (continue) Set $\varphi(x,t) = \xi_R(x)\varphi_n(x,t)$ as a test function in (9.10), where $\xi_R(r) = \xi(|x|/R)$ and $\xi(r) \in C^\infty(\mathbf{R})$ satisfies that $0 \le \xi(r) \le 1$ for $r \ge 0$, $\xi(r) = 0$ for $r \ge 1$ and $\xi(r) = 1$ for $0 \le r \le 1/2$.

Then

$$\begin{aligned}
&\int_{\mathbf{R}^{\mathbf{N}}} (u(\tau) - v(\tau))\chi\, dx \\
&\quad \le \lambda \int\int_{Q_\tau} (u - v)\varphi_n \xi_R\, dxdt \\
&\qquad + \int\int_{Q_\tau} (u - v)\xi_R\{(\tilde{\eta} - \eta_n)\triangle\varphi_n - (\tilde{h} - h_n)a \cdot \nabla\varphi_n\}\, dxdt \\
&\qquad + \int\int_{Q_\tau} (u - v)\{2\tilde{\eta}\nabla\varphi_n \cdot \nabla\xi_R + \tilde{\eta}\varphi_n\triangle\xi_R - \tilde{h}a \cdot \nabla\xi_R\varphi_n\}\, dxdt \\
&\qquad + \int_{\mathbf{R}^{\mathbf{N}}} (u(x,0) - v(x,0))\xi_R\varphi_n(0)\, dx \\
&\qquad + \int\int_{Q_\tau} (u - v)\tilde{k}\xi_R\varphi_n\, dxdt. \qquad (9.21)
\end{aligned}$$

We note, by Lemma 9.4,

$$\begin{aligned}
&\|(\tilde{\eta} - \eta_n)\triangle\varphi_n\|_{L^1(Q_{R,\tau})} + \|(\tilde{h} - h_n)a \cdot \nabla\varphi_n\|_{L^1(Q_{\tau,R})} \\
&\quad \le \|(\tilde{\eta} - \eta_n)/\sqrt{\eta_n}\|_{L^2(Q_{\tau,R})}\|\sqrt{\eta_n}\triangle\varphi_n\|_{L^2(Q_{\tau,R})} \\
&\qquad\qquad + |a|\|\tilde{h} - h_n\|_{L^2(Q_{\tau,R})}\|\nabla\varphi_n\|_{L^2(Q_{\tau,R})} \\
&\qquad\qquad\qquad \to 0 \quad (\text{as } n \to \infty).
\end{aligned}$$

Hence, if $n \to \infty$ in (9.21) we obtain by the Schwarz's inequality,

$$\begin{aligned}
&\int_{\mathbf{R}^{\mathbf{N}}} (u(\tau) - v(\tau))\chi\, dx \\
&\quad \le \int_{\mathbf{R}^{\mathbf{N}}} [u(x,0) - v(x,0)]^+\, dx + (\lambda + K)\int\int_{Q_\tau} [u - v]^+\, dxdt \\
&\qquad + \frac{C}{R}\|u - v\|_{L^2(Q_\tau)}\|\nabla\varphi_n\|_{L^2(Q_{\tau,R})} + \frac{C}{R}\|u - v\|_{L^1(Q_\tau)}(1 + 1/R) \\
&\hspace{30em} (9.22)
\end{aligned}$$

where $K = \sup_{Q_\tau} \tilde{k}$ and $[u]^+ = \max\{u, 0\}$. Here we used (9.12),

$$|\nabla\xi_R(x)| \le \frac{\|\xi'\|_\infty}{R} \qquad (9.23)$$

and

$$|\triangle\xi_R(x)| \le \frac{\|\xi''\|_\infty/R + 2(N-1)\|\xi'\|_\infty}{R}. \qquad (9.24)$$

Note $u - v \in L^1(Q_\tau) \cap L^\infty(Q_\tau)$. Then, if $R \to \infty$, we get

$$\int_{\mathbf{R^N}} (u(\tau) - v(\tau))\chi\, dx \qquad (9.25)$$
$$\leq \int_{\mathbf{R^N}} [u(x,0) - v(x,0)]^+\, dx + (\lambda + K) \int\int_{Q_\tau} [u - v]^+\, dxdt$$

for any $\chi \in C_0^\infty(\mathbf{R^N})$ satisfying $0 \leq \chi \leq 1$, and hence we have

$$\int_{\mathbf{R^N}} [u(t) - v(t)]^+\, dx \qquad (9.26)$$
$$\leq \int_{\mathbf{R^N}} [u(x,0) - v(x,0)]^+\, dx + (\lambda + K) \int_0^t \int_{\mathbf{R^N}} [u - v]^+\, dxdt$$

for $t \in [0, \tau]$, which leads to

$$\int_{\mathbf{R^N}} [u(\tau) - v(\tau)]^+\, dx \leq e^{(\lambda+K)\tau} \int_{\mathbf{R^N}} [u(x,0) - v(x,0)]^+\, dx. \qquad (9.27)$$

Thus, when $u(x,0) \leq v(x,0)$, we have $[u(\tau) - v(\tau)]^+ = 0$, that is, $u(x,\tau) \leq v(x,\tau)$ for $\tau \in [0, T)$. The proof is complete. □

References

[1] Aguirre J. and Escobedo M., *On the blow-up of solutions of a convective reaction diffusion equation.* Proc. Royal Soc. Edinburgh **123A** (1993), 433–460.

[2] Aronson D.G., Crandall M.G. and Peletier L.A., *Stabilization of solutions of a degenerate nonlinear diffusion problem.* Nonlinear Anal. **6** (1982), 1001–1022.

[3] Bertsch M., Kersner R. and Peletier L.A., *Positivity versus localization in degenerate diffusion equations.* Nonlinear Anal. **9** (1985), 987–1008.

[4] DiBenedetto E., *Continuity of weak solutions to a general porous medium equation.* Indiana Univ. Math. J. **32** (1983), 83–118.

[5] Escobedo M., Vázquez J.L. and Zuazua E., *A diffusion-convection equation in several space dimensions.* Indiana Univ. Math. J. **42** (1993), 1413–1440.

[6] Escobedo M. and Zuazua E., *Large time behavior for convection-diffusion equations in* $\mathbf{R}^N$. J. Func. Anal. **100** (1991), 119–161.

[7] Friedman A. and Kamin S., *The asymptotic behavior of gas in a n-dimensional porous medium.* Trans. Amer. Math. Soc. **262** (1980), 551–563.

[8] Friedman A. and Lacey A.A., *Blowup of solutions of semilinear parabolic equations.* J. Math. Anal. Appl. **132** (1988), 171–186.

[9] Fujita H., *On the blowing up of solutions of the Cauchy problem for* $u_t = \triangle u + u^{1+\sigma}$. J. Fac. Sci. Univ. Tokyo **13** (1966), 109–124.

[10] Galaktionov V.A., *Blow-up for quasilinear heat equations with critical Fujita's exponents.* Proc. Roy. Soc. Edinburgh **124A** (1994), 517–525.

[11] Galaktionov V.A., Kurdyumov S.P., Mikhailov A.P. and Samarskii A.A., *Unbounded solutions of the Cauchy problem for the parabolic equation* $u_t = \nabla(u^\sigma \nabla u) + u^\beta$. Soviet Phys. Dokl. **25** (1980), 458–459.

[12] Gilding B.H., *A nonlinear degenerate parabolic equation.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. **4** (1977), 393–432.

[13] Hayakawa K., *On nonexistence of global solutions of some semilinear parabolic equations.* Proc. Japan Acad. **49** (1973), 503–505.

[14] Hui K.M., *Singular limit of solutions of* $u_t = \triangle u^m - A \cdot \nabla(u^q/q)$ *as* $q \to \infty$. Trans. Amer. Math. Soc. **347** (1991), 1687–1712.

[15] Imai T. and Mochizuki K., *On blow-up of solutions for quasilinear degenerate parabolic equations.* Publ. RIMS, Kyoto Univ. **27** (1991), 695–709.

[16] Imai T., Mochizuki K. and Suzuki R., *On blow-up problems for quasilinear degenerate parabolic equations with convection.* Nonlinear Mathematical Problems in Industry, GAKUTO International Series, Mathematical Sciences and Applications **2** (1993), 435–448.

[17] Kawanago T., *Existence and behavior of solutions for* $u_t = \triangle u^m + u^\ell$. Advances Math., to appear.

[18] Kawanago T., *The behavior of solutions of quasi-linear heat equations.* Osaka J. Math. **27** (1990), 769–796.

[19] Ladyzenskaja O.A., Solonikov V.A. and Ural'ceva N.N., *Linear and Quasilinear Equations of Parabolic Type.* Transl. Math. Monographs, 23, AMS Providence, R. I., 1968.

[20] Levine H.A., *The role of critical exponents in blowup theorems.* SIAM Review **32** (1990), 262–288.

[21] Levine H.A., Payne L.E., Sacks P. and Straughan B., *Analysis of a convective reaction-diffusion equation* (II). SIAM, J. Math. Anal. **20** (1989), 133–147.

[22] Lortz D., Meyer-Spasche R. and Stredulinsky E.W., *Asymptotic behavior of the solutions of certain parabolic equations.* Comm. Pure Appl. Math. **37** (1984), 677–703.

[23] Mochizuki K. and Mukai K., *Existence and nonexistence of global solutions to fast diffusions with source.* Methods and Applications of Analysis **2** (1995), 92 –102.

[24] Mochizuki K. and Suzuki R., *Critical exponent and critical blow-up for quasilinear parabolic equations.* Israel J. Math. **98** (1997), 141–156.

[25] Ohara Y., L^∞*-estimate of solutions of some nonlinear degenerate parabolic equations.* Nonlinear Anal. T.M.A. **18** (1992), 413–426.

[26] Oleinik O.A., Kalashnikov A.S. and Chzou Yui-Lin, *The Cauchy problem and boundary problems for equations of the type of nonlinear filtration.* Izv. Akad. Nauk. SSSR Ser. Math. **22** (1958), 667–704, (Russian).

[27] Pierre M., *Uniqueness of the solutions of* $u_t - \triangle\varphi(u) = 0$ *with initial datum a measure.* Nonlinear Anal. T.M.A. **6** (1982), 175–187.

[28] Suzuki R., *Blow-up solutions of quasilinear degenerate parabolic equations with convection.* SIAM Math. Anal. **26** (1995), 77–97.

[29] Suzuki R., *Critical blow-up for quasilinear parabolic equations in exterior domains.*

Tokyo J. Math. **19** (1996), 397–409.
[30] Weissler F.B., *Existence and nonexistence of global solutions for a semilinear heat equation.* Israel J. Math. **38** (1981), 29–40.

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