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Perfect braided crossed modules and their mod-q analogues

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Abstract. In this paper, we consider the extension theory of braided crossed modules. In particular, we prove the braided version of Norrie's theorem and its mod-q analogues.

Key words: crossed module, braided crossed module, mod-q non-Abelian tensor product.

1. Introduction

Crossed modules are known in many areas. For example, in non-Abelian homological algebra, crossed modules play the role of coefficients for degree two cohomology groups (see [1]). Alternatively, Brown and Spencer [8] obtained certain crossed modules as the fundamental groupoids of topological groups.

Higher dimensional groupoids are known too. For example, Brown and Higgins [5] defined the fundamental double groupoid of a pair of spaces, and Loday [16] developed the point of view to the fundamental cat^n-group ΠX of a n-cube of spaces X. Among other results, he proved the equivalence between cat^2-groups and crossed squares, and braided crossed modules appeared as a special case of crossed squares. In the work of Bullejos and Cegarra [9], braided crossed modules were used as coefficients for certain degree three non-Abelian cohomology groups. More generally, Breen [1] considered, as the objects of degree three non-Abelian cohomology groups, the extensions of the form:

$$1 \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow k,$$

where $\mathcal{G}$, $\mathcal{H}$ are crossed modules and k is a group. Thus it is quite natural to consider the case where k is also a crossed module, braided crossed module and so on.

By use of the Brown-Loday non-Abelian tensor product of groups, Norrie [18] determined the universal central extensions of perfect crossed

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modules. The Brown-Loday non-Abelian tensor product of groups was extended to mod-q tensor product by D. Conduché and C. Rodriguez-Fernández, and Doncel-Juárez and Grandjeán L.-Valcárcel used this to obtain the mod-q analogue of Norrie's theorem.

In this paper, we shall consider the extension theory of braided crossed modules and prove the braided version of Norrie's theorem and its mod-q analogues.

2. Preliminaries

We shall recall some definitions and properties of crossed modules and braidings on them.

Definition 1 Let N and G be groups together with a homomorphism $\partial : N \longrightarrow G$. This $\partial : N \longrightarrow G$ is called a *crossed module* if G acts on N and satisfies the following conditions:

(1) $\partial({}^g n) = g\partial(n)g^{-1}$, $g \in G$, $n \in N$,

(2) ${}^{\partial(n)}n' = nn'n^{-1}$, $n, n' \in N$.

Example 1. For a group G, the identity map $G \longrightarrow G$ together with the action ${}^g g' = gg'g^{-1}$ defines a crossed module.

Definition 2 Let (M, P, ∂), (N, G, ∂') be crossed modules. A *crossed module morphism* $(\varphi, \psi) : (M, P) \longrightarrow (N, G)$, is a pair of group homomorphisms, $\varphi : M \longrightarrow N$ and $\psi : P \longrightarrow G$, such that

(1) $\psi\partial = \partial'\varphi$,

(2) $\varphi({}^g n) = {}^{\psi(g)}\varphi(n)$, $g \in P$, $n \in M$.

When φ and ψ are surjective, the morphism is called an extension.

Definition 3 For a non-negative integer q, the q-center of a crossed module $N \longrightarrow G$ is the crossed module

$$(N^G)^q \longrightarrow Z(G)^q \cap St_G(N), \quad \text{where}$$
$$(N^G)^q = \{n \in N; n^q = 1, {}^g n = n, g \in G\}$$
$$Z^q(G) = \{g \in Z(G); g^q = 1\}$$

In particular, we call the 0-center the center of $N \longrightarrow G$.

Definition 4 An extension of a crossed module is called *q-central* if the crossed module $ker\varphi \longrightarrow ker\psi$ is contained in the q-center of the crossed module $N \longrightarrow G$. In particular, we call the 0-centeral extension the centeral extension.

Definition 5 When $N \longrightarrow G$ is a crossed module, the q-commutator crossed module is defined as a crossed module

$$D^q_G(N) \longrightarrow [G,G]^q$$

where $D^q_G(N)$ is the subgroup of N generated by

$$\{{}^g n n^{-1} r^q; g \in G, n, r \in N\}$$

and $[G,G]^q$ is the subgroup of G generated by

$$\{[g,h]k^q; g,h,k \in G\}$$

In particular, we call the 0-commutator crossed module the commutator crossed module.

Definition 6 A crossed module $N \longrightarrow G$ is called *q-perfect* if it coincides with the q-commutator crossed module. In particular, we call the 0-perfect crosed module the perfect crosed module.

Based on the earlier works of Dennis [12] and Miller [17], Brown and Loday [6] defined the notion of non-Abelian tensor product $M \otimes N$ of two crossed modules. Later, the notion of mod-q exterior product of groups, for a non-negative integer q, was introduced by Ellis [14], and Brown [3] defined the mod-q non-Abelian tensor product $G \otimes^q G$ of group G.

The following definition of the mod-q non-Abelian tensor product of crossed modules is due to Conduché and Rodríguez-Fernández [11].

Definition 7 Let (M,G,∂), (N,G,∂') be two crossed modules and q a non-negative integer. Then the tensor product $M \otimes^q N$ is defined as a group generated by the symbols

$$a \otimes^q b (a \in M, b \in N) \quad \text{and} \quad \{k\}(k \in M \times_G N)$$

with the following relations:

(1) $a \otimes^q bc = (a \otimes^q b)({}^b a \otimes^q {}^b c)$,

(2) $ab \otimes^q c = ({}^a b \otimes^q {}^a c)(a \otimes^q c)$,

(3) $\{k\}(a \otimes^q b)\{k\}^{-1} = {}^{\alpha(k)^q} a \otimes^q {}^{\alpha(k)^q} b$,

(4) $[\{k\},\{h\}] = \pi_1(k)^q \otimes^q \pi_2(h)^q$,

(5) $\{kh\} = \{k\}(\Pi(\pi_1(k)^{-1} \otimes^q ({}^{\alpha(k)^{1-q+i}}\pi_2(h))^i))\{\mathrm{h}\}$,
(6) $\{(a^b a^{-1}, {}^a b b^{-1})\} = (a \otimes^q b)^q$
where $\alpha = \partial \circ \pi_1$.

Note that the Brown-Loday non-Abelian tensor product $M \otimes N$ can be regarded as the special case where the generators are just $a \otimes^0 b$ ($a \in M, b \in N$) and the relations are just (1) and (2). Besides, it was shown in [6] that, for a group G, the following identities hold in $G \otimes G$:

(a) $(a \otimes b)(c \otimes d)(a \otimes b)^{-1} = {}^{[a,b]}c \otimes {}^{[a,b]}d$,
(b) $[a, b] \otimes c = (a \otimes b)({}^c a \otimes {}^c b)$,
(c) $a \otimes [b, d] = ({}^a b \otimes {}^a c)(b \otimes c)^{-1}$,

for all $a, b, c \in G$, $[a, b] = aba^{-1}b^{-1}$.

We next consider braidings on crossed modules.

Definition 8 A braiding on a crossed module $\partial : N \longrightarrow G$ is a map $\{\ ,\ \} : G \times G \longrightarrow N$ (bracket operation) satisfying the following conditions:

(1) $\partial\{a, b\} = aba^{-1}b^{-1}$
(2) $\{\partial(n), b\} = n^b n^{-1}$
(3) $\{a, \partial(n)\} = {}^a n n^{-1}$
(4) $\{a, bc\} = \{a, b\}^b\{a, c\}$
(5) $\{ab, c\} = {}^a\{b, c\}\{a, c\}$, $a, b, c \in G$, $n \in N$.

Example 2. There are canonical braidings on the crossed modules id: $G \longrightarrow G$ and $G \otimes G \longrightarrow G, a \otimes b \longmapsto [a, b]$ by the following maps:

$G \times G \longrightarrow G$, $(a, b) \longmapsto [a, b] = aba^{-1}b^{-1}$,
$G \times G \longrightarrow G \otimes G$, $(a, b) \longmapsto a \otimes b$.

Definition 9 A morphism between two braided crossed modules is defined as a crossed module morphism which preserves the braiding structures. In particular, a q-central extension of a braided crossed module is a q-central extension of the underlying crossed module which preserves the braiding structures.

3. Canonical braidings and their universalities

To construct new braidings, we start from the following observation:

Proposition 1 *If a crossed module $N \xrightarrow{\partial} G$ has a braiding $\{\ ,\ \}$, then there is a group homomorphism $G \otimes G \xrightarrow{f} N$, $a \otimes b \longmapsto \{a, b\}$.*

Proof. Let us check that f preserves the defining relations in $G \otimes G$. By the definitions, we have

$$f(a \otimes bc) = \{a, bc\} = \{a, b\}^{b}\{a, c\},$$
$$f(a \otimes b)f({}^{b}a \otimes {}^{b}c) = \{a, b\}\{{}^{b}a, {}^{b}c\}.$$

But by a result of Conduché [10], any braiding is *equivariant* (i.e., ${}^{a}\{b, c\} = \{{}^{a}b, {}^{a}c\}$), so that $f(a \otimes bc) = f(a \otimes b)f({}^{b}a \otimes {}^{b}c)$. The other relation can be proved by the same computation. □

We next consider the q-tensor analogues. The main difference is the existence of the elements $\{k\}$, and to construct a well behaved map on $G \otimes^{q} G$, we assume that crossed modules $N \longrightarrow G$ are q-central extensions of G.

Proposition 2 *When a crossed module $\partial : N \longrightarrow G$ is a q-central extension of G and has a braiding $\{\ ,\ \}$, there is a group homomorphism $f : G \otimes^{q} G \longrightarrow N$, $a \otimes b \longmapsto \{a, b\}$, $\{k\} \longmapsto s(k)^{q}$ (s is a section of ∂).*

Proof. We have to check that f preserves the relations (3)–(6) in mod-q tensor product. We first consider the relation (3). Then we have $f(\{k\}(a \otimes^{q} b)\{k\}^{-1}) = s(k)^{q}\{a, b\}s(k)^{-q} = {}^{k^{q}}\{a, b\} = \{{}^{k^{q}}a, {}^{k^{q}}b\} = f({}^{k^{q}}a \otimes^{q} {}^{k^{q}}b)$. We next consider the relation (4). Then we have $f([\{k\}, \{h\}]) = [s(k)^{q}, s(h)^{q}] = {}^{s(k)^{q}}s(h)^{q}(s(h)^{q})^{-1} = {}^{k^{q}}s(h)^{q}(s(h)^{q})^{-1} = \{k^{q}, h^{q}\}$. For the relation (5), we have $f(\{kh\}) = s(kh)^{q} = (s(k)s(h))^{q} = s(k)^{q}(\overset{q-1}{\Pi} [(s(k)^{-1}, ({}^{(k)^{1-q+i}}h)^{i}])s(h)^{q} = s(k)^{q}(\overset{q-1}{\Pi} \{k^{-1}, ({}^{(k)^{1-q+i}}h)^{i}\})s(h)^{q}$. Finally, we consider the relation (6). Then we have f $(\{(k^{h}k^{-1}, {}^{k}hh^{-1})\}) = s([k, h])^{q}$, and because s([k,h]) and {k,h} have the same image under ∂, $s([k, h])^{q}$ coincides with $\{k, h\}^{q}$. □

We proceed to construct a canonical braiding on $\rho : N \otimes G \longrightarrow G \otimes G$ when $N \longrightarrow G$ is braided with a braiding $\{\ ,\ \}$. Define $\underline{\{\ ,\ \}} : G \otimes G \times G \otimes G \longrightarrow N \otimes G$ by

$$\underline{\{\ ,\ \}} : (a \otimes b, c \otimes d) \longmapsto \{a, b\} \otimes [c, d].$$

Then we have the following proposition:

Proposition 3 $\underline{\{\ ,\ \}}$ *satisfies the braiding conditions.*

Proof. The proof is by computations:

We first consider the identity (1). If we take $a = a \otimes b$, $b = c \otimes d$, we have $\rho(\underline{\{a \otimes b, c \otimes d\}}) = \rho(\{a,b\} \otimes [c,d]) = \partial\{a,b\} \otimes [c,d] = [a,b] \otimes [c,d]$, so that we need the following identity:

$$(a \otimes b)(c \otimes d)(a \otimes b)^{-1}(c \otimes d)^{-1} = [a,b] \otimes [c,d],$$

but this is the product of (a) and (b) in page 4.

The identities (2) and (3) are proved by a result in Brown, Loday [6]. Alternatively, one can prove them using a technique which will be described in Lemma 1.

We next consider the identity (4). If we take $a = a \otimes b$ and $bc = (c \otimes d)(c' \otimes d')$, we have $\underline{\{a \otimes b, (c \otimes d)(c' \otimes d')\}} = \{a,b\} \otimes [c,d][c',d']$. On the other hand, we have $\underline{\{a \otimes b, c \otimes d\}}^{c \otimes d}\underline{\{a \otimes b, c' \otimes d'\}} = (\{a,b\} \otimes [c,d])^{c \otimes d}(\{a,b\} \otimes [c',d') = (\{a,b\} \otimes [c,d])({}^{[c,d]}\{a,b\} \otimes {}^{[c,d]}[c',d']) = \{a,b\} \otimes [c,d][c',d']$.

Finally, we consider the identity (5). If we take $ab = (a \otimes b)(a' \otimes b')$ and $c = c \otimes d$, we have $\underline{\{(a \otimes b)(a' \otimes b'), c \otimes d\}} = \{a,b\}\{a',b'\} \otimes [c,d]$. On the other hand, ${}^{a \otimes b}\underline{\{a' \otimes b', c \otimes d\}\{a \otimes b, c \otimes d\}} = {}^{a \otimes b}(\{a',b'\} \otimes [c,d])(\{a,b\} \otimes [c,d]) = ({}^{[a,b]}\{a',b'\} \otimes {}^{[a,b]}[c,d])(\{a,b\} \otimes [c,d]) = ({}^{\{a,b\}}\{a',b'\} \otimes {}^{\{a,b\}}[c,d])(\{a,b\} \otimes [c,d]) = \{a,b\}\{a',b'\} \otimes [c,d]$. □

Remark 1. In (4), (5) the property $\partial(\{a,b\}) = [a,b]$ and ${}^{\partial(n)}n' = nn'n^{-1}$ were used.

When a crossed modules $N \longrightarrow G$ is a q-central extension of G and equipped with a braiding $\{\ ,\ \}$, one can use Proposition 2 to define a canonical braiding $\underline{\{\ ,\ \}}^q$ on $N \otimes^q G \longrightarrow G \otimes^q G$.

Before checking the braiding conditions, we prove the next lemma.

Lemma 1 *In $N \otimes^q G$, the next identities hold*:

(a) $a^b a^{-1} \otimes^q h^q = (a \otimes^q b)({}^{h^q}a \otimes^q {}^{h^q}b)^{-1}$,

(b) $\{n\}^q \otimes^q [a,b] = \{n\}\{{}^{[a,b]}n\}^{-1}$,

(c) $n^q \otimes^q h^q = \{n\}\{{}^{h^q}n\}^{-1}$.

Proof. Recall that for two crossed modules (M, G, ∂) and (N, G, ∂'), Doncel-Juárez and Grandjeán L.-Valcárcel constructed the following crossed module $\rho : M \otimes^q N \longrightarrow G \otimes^q G$:

$$\rho(m \otimes n) = \partial(m) \otimes \partial'(n), \rho(\{k\}) = \{\partial(\pi_1(k))\}$$

$$^{(a\otimes b)}(m\otimes n) = {}^{[a,b]}m \otimes {}^{[a,b]}n,\ {}^{(a\otimes b)}(\{k\}) = \{{}^{[a,b]}k\},$$
$$^{\{h\}}(m\otimes n) = {}^{h^q}m \otimes {}^{h^q}n,\ {}^{\{h\}}(\{k\}) = \{{}^{h^q}k\},$$

($[a,b] = aba^{-1}b^{-1}, a, b, h \in G, m \in M, n \in N, k \in M \times_G N, \pi_1 : M \times_G N \longrightarrow M$), and proved that $N \otimes^q G \longrightarrow G \otimes^q G$ becomes the universal central extension of a crossed module $N \longrightarrow G$.

To prove the identities $(a) \sim (c)$, we use the universality of $N\otimes^q G$, and show that, for any q-central extension (X_1, X_2, ∂') of (N, G, ∂), the unique map $\varphi_1 : N\otimes^q G \longrightarrow X_1$ defined by $\varphi_1(n\otimes^q g) = s_1(n)^{s_2(g)}s_1(n)^{-1}, \varphi_1(\{h\}) = s_1(h)^q$, where s_1 and s_2 are sections of $\psi_1 : X_1 \to N$ and $\psi_2 : X_2 \to G$ respectively, preserves the relations.

We first check the identity (a). By the definition, we have

$$\varphi_1(a^ba^{-1} \otimes^q h^q) = s_1(a^ba^{-1})^{s_2(h^q)}s_1(a^ba^{-1})^{-1}.$$

But because $s_1(a^ba^{-1})^{s_2(h^q)}s_1(a^ba^{-1})^{-1}$ has a form x^yx^{-1} in X_1, we can change $s_1(a^ba^{-1})$ to $s_1(a)^{s_2(b)}s_1(a)^{-1}$. Then we have

$$\begin{aligned} &s_1(a^ba^{-1})^{s_2(h^q)}s_1(a^ba^{-1})^{-1} \\ &\quad = (s_1(a)^{s_2(b)}s_1(a)^{-1})^{s_2(h^q)}(s_1(a)^{s_2(b)}s_1(a)^{-1})^{-1}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &\varphi_1((a\otimes b)({}^{h^q}a \otimes^q {}^{h^q}b)^{-1}) \\ &\quad = (s_1(a)^{s_2(b)}s_1(a)^{-1})\varphi_1({}^{h^q}a \otimes^q {}^{h^q}b)^{-1}) \\ &\quad = (s_1(a)^{s_2(b)}s_1(a)^{-1})(s_1({}^{h^q}a)^{s_2({}^{h^q}b)}s_1({}^{h^q}a)^{-1})^{-1}. \end{aligned}$$

Hence we should prove the formula:

$$^{s_2(h^q)}(s_1(a)^{s_2(b)}s_1(a)^{-1})^{-1} = (s_1({}^{h^q}a)^{s_2({}^{h^q}b)}s_1({}^{h^q}a)^{-1})^{-1},$$

but notice that the latter has the form $(x^yx^{-1})^{-1}$. Thus we can replace $s_1({}^{h^q}a)$ by ${}^{s_2(h^q)}s_1(a)$ and $s_2({}^{h^q}b)$ by $s_2(h^q)s_2(b)s_2(h^q)^{-1}$.

We next check the identity (b). By the definition, we have

$$\begin{aligned} \varphi_1(\{n\}^q \otimes^q [a,b]) &= s_1(n^q)^{s_2([a,b])}s_1(n^q)^{-1} \\ &= (s_1(n)^q)^{s_2([a,b])}(s_1(n)^q)^{-1}. \end{aligned}$$

On the other hand, we have

$$\varphi_1(\{n\}\{{}^{[a,b]}n\}^{-1}) = s_1(n)^q(s_1({}^{[a,b]}n)^q)^{-1}.$$

But because ${}^{s_2([a,b])}s_1(n)$ and $s_1({}^{[a,b]}n)$ have the same image by $\psi_1 : X_1 \longrightarrow N$, one can see that, by the property of q-central extensions of a crossed module, ${}^{s_2([a,b])}(s_1(n)^q)^{-1}$ coincides with $(s_1({}^{[a,b]}n)^q)^{-1}$.

Finally, we check the identity (c). By the definition, we have

$$\varphi_1(n^q \otimes^q h^q) = s_1(n^q)^{s_2(h^q)}s_1(n^q)^{-1} = s_1(n)^q({}^{s_2(h^q)}s_1(n))^{-q}.$$

On the other hand, we have

$$\varphi_1(\{n\}\{{}^{h^q}n\}^{-1}) = s_1(n)^q s_1({}^{h^q}n)^{-q}.$$

But one can easily see that ${}^{s_2(h^q)}s_1(n)$ and $s_1({}^{h^q}n)$ have the same image by ψ_1. Thus the result follows. □

Proposition 4 $\underline{\{\ ,\ \}}^q$ *becomes a braiding on* $N \otimes^q G \longrightarrow G \otimes^q G$.

Proof. By the end of this proof, we denote $\{\ ,\ \}^q$ by $\{\ ,\ \}$. When the elements $\{k\}$ do not appear in the relations, they are derived from the results for $\underline{\{\ ,\ \}}$. So we consider the case where the elements $\{k\}$ are appearing in the relations.

We first consider the relation (1). If we take $a = \{k\}$ and $b = c \otimes^q d$, we have $\rho\underline{\{\{k\}, c \otimes^q d\}} = \rho(s(k)^q \otimes^q \{c,d\}') = k^q \otimes^q [c,d]$. On the other hand, we have $\{k\}(c \otimes^q d)\{k\}^{-1}(c \otimes^q d)^{-1} = ({}^{k^q}c \otimes^q {}^{k^q}d)(c \otimes^q d)^{-1}$. Hence we need the identity:

$$k^q \otimes^q [c,d] = ({}^{k^q}c \otimes^q {}^{k^q}d)(c \otimes^q d)^{-1},$$

but this is the formula (c) applid to mod-q tensor product with a $= k^q$, b = c, c = d.

We next consider the relation (2). If we take $n = a \otimes^q b$ and $b = \{h\}$, then by the definition we have $\underline{\{\partial(a) \otimes^q b, \{h\}\}} = \{\partial(a), b\} \otimes^q h^q = a^b a^{-1} \otimes^q h^q$. On the other hand, we have $(a \otimes^q b)^{\{h\}}(a \otimes^q b)^{-1} = (a \otimes^q b)({}^{h^q}a \otimes^q {}^{h^q}b)^{-1}$. Thus by Lemma 1 (a), they coincide. If we take $n = \{n\}$ and $b = a \otimes^q b$, then we have $\underline{\{\rho\{n\}, a \otimes^q b\}} = n^q \otimes^q [a,b]$. On the other hand, we have $\{n\}^{a \otimes^q b}\{n\}^{-1} = \{n\}\{{}^{[a,b]}n\}^{-1}$. Thus by Lemma 1 (b), they coincide. If we take $n = \{n\}$ and $b = \{h\}$, we have $\underline{\{\rho\{n\}, \{h\}\}} = n^q \otimes^q h^q$. On the other hand, we have $\{n\}^{\{h\}}\{n\}^{-1} = \{n\}\{{}^{h^q}n\}^{-1}$. Thus by Lemma 1 (c), they coincide.

The relation (3) follows by the same computations.

We next consider the relation (4). If we take $a = \{k\}$ and $bc =$

$(a \otimes^q b)(c \otimes^q d)$, we have $\underline{\{\{k\}, (a \otimes^q b)(c \otimes^q d)\}} = s(k)^q \otimes^q [a,b][c,d]$. On the other hand, we have $\underline{\{\{k\}, a \otimes^q b\}}^{(a\otimes^q b)}\underline{\{\{k\}, c \otimes^q d\}} = (s(k)^q \otimes^q [a,b])$ $({}^{(a\otimes b)}(s(k)^q \otimes^q [c,d])) = (s(k)^q \otimes^q [a,b])({}^{[a,b]}s(k)^q \otimes^q {}^{[a,b]}[c,d]) = s(k)^q \otimes^q [a,b][c,d]$. If we take $a = \{k\}$ and $bc = \{h\}(c \otimes^q d)$, we have $\underline{\{\{k\}, \{h\}(c \otimes^q d)\}} = s(k)^q \otimes^q s(h)^q[c,d]$. On the other hand, we have $(\underline{\{\{k\}, \{h\}\}})({}^{\{h\}}\underline{\{\{k\}, c \otimes^q d\}}) = (s(k)^q \otimes^q s(h)^q)({}^{\{h\}}(s(k)^q \otimes^q [c,d]) = (s(k)^q \otimes^q s(h)^q)({}^{h^q}s(k)^q \otimes^q {}^{h^q}[c,d]) = s(k)^q \otimes^q s(h)^q[c,d]$. If we take $a = \{k\}$ and $bc = (c \otimes^q d)\{h\}$, we have $\underline{\{\{k\}, (c \otimes d)\{h\}\}} = s(k)^q \otimes^q [c,d]s(h)^q$. On the other hand, we have $\underline{\{\{k\}, c \otimes^q d\}}^{(c\otimes^q d)}\underline{\{\{k\}, \{h\}\}} = (s(k)^q \otimes^q [c,d])^{c\otimes d}(s(k)^q \otimes^q s(h)^q) = (s(k)^q \otimes^q [c,d])({}^{[c,d]}s(k)^q \otimes^q {}^{[c,d]}s(h)^q) = s(k)^q \otimes^q [c,d]s(h)^q$.

(5) Omitted. □

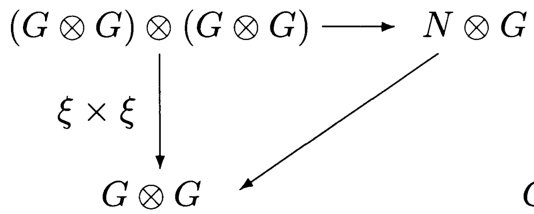
We have so far been concerned with constructing canonical braidings on the crossed modules $N \otimes G \longrightarrow G \otimes G$ and $N \otimes^q G \longrightarrow G \otimes^q G$. Since it is known that $N \otimes G \longrightarrow G \otimes G$ ($N \otimes^q G \longrightarrow G \otimes^q G$) are the universal ($q$-universal) central extensions of perfect (q-perfect) crossed modules $N \longrightarrow G$, it is quite natural to consider their braided version.

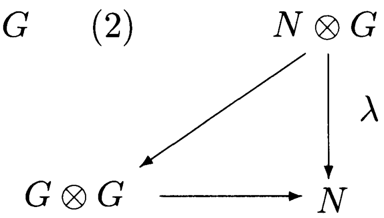
The next proposition shows that the canonical braidings $\underline{\{\ ,\ \}}$ on the crossed modules $N \otimes G \longrightarrow G \otimes G$ are compatible with $\{\ ,\ \}$.

Proposition 5 *The next diagram becomes commutative.*

$$\begin{array}{ccc} (G \otimes G) \otimes (G \otimes G) & \longrightarrow & N \otimes G \\ {\scriptstyle \xi \times \xi}\downarrow & & \downarrow{\scriptstyle \lambda} \\ G \otimes G & \longrightarrow & N \end{array}$$

Proof. It is enough to show that the next diagrams commute:

(1) 

(2) 

The diagram (1) becomes commutative because of the braiding condition (1). The triangle (2) also becomes commutative by the braiding condition (2) for $\{\ ,\ \}$. □

Thus we know that the braided crossed module $(N \otimes G \longrightarrow G \otimes G, \underline{\{\ ,\ \}})$ is an extension of $(N \longrightarrow G, \{\ ,\ \})$. Furthermore, this braiding has a universal property.

Theorem 1 *If $(N \longrightarrow G, \{\ ,\ \})$ is a perfect braided crossed module, and $(X_1 \xrightarrow{\Omega} X_2, \{\ ,\ \}')$ is a central extension of it with a compatible braiding, then the next diagram becomes commutative.*

$$\begin{array}{ccc}
(G \otimes G) \times (G \otimes G) & \xrightarrow{\underline{\{\,,\,\}}} & N \otimes G \\
\downarrow & & \downarrow \\
X_2 \times X_2 & \xrightarrow{\{\,,\,\}'} & X_1
\end{array}$$

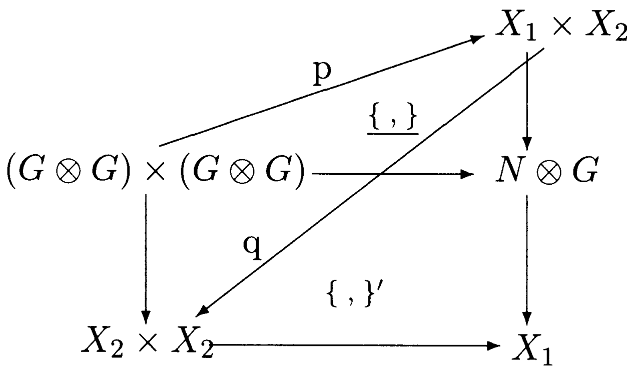
Proof. Define

$r : G \otimes G \longrightarrow X_1$ to be r $= \{\ ,\ \}' \circ s_2$ (by choosing a section $s_2 : G \longrightarrow X_2$ and extending it on $G \otimes G$),

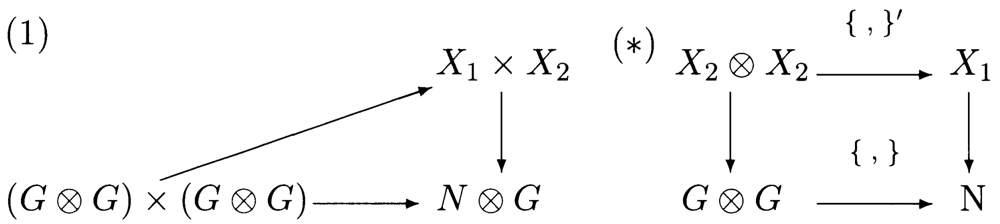
$t : G \otimes G \longrightarrow X_2$, $a \otimes b \longmapsto [s_2(a), s_2(b)]$, by the same s_2,

p $= r \times t$, q $= \Omega \times id$.

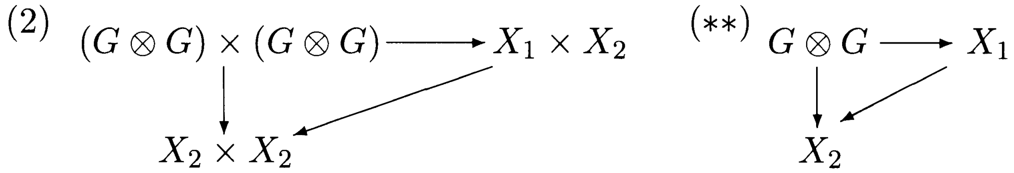
Let us consider the next diagram and show that each triangle commutes.



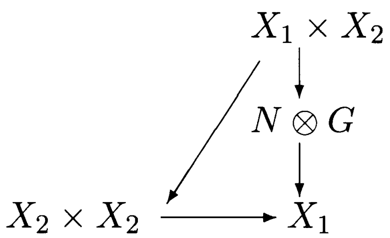
By the definitions, the diagram (1) becomes naturally commutative because the diagram $(*)$ is commutative.



The next diagram (2) also becomes commutative because the diagram (∗∗) is commutative by the braiding condition (2) and the choice of r.



Finally let us see the next diagram commutes.



It follows again by the braiding condition (2) and the constructions. □

Corollary 1 *If* $(N \longrightarrow G, \{\ ,\ \})$ *is a q-perfect braided crossed module with N being a q-central extension of G, then* $(N \otimes^q G \longrightarrow G \otimes^q G, \underline{\{\ ,\ \}}^q)$ *becomes the universal q-central extension of it.*

It follows because we can construct the similar maps by $r(\{k\}) = (s_1 \circ s(k))^q$ and $t(\{k\}) = (\omega \circ s_1 \circ s(k))^q$.

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