

A note on the Poincaré polynomial of an arrangement

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Abstract. Let $V = \mathbb{K}^\ell$ be a vector space, where $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . A *hyperplane* in V is an affine subspace of dimension $\ell - 1$. An *arrangement* \mathcal{A} is a finite set of hyperplanes in V . Let $L = L(\mathcal{A})$ be the set of intersections of the hyperplanes of \mathcal{A} , partially ordered by reverse inclusion. Let μ be the Möbius function on L , and define a rank function on L by $r(X) = \ell - \dim X$. The *Poincaré polynomial* on \mathcal{A} is given by

$$\pi(\mathcal{A}, t) = \sum_{X \in L} \mu(X)(-t)^{r(X)}.$$

For $X \in L$, define the combinatorial sum

$$p(X) = (-1)^{r(X)} \sum_{X \leq Z} \mu(Z)r(Z).$$

Both the Poincaré polynomial and the quantity $p(X)$ have physical interpretations in certain cases (see the work of Zaslavsky and Varchenko, respectively).

In this paper, we prove an identity involving the Poincaré polynomial and $p(X)$ and show two applications which have connections to the work of Varchenko. The first is a chamber-counting result with an interpretation when $\mathbb{K} = \mathbb{R}$, the second a result related to the Euler beta function, defined by Varchenko when $\mathbb{K} = \mathbb{C}$.

Key words: arrangement, hyperplane, Poincaré polynomial.

1. Introduction

Let \mathbb{K} be a field, and let V be a vector space over \mathbb{K} of dimension ℓ . A *hyperplane* H in V is an affine subspace of dimension $(\ell - 1)$. An *arrangement* \mathcal{A} is a finite set of hyperplanes in V . When we wish to emphasize the dimension of V , we call \mathcal{A} an ℓ -arrangement. When we wish to emphasize the vector space itself, we write (\mathcal{A}, V) to denote the arrangement.

We refer to [3] for terminology and basic results. Let $L = L(\mathcal{A})$ be the set of intersections of the hyperplanes of \mathcal{A} , partially ordered by reverse inclusion. We may define a rank function on the elements (*edges*) of L by $r(X) = \text{codim } X = \ell - \dim X$. We may also define a meet and a join operation on $L(\mathcal{A})$ which give it the properties of a geometric poset.

Let $X \wedge Y = \cap\{Z \in L \mid X \cup Y \subseteq Z\}$ be the *join* of X and Y , and let $X \vee Y = X \cap Y$ be their *meet*. V is the unique minimal element of the poset. If the intersection of all of the elements of \mathcal{A} is nonempty, then we say that \mathcal{A} is *central*. In this case, $L(\mathcal{A})$ is a geometric lattice with a unique maximal element $T = T(\mathcal{A})$ given by the intersection of the hyperplanes of \mathcal{A} . Whether the arrangement is central or not, the maximal elements of $L(\mathcal{A})$ have the same rank, and we may define the rank of the arrangement $r(\mathcal{A})$ as the rank of a maximal element of $L(\mathcal{A})$.

Let μ be the Möbius function of L . That is, define $\mu : L \times L \rightarrow \mathbb{Z}$ by

$$\begin{aligned} \mu(X, X) &= 1 \quad \text{if } X \in L, \\ \sum_{X \leq Z \leq Y} \mu(X, Z) &= 0 \quad \text{if } X, Y, Z \in L \text{ and } X < Y, \\ \mu(X, Y) &= 0 \quad \text{otherwise.} \end{aligned}$$

Note that for fixed X , the values of $\mu(X, Y)$ may be computed recursively. For $X \in L$, define $\mu(X) = \mu(V, X)$.

Define the *Poincaré polynomial* of \mathcal{A} by

$$\pi(\mathcal{A}, t) = \sum_{X \in L} \mu(X) (-t)^{r(X)}.$$

The Poincaré polynomial is a degree $r(\mathcal{A})$ polynomial in t with nonnegative coefficients.

In this paper, we prove an identity involving the Poincaré polynomial which has useful applications, particularly in the work of Varchenko. In the next section, we define some terminology necessary for the statement of the main theorem. In Section 3, we discuss some of the applications to Varchenko's work. In Section 4, we give a proof of the main theorem, which is lattice-theoretic in nature. An example of the theorem and its applications is given in Section 5.

2. Necessary constructions and statement of the main theorem

The Poincaré polynomial is one of the most important combinatorial invariants of an arrangement, and its properties have been extensively studied. Zaslavsky, in [7], showed that the Poincaré polynomial can be used as a counting function in the case when the underlying field \mathbb{K} is the real numbers (i.e. \mathcal{A} is a *real* arrangement). In this case, the hyperplanes of \mathcal{A} separate V into open disjoint convex *chambers*. That is, the complement of

the union of the hyperplanes is a union of open, disjoint subsets of V . Let $\mathcal{C}(\mathcal{A})$ and $\mathcal{B}(\mathcal{A})$ be the collections of chambers and bounded chambers of \mathcal{A} , respectively. Zaslavsky gives us a means of counting these chambers:

Theorem 2.1 *Let \mathcal{A} be a real arrangement. Then*

- (i) $|\mathcal{C}(\mathcal{A})| = \pi(\mathcal{A}, 1)$.
- (ii) *If the maximal elements of the intersection poset are points (i.e. \mathcal{A} is essential), then $|\mathcal{B}(\mathcal{A})| = (-1)^{r(\mathcal{A})} \pi(\mathcal{A}, -1)$.*

In order to state the basic result, we need several additional constructions involving arrangements. Except where noted, we again refer to [3] for the definitions.

Let (\mathcal{A}, V) be an arrangement. If $\mathcal{B} \subseteq \mathcal{A}$ is a subset, then \mathcal{B} is called a *subarrangement*. For $X \in L(\mathcal{A})$ define a subarrangement \mathcal{A}_X of \mathcal{A} by

$$\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subseteq H\}.$$

Note that \mathcal{A}_V is an empty arrangement. If $X \neq V$, \mathcal{A}_X is a central arrangement, and X is the intersection of all the planes in \mathcal{A}_X . \mathcal{A}_X is sometimes called the *localization of \mathcal{A} to X* .

We may also define an arrangement (\mathcal{A}^X, X) in X by

$$\mathcal{A}^X = \{X \cap H \mid H \in \mathcal{A} \setminus \mathcal{A}_X \text{ and } X \cap H \neq \emptyset\}.$$

We call \mathcal{A}^X the *restriction of \mathcal{A} to X* .

Two other constructions which will prove useful are the inverse operations of coning and deconing. In order to describe these operations, we first give coordinates to vector space V .

Let V^* be the dual space of V , the space of linear forms on V . Let $S = S(V^*)$ be the symmetric algebra of V^* . Choose a basis $\{e_1, \dots, e_\ell\}$ in V , and let $\{x_1, \dots, x_\ell\}$ be the dual basis in V^* so that $x_i(e_j) = \delta_{i,j}$. $S(V^*)$ can be identified with the polynomial algebra $S = \mathbb{K}[x_1, \dots, x_\ell]$. Each hyperplane $H \in \mathcal{A}$ is the kernel of a polynomial α_H of degree 1 defined up to a constant. The product $Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} \alpha_H$ is called a *defining polynomial of \mathcal{A}* .

An affine ℓ -arrangement \mathcal{A} defined by $Q(\mathcal{A}) \in S$ gives rise to a central $(\ell + 1)$ -arrangement $c\mathcal{A}$ called the *cone over \mathcal{A}* . Let $\hat{Q} \in \mathbb{K}[x_0, x_1, \dots, x_\ell]$ be the polynomial $Q(\mathcal{A})$ homogenized, and let $c\mathcal{A}$ be given by the defining polynomial $Q(c\mathcal{A}) = x_0 \hat{Q}$. Note that $|c\mathcal{A}| = |\mathcal{A}| + 1$. We call $H_0 = \ker(x_0)$ the *additional hyperplane*. If $H \in \mathcal{A}$ is defined by the linear form α_H ,

then cH is the hyperplane in $c\mathcal{A}$ defined by the homogenization of α_H . Moreover, if X is any element of $L(\mathcal{A})$, we write $X = H_1 \cap \cdots \cap H_p$, where $H_i \in \mathcal{A}$. Define $cX = cH_{i_1} \cap \cdots \cap cH_{i_p} \in L(c\mathcal{A})$. cX is independent of the representation of X as an intersection of hyperplanes. In this way, the poset $L(\mathcal{A})$ naturally embeds in $L(c\mathcal{A})$.

There is an inverse operation to the coning construction. A nonempty central $(\ell + 1)$ -arrangement \mathcal{A} gives rise to an ℓ -arrangement $d\mathcal{A}$, which is in general not central, by the *deconing* construction. Choose a *distinguished* hyperplane $H_0 \in \mathcal{A}$. Choose coordinates so that $H_0 = \ker(x_0)$. Let $Q(\mathcal{A}) \in \mathbb{K}[x_0, x_1, \dots, x_\ell]$ be a defining polynomial for \mathcal{A} . The defining polynomial $Q(d\mathcal{A})$ is obtained by substituting 1 for x_0 in $Q(\mathcal{A})$. It should be pointed out that the arrangement $d\mathcal{A}$ depends on the choice of the distinguished hyperplane H_0 . Deconing with respect to different hyperplanes gives different arrangements with (possibly) different intersection posets.

The following proposition relates the Poincaré polynomials of an arrangement and its cone:

Proposition 2.2 [3] *Let \mathcal{A} be an affine arrangement with cone $c\mathcal{A}$. Then*

$$\pi(c\mathcal{A}, t) = (1 + t)\pi(\mathcal{A}, t).$$

Equivalently, Proposition 2.2 states that for any central arrangement \mathcal{A} and $H_0 \in \mathcal{A}$, $\pi(d\mathcal{A}, t) = \frac{\pi(\mathcal{A}, t)}{1+t}$. In particular, the Poincaré polynomial of the deconed arrangement does *not* depend on the distinguished hyperplane.

A fact that will prove useful for our purposes is that the operations of coning and restricting commute.

Lemma 2.3 *For any $X \in L(\mathcal{A})$, the intersection posets $L((c\mathcal{A})^X)$ and $L(c(\mathcal{A}^X))$ are naturally isomorphic.*

Proof. Let H_0 be the additional hyperplane in $c\mathcal{A}$. Then $H_0 \cap X$ serves as the additional hyperplane in $c(\mathcal{A}^X)$. \square

Finally, let \mathcal{A} be an arrangement. Define the combinatorial sum $p : L(\mathcal{A}) \rightarrow \mathbb{Z}$ by

$$p(X) = (-1)^{r(X)} \sum_{Z \leq X} \mu(Z)r(Z).$$

The sum is over all elements $Z \in L(\mathcal{A})$ which lie below X in the poset.

$p(X)$ is also known as the *beta invariant* of the matroid $L(\mathcal{A}_X)$. Crapo

first defined the beta invariant in [1], within the context of matroid theory (See Theorem III of [1], for example). We use the notation $p(X)$ for the beta invariant to agree with Varchenko in [5] and [6] and to avoid confusion when we consider the Euler beta function in Section 3.

In [5], Varchenko gives an interpretation of $p(X)$ when \mathcal{A} is a real arrangement. Every element $X \in L(\mathcal{A})$ of codimension r is associated to an arrangement in $(r - 1)$ dimensional projective space. Namely, let N be an r dimensional normal to X , and consider the arrangement obtained by projecting the hyperplanes of \mathcal{A}_X onto N . This yields an arrangement $\tilde{\mathcal{A}}_X$, which has the same intersection lattice as \mathcal{A}_X but is essential, with all the new hyperplanes passing through the point $v = X \cap N$. Now consider the usual antipodal map $s : S^r \rightarrow \mathbb{R}P^{r-1}$. The planes of $\tilde{\mathcal{A}}_X$ intersect the unit sphere S^r transversally, and under s , $\tilde{\mathcal{A}}_X$ is mapped to a projective arrangement $P\mathcal{A}_X$ in $\mathbb{R}P^{r-1}$. $P\mathcal{A}_X$ is referred to as the *projective localization of \mathcal{A} at X* .

A chamber in the complement of an arrangement is said to be *bounded away from* (or *bounded relative to*) a given hyperplane if the closure of the chamber does not intersect the hyperplane.

Proposition 2.4

- (i) For any arrangement \mathcal{B} in a real projective space, the number of the chambers which are bounded away from a given hyperplane $H_0 \in \mathcal{B}$ does not depend on H_0 . In particular,
- (ii) If \mathcal{A} is an arrangement in real affine space with $X \in L(\mathcal{A})$, then in the projective localization $P\mathcal{A}_X$, the number of chambers which are bounded away from a given hyperplane is $p(X)$.

Varchenko gives a proof of Proposition 2.4 in [5]. We give a slightly different proof here. Any arrangement in real projective $(r - 1)$ dimensional space gives rise to an affine arrangement in \mathbb{R}^{r-1} by deleting the hyperplane at infinity. Then the number of chambers bounded away from a given hyperplane in the projective arrangement can be determined by counting the number of bounded chambers in the associated real affine arrangement: the bounded chambers in the real arrangement are those chambers which are bounded away from the hyperplane at infinity.

In the case of the projective localization $P\mathcal{A}_X$, the number of chambers bounded away from a given hyperplane is exactly $p(X)$. Recalling our definition of the deconing construction given earlier, choose $H_0 \in \mathcal{A}_X$. Then the

operation of deconing with respect to H_0 may be viewed as first projectivizing \mathcal{A}_X , making the image of H_0 the hyperplane at infinity, then removing this hyperplane at infinity. Then the number of chambers bounded away from a given hyperplane in $P\mathcal{A}_X$ is equivalent to the number of bounded chambers in $d\mathcal{A}_X$. From Zaslavsky's work, this number is

$$\begin{aligned} & (-1)^{r(X)-1} \pi(d\mathcal{A}_X, -1) \\ &= (-1)^{r(X)-1} \lim_{t \rightarrow -1} \frac{\pi(\mathcal{A}_X, t)}{1+t} \end{aligned} \quad (2.5)$$

$$= (-1)^{r(X)-1} \left. \frac{d}{dt} \pi(\mathcal{A}_X, t) \right|_{t=-1} \quad (2.6)$$

$$= (-1)^{r(X)-1} \left. \frac{d}{dt} \left(\sum_{Z \leq X} \mu(Z) (-t)^{r(Z)} \right) \right|_{t=-1} \quad (2.7)$$

$$= (-1)^{r(X)} \sum_{Z \leq X} \mu(Z) r(Z) = p(X). \quad (2.8)$$

Since $\pi(d\mathcal{A}_X, -1)$ does not depend upon the distinguished hyperplane, the quantities in 2.5 through 2.8 are independent of H_0 .

In the proof of Proposition 2.4, The equalities of 2.5–2.8 give us two more facts about $p(X)$.

Corollary 2.9 *For any $X \in L(\mathcal{A})$, $p(X)$ is a nonnegative integer.*

Corollary 2.9 was proven by Crapo in Theorem II of [1].

The equalities of 2.5–2.8 can also be found in the proof of [4, Proposition 4]. In that proposition, Schectman, Terao, and Varchenko show the following:

Proposition 2.10 *Let \mathcal{A} be an arrangement in complex projective space. Then, up to sign, $p(X)$ is the Euler characteristic of the complement of \mathcal{A}_X in $\mathbb{C}P^\ell$.*

Proof. The Euler characteristic of the complement of \mathcal{A}_X is $\left| \frac{d}{dt} \pi(\mathcal{A}_X, t) \right|_{t=-1}$ up to a sign. \square

Schechtman, Varchenko, and Terao, in [4] discuss the beta invariant $p(X)$ in the context of decomposable arrangements and dense edges of an arrangement. They also give necessary and sufficient conditions for determining when $p(X)$ is identically zero.

With all our definitions, we can now state our main result:

Theorem 2.11 *Let \mathcal{A} be a central arrangement. Then for any $H_0 \in \mathcal{A}$,*

$$\sum_{H_0 \leq X} t^{r(X)-1} \pi(\mathcal{A}^X, t) p(X) = \pi(d\mathcal{A}, t). \tag{2.12}$$

3. Applications

3.1. Varchenko’s bilinear form

In [6], Varchenko establishes a determinant formula for a bilinear form defined on real arrangements. As part of his proof, he establishes the following counting formula:

Proposition 3.1 *Let \mathcal{A} be a central real arrangement, and let $H_0 \in \mathcal{A}$. For $X \in \mathcal{A}$, let $n(X) = |\mathcal{C}(\mathcal{A}^X)|$ denote the number of chambers in the complement of \mathcal{A}^X . Then*

$$2 \sum_{H_0 \leq X} n(X) p(X) = n,$$

where n is the number of chambers in the complement of \mathcal{A} .

Varchenko uses a counting argument to prove this proposition. However, one can see that the proposition is also a direct result of Theorem 2.11 in the specific case when $t = 1$. In particular, note that the quantities $n(X)$, $p(X)$ and n in the lemma may be obtained combinatorially from the intersection poset. Although the result has a pleasant physical interpretation in the case of real arrangements, the identity is nevertheless true for all central arrangements.

One might be tempted to consider in a similar way, the specific case when $t = -1$. However, when \mathcal{A} is a central arrangement, then for any $X \in L(\mathcal{A})$, \mathcal{A}^X is also central. As a result, when $X \neq T(\mathcal{A})$, \mathcal{A}^X has no bounded chambers in its complement, and hence $\pi(\mathcal{A}^X, -1) = 0$. On the other hand, $\pi(\mathcal{A}^T, -1) = 1$, and Equation 2.12 reduces to a restatement of the definition of $p(T)$:

$$p(T) = (-1)^{r(T)-1} \pi(d\mathcal{A}, -1).$$

In the next section, we look at an application of Theorem 2.11 in the case when $t = -1$, after rearranging terms slightly.

3.2. The Euler beta function

In [5], Varchenko defines a function related to his work on hypergeometric functions. Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an affine arrangement in \mathbb{C}^ℓ . Choose a system of weights $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ for \mathcal{A} so that each λ_i is the weight for the corresponding H_i . Define a new arrangement \mathcal{A}_∞ in $\mathbb{C}P^\ell$ obtained by adjoining the hyperplane at infinity to \mathcal{A} . That is, let $\mathcal{A}_\infty = \{\overline{H}_1, \dots, \overline{H}_n, H_\infty\}$, where \overline{H}_i is the image of H_i in $\mathbb{C}P^\ell$ under the natural inclusion. In this paper, we will not distinguish between a plane H_i and its image \overline{H}_i in \mathcal{A}_∞ . Give hyperplane H_∞ the weight $\lambda_\infty = -\sum_{i=1}^n \lambda_i$. Let $L = L(\mathcal{A})$ and $L_\infty = L(\mathcal{A}_\infty)$. Note that L embeds naturally in L_∞ , and that L_∞ is the lattice obtained by removing the top element $T(c\mathcal{A})$ from the lattice $L(c\mathcal{A})$.

Extend our weight system λ to a function $\lambda : L_\infty \rightarrow \mathbb{C}$ by

$$\lambda(X) = \sum_{\substack{H_i \leq X, \\ i \in \{1, \dots, n, \infty\}}} \lambda_i.$$

For $X \in L$, define $\delta(X) = |\pi(\mathcal{A}^X, -1) \frac{d}{dt} \pi(\mathcal{A}_X, t)|_{t=-1}$. δ is a map from L to the nonnegative integers. The definition of $\delta(X)$ can be extended to L_∞ , by extending the definition of $\pi(\mathcal{A}^X, t)$. For $X \in L_\infty \setminus L$, there exists some $i, 1 \leq i \leq n$ such that $H_i \not\leq X$. Decompose $c\mathcal{A}$ with respect to hyperplane H_i . We agree to let $\pi(\mathcal{A}^X, t) = \pi((d(c\mathcal{A}))^X, t)$. Note that for $X \in L_\infty \setminus L$, $\pi(\mathcal{A}^X, t)$ is independent of the chosen $H_i \not\leq X$, and so it is well defined. Since $p(X)$ depends on \mathcal{A}_X , it is clearly defined on L_∞ . Thus δ can be viewed as a map on L_∞ .

The Euler beta function is then given as

$$\beta(\mathcal{A}, \lambda) = \prod_{X \in L} \Gamma(\lambda(X) + 1)^{\delta(X)} \prod_{X \in L_\infty \setminus L} \Gamma(-\lambda(X) + 1)^{-\delta(X)} \quad (3.2)$$

It has been observed that for each i , $\beta(\mathcal{A}, \lambda)$ contains the same number of λ_i 's in the numerator and denominator. More precisely, for each $H_i \leq X$, λ_i will appear in the numerator of $\beta(\mathcal{A}, \lambda)$ with *multiplicity* $\delta(X)$. Similarly, for every $H_i \not\leq X$, with $X \in L_\infty \setminus L$, λ_i will appear in the denominator with multiplicity $\delta(X)$. Our observation asserts that for each i , the total multiplicity of λ_i in the numerator is the same as in the denominator.

Proposition 3.3 *Let $\mathcal{A}, \mathcal{A}_\infty, \delta, \lambda$ be as above. Then for each $i, 1 \leq i \leq n$,*

$$\sum_{\substack{H_i \leq X \\ X \in L}} \delta(X) = \sum_{\substack{H_i \not\leq X, \\ X \in L_\infty \setminus L}} \delta(X).$$

In [2], Loeser gives a case-by-case proof of this proposition. Theorem 2.11 provides us with an alternate, case-free proof.

It can first be seen that Proposition 3.3 is equivalent to the statement that the quantity

$$\sum_{\substack{X \in L_\infty, \\ H \leq X}} \delta(X) \tag{3.4}$$

is independent of $H \in \mathcal{A}_\infty$.

If we know that (3.4) is true, then we have that for any $H_i \in \mathcal{A}$,

$$\sum_{\substack{X \in L_\infty, \\ H_i \leq X}} \delta(X) = \sum_{\substack{X \in L_\infty, \\ H_\infty \leq X}} \delta(X)$$

Notice that $H_\infty \leq X$ if and only if $X \in L_\infty \setminus L$. Thus we have

$$\sum_{\substack{X \in L_\infty, \\ H_i \leq X}} \delta(X) = \sum_{X \in L_\infty \setminus L} \delta(X) \tag{3.5}$$

If we now subtract the quantity

$$\sum_{\substack{X \in L_\infty \setminus L \\ H_i \leq X}} \delta(X),$$

from both sides of Equation 3.5 we will have proven Proposition 3.3. It remains to prove (3.4). We shall prove a somewhat stronger proposition. Define a polynomial $\delta_t(X)$ on L_∞ by

$$\delta_t(X) = t^{r(X)-1} \pi(\mathcal{A}^X, t) p(X).$$

Then notice that the $\delta(X)$ of Proposition 3.3 is $\delta_{-1}(X)$. We shall prove the following proposition, of which Proposition 3.3 is an immediate consequence.

Proposition 3.6 *Let $\mathcal{A}, \mathcal{A}_\infty, \delta$, and λ be as above. Then*

$$\sum_{\substack{H \leq X, \\ X \in L_\infty}} \delta_t(X)$$

is independent of $H \in \mathcal{A}_\infty$.

Proof. Let $H = H_i$, where $i \in \{1, \dots, n, \infty\}$. Although \mathcal{A} is an affine arrangement, $c\mathcal{A}$ is central, and we may apply Theorem 2.11 to obtain

$$\sum_{\substack{H \leq X, \\ X \in L(c\mathcal{A})}} t^{r(X)-1} \pi((c\mathcal{A})^X, t) p(X) = \pi(\mathcal{A}, t) \quad (3.7)$$

On the left hand side of Equation 3.7, the term which occurs when $X = T = T(c\mathcal{A})$ is $t^{r(T)-1}p(T)$. We subtract this term from both sides to obtain

$$\sum_{\substack{H \leq X, \\ X \in L(c\mathcal{A}), \\ X \neq T}} t^{r(X)-1} \pi((c\mathcal{A})^X, t) p(X) = \pi(\mathcal{A}, t) - t^{r(T)-1}p(T). \quad (3.8)$$

Now, using Proposition 2.2 and Lemma 2.3, we find that for $X \neq T$, $\pi((c\mathcal{A})^X, t) = (1+t)\pi(\mathcal{A}^X, t)$. Dividing both sides of Equation 3.8 by $1+t$ gives

$$\sum_{\substack{H \leq X, \\ X \in L(c\mathcal{A}), \\ X \neq T}} t^{r(X)-1} \pi(\mathcal{A}^X, t) p(X) = \frac{\pi(\mathcal{A}, t) - t^{r(T)-1}p(T)}{1+t}.$$

L_∞ is naturally isomorphic to $L(c\mathcal{A})$ with the top element T removed. Thus,

$$\sum_{\substack{H \leq X, \\ X \in L_\infty}} t^{r(X)-1} \pi(\mathcal{A}^X, t) p(X) = \frac{\pi(\mathcal{A}, t) - t^{r(T)-1}p(T)}{1+t}. \quad (3.9)$$

Note that the left hand side of Equation 3.9 is

$$\sum_{\substack{H \leq X, \\ X \in L_\infty}} \delta_t(X).$$

Moreover, the right hand side of the equation is independent of $H \in \mathcal{A}$, and hence so is the left hand side. \square

4. Proof of the Main Theorem

Let $H_0 \in \mathcal{A}$ be our distinguished hyperplane. To prove Theorem 2.11, we shall work on the left hand side (LHS) and right hand side (RHS) of Equation 2.12 separately. Since both sides of the equation are polynomials in variable t , we shall simplify the left hand side, then show that the coefficients of the corresponding powers of t are the same on both sides.

Using our definitions of $\pi(\mathcal{A}^X, t)$ and $p(X)$, we have that the left hand side is

$$\begin{aligned}
 \text{LHS} &= \sum_{H_0 \leq X} t^{r(X)-1} \pi(\mathcal{A}^X, t) p(X) \\
 &= \sum_{H_0 \leq X} t^{r(X)-1} \left(\sum_{Y \geq X} \mu(X, Y) (-t)^{r(Y)-r(X)} \right) \left((-1)^{r(X)} \theta_X \right) \\
 &\quad \left(\text{where } \theta_X = \sum_{Z \leq X} \mu(Z) r(Z) \right) \\
 &= - \sum_{H_0 \leq X} \sum_{Y \geq X} (-t)^{r(X)-1} \mu(X, Y) (-t)^{r(Y)-r(X)} \theta_X \\
 &= - \sum_{H_0 \leq X} \sum_{Y \geq X} (-t)^{r(Y)-1} \mu(X, Y) \theta_X.
 \end{aligned}$$

Interchanging the order of summation, we get

$$\begin{aligned}
 \text{LHS} &= - \sum_{H_0 \leq Y} \sum_{H_0 \leq X \leq Y} (-t)^{r(Y)-1} \mu(X, Y) \theta_X \\
 &= - \sum_{H_0 \leq Y} \sum_{H_0 \leq X \leq Y} (-t)^{r(Y)-1} \mu(X, Y) \sum_{Z \leq X} \mu(Z) r(Z) \\
 &= - \sum_{H_0 \leq Y} (-t)^{r(Y)-1} \sum_{H_0 \leq X \leq Y} \sum_{Z \leq X} \mu(X, Y) \mu(Z) r(Z)
 \end{aligned}$$

Interchanging the order of the last two sums yields

$$\begin{aligned}
 \text{LHS} &= - \sum_{H_0 \leq Y} (-t)^{r(Y)-1} \sum_{Z \leq Y} \sum_{\substack{Z \leq X \leq Y, \\ H_0 \leq X}} \mu(X, Y) \mu(Z) r(Z) \\
 &= - \sum_{H_0 \leq Y} (-t)^{r(Y)-1} \sum_{Z \leq Y} \mu(Z) r(Z) \sum_{Z \vee H_0 \leq X \leq Y} \mu(X, Y).
 \end{aligned}$$

From [3, Lemma 2.38], we know that the last sum,

$$\sum_{Z \vee H_0 \leq X \leq Y} \mu(X, Y) = \begin{cases} 0 & \text{if } Z \wedge H_0 \neq Y \\ 1 & \text{if } Z \wedge H_0 = Y \end{cases}.$$

The only occurrence of a nonzero value in that final sum is when the variable Y in the second sum is $Z \vee H_0$. In that case the value of the sum is 1. Thus, we have that

$$\begin{aligned} \text{LHS} &= - \sum_{H_0 \leq Y} (-t)^{r(Y)-1} \sum_{Z \vee H_0 = Y} \mu(Z) r(Z) \\ &= - \sum_{H_0 \leq Y} \sum_{Z \vee H_0 = Y} (-t)^{r(Y)-1} \mu(Z) r(Z) \end{aligned}$$

Interchanging the order of summation one more time, we have

$$\begin{aligned} \text{LHS} &= - \sum_{Z \in L(\mathcal{A})} \sum_{Y = Z \vee H_0} (-t)^{r(Y)-1} \mu(Z) r(Z) \\ &= - \sum_{Z \in L(\mathcal{A})} (-t)^{r(Z \vee H_0)-1} \mu(Z) r(Z) \end{aligned}$$

Now, notice that

$$\begin{aligned} Z \vee H_0 &= Z \cap H_0 \\ &= \begin{cases} Z & \text{if } Z \subseteq H_0 \text{ (i.e. } H_0 \leq Z) \\ \text{a rank } r(Z) + 1 \text{ element} & \text{if } Z \not\subseteq H_0 \text{ (i.e. } H_0 \not\leq Z) \end{cases} \end{aligned}$$

So the rank of $Z \vee H_0$ can be given by

$$r(Z \vee H_0) = \begin{cases} r(Z) & \text{if } H_0 \leq Z \\ r(Z) + 1 & \text{if } H_0 \not\leq Z \end{cases}.$$

As a result, we have that

$$\text{LHS} = - \left(\sum_{H_0 \leq Z} (-t)^{r(Z)-1} \mu(Z) r(Z) + \sum_{H_0 \not\leq Z} (-t)^{r(Z)+1} \mu(Z) r(Z) \right).$$

As we stated at the start of the section, both the left hand side of our equation (computed above) and the right hand side are polynomials in the indeterminate t . We shall show that these polynomials are the same by showing that the corresponding coefficients of the t^i terms are equal.

On the left hand side, we have that the coefficient of t^i is

$$\begin{aligned}
 & - \left(\sum_{\substack{H_0 \leq Z, \\ r(Z)-1=i}} (-1)^{r(Z)-1} \mu(Z)r(Z) + \sum_{\substack{H_0 \not\leq Z, \\ r(Z)=i}} (-1)^{r(Z)} \mu(Z)r(Z) \right) \\
 & = - \left(\sum_{\substack{H_0 \leq Z, \\ r(Z)=i+1}} (-1)^i \mu(Z)(i+1) + \sum_{\substack{H_0 \not\leq Z, \\ r(Z)=i}} (-1)^i \mu(Z)i \right) \\
 & = (-1)^{i+1} \left((i+1) \sum_{\substack{H_0 \leq Z, \\ r(Z)=i+1}} \mu(Z) + i \sum_{\substack{H_0 \not\leq Z, \\ r(Z)=i}} \mu(Z) \right). \tag{4.1}
 \end{aligned}$$

Let $\mathcal{B} = (\{0\}, \mathbb{K})$ be the nonempty central 1-arrangement. Consider the product arrangement $(d\mathcal{A}) \times \mathcal{B}$ in $\mathbb{K}^{\ell-1} \oplus \mathbb{K} = \mathbb{K}^\ell$ given by

$$(d\mathcal{A}) \times \mathcal{B} = \{H \oplus \mathbb{K} \mid H \in d\mathcal{A}\} \cup \{\mathbb{K}^{\ell-1} \oplus \{0\}\}.$$

The posets $L((d\mathcal{A}) \times \mathcal{B})$ and $L(d\mathcal{A}) \times L(\mathcal{B})$ are naturally isomorphic (see [3, Proposition 2.14]) and thus the partial orders on $L(d\mathcal{A})$ and $L(\mathcal{B})$ induce a natural partial order on $L((d\mathcal{A}) \times \mathcal{B})$. Let $\hat{\mu}$ be the Möbius function on $L((d\mathcal{A}) \times \mathcal{B})$.

Now, if $Z \in L(\mathcal{A})$ with $Z \geq H_0$, then Proposition 2.43 of [3] (applied to the arrangement $d\mathcal{A}$) tells us that

$$\mu(Z) = \sum_{\substack{X \in L(d\mathcal{A}) \\ cX \cap H_0 = Z}} \hat{\mu}(X \oplus \{0\}).$$

On the other hand, if $Z \in L(\mathcal{A})$ with $Z \not\geq H_0$, then Proposition 2.43 of [3] tells us that $\mu(Z) = \hat{\mu}(X \oplus \mathbb{K})$, where $X \in L(d\mathcal{A})$ and $Z = cX$.

Thus, from Equation 4.1, the coefficient of t^i on the left hand side is

$$\text{LHS} = \tag{4.2}$$

$$(-1)^{i+1} \left((i+1) \sum_{\substack{Z \in L(\mathcal{A}), \\ H_0 \leq Z, \\ r(Z)=i+1}} \sum_{\substack{X \in L(d\mathcal{A}), \\ cX \cap H_0 = Z}} \hat{\mu}(X \oplus \{0\}) + i \sum_{\substack{X \in L(d\mathcal{A}), \\ r(X)=i}} \mu(X \oplus \mathbb{K}) \right).$$

Note that $\hat{\mu}(X \oplus \{0\}) = -\mu_d(X)$ and $\hat{\mu}(X \oplus \mathbb{K}) = \mu_d(X)$, where μ_d is the Möbius function on $L(d\mathcal{A})$.

Then, Equation 4.2 gives us

$$\text{LHS} = (-1)^{i+1} \left(- (i+1) \sum_{\substack{Z \in L(\mathcal{A}), \\ H_0 \leq Z, \\ r(Z)=i+1}} \sum_{\substack{X \in L(d\mathcal{A}), \\ cX \cap H_0 = Z}} \mu_d(X) + i \sum_{\substack{X \in L(d\mathcal{A}), \\ r(X)=i}} \mu_d(X) \right).$$

One can check that

$$\bigcup_{\substack{Z \in L(\mathcal{A}), \\ H_0 \leq Z, \\ r(Z)=i+1}} \{X \in L(d\mathcal{A}) \mid cX \cap H_0 = Z\} = \{X \in L(d\mathcal{A}) \mid r(X) = i\}.$$

Thus, we have

$$\begin{aligned} \text{LHS} &= (-1)^{i+1} \left(- (i+1) \sum_{\substack{X \in L(d\mathcal{A}), \\ r(X)=i}} \mu_d(X) + i \sum_{\substack{X \in L(d\mathcal{A}), \\ r(X)=i}} \mu_d(X) \right) \\ &= (-1)^i \sum_{\substack{X \in L(d\mathcal{A}), \\ r(X)=i}} \mu_d(X). \end{aligned}$$

This final quantity is the coefficient of t^i in

$$\pi(d\mathcal{A}, t) = \sum_{X \in L(d\mathcal{A})} \mu_d(X) (-t)^{r(X)}.$$

The proof is complete.

5. Example

Consider the arrangement \mathcal{A} in $V = \mathbb{K}^2$ given by

$$Q(\mathcal{A}) = (x_2 - 1)(x_2 + 1)(x_1 + 1)(x_1 - 1)(x_1 + x_2)(x_1 - x_2).$$

In the case when $\mathbb{K} = \mathbb{R}$, \mathcal{A} can be viewed as six lines intersecting in the plane.

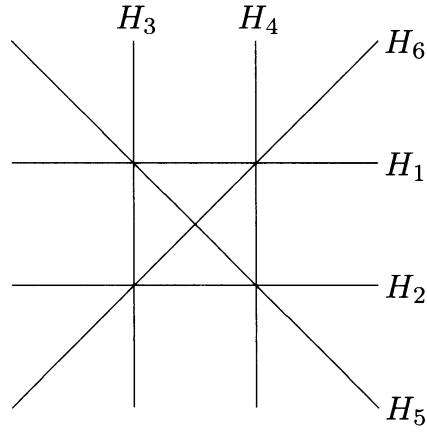


Fig. 5.1. Arrangement \mathcal{A} when $\mathbb{K} = \mathbb{R}$.

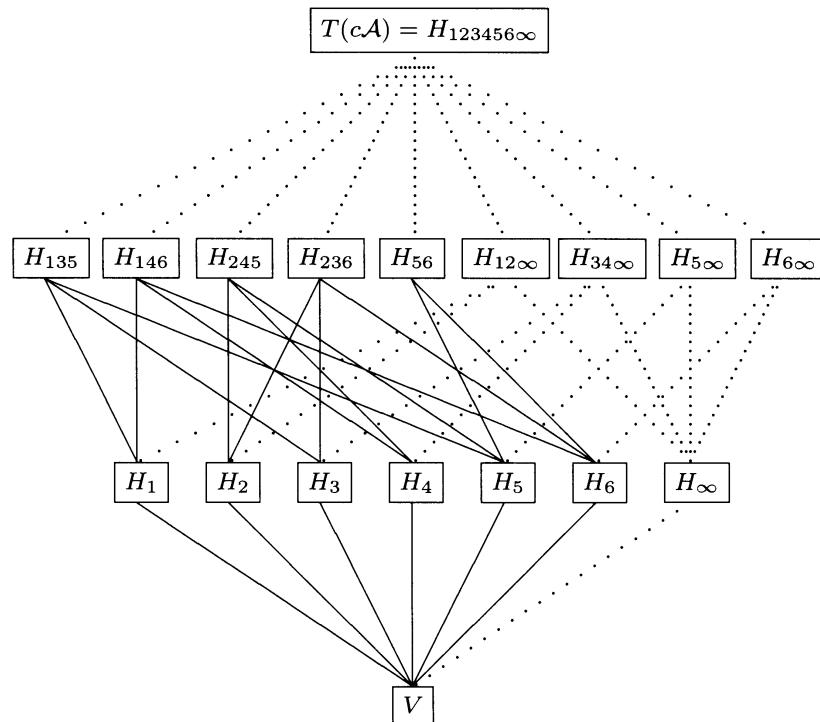


Fig. 5.2. The Hasse diagram for $L(c\mathcal{A})$.

\mathcal{A} is not a central arrangement, but we shall use it in this example to illustrate the applications of Theorem 2.11. We shall use the central arrangement $c\mathcal{A}$ to demonstrate the main theorem directly, in the context of Equation 3.7. $c\mathcal{A}$ is the arrangement in \mathbb{K}^4 given by

$$Q(c\mathcal{A}) = (x_2 - x_0)(x_2 + x_0)(x_1 + x_0)(x_1 - x_0)(x_1 + x_2)(x_1 - x_2)x_0.$$

We let $H_\infty = \ker x_0$ denote the additional hyperplane.

We use a *Hasse diagram* to depict the intersection poset of the arrangement. The vertices of the Hasse diagram are labeled by the elements of the poset, arranged on levels according to rank. An edge is drawn between a rank r element X and a rank $r+1$ element if and only if $X < Y$ (i.e. $Y \subset X$). Figure 5.2 gives the Hasse diagram for $L(c\mathcal{A})$. For ease in identifying elements of the poset, we let $H_{i_1 \dots i_p} = H_{i_1} \cap \dots \cap H_{i_p}$. The Hasse diagram of $L(\mathcal{A})$ appears as a sublattice of $L(c\mathcal{A})$ and is indicated by solid lines in the

X	$r(X)$	$\mu(X)$	$p(X)$	$\pi(\mathcal{A}^X, t)$	$\delta_t(X)$	$\delta(X)$	$\pi((c\mathcal{A})^X, t)$
V	0	1	0	$9t^2 + 6t + 1$	0	0	$9t^3 + 15t^2 + 7t + 1$
H_1	1	-1	1	$2t + 1$	$2t + 1$	1	$2t^2 + 3t + 1$
H_2	1	-1	1	$2t + 1$	$2t + 1$	1	$2t^2 + 3t + 1$
H_3	1	-1	1	$2t + 1$	$2t + 1$	1	$2t^2 + 3t + 1$
H_4	1	-1	1	$2t + 1$	$2t + 1$	1	$2t^2 + 3t + 1$
H_5	1	-1	1	$3t + 1$	$3t + 1$	2	$3t^2 + 4t + 1$
H_6	1	-1	1	$3t + 1$	$3t + 1$	2	$3t^2 + 4t + 1$
H_∞	1	-1	1	$3t + 1$	$3t + 1$	2	$3t^2 + 4t + 1$
H_{135}	2	2	1	1	t	1	$t + 1$
H_{146}	2	2	1	1	t	1	$t + 1$
H_{245}	2	2	1	1	t	1	$t + 1$
H_{236}	2	2	1	1	t	1	$t + 1$
H_{56}	2	1	0	1	0	0	$t + 1$
$H_{12\infty}$	2	2	1	1	t	1	$t + 1$
$H_{34\infty}$	2	2	1	1	t	1	$t + 1$
$H_{5\infty}$	2	1	0	1	0	0	$t + 1$
$H_{6\infty}$	2	1	0	1	0	0	$t + 1$
$T(c\mathcal{A})$	3	-9	4	-	-	-	1

Table 5.3. Values of $\mu(X)$, $p(X)$, $\pi(\mathcal{A}^X, t)$, $\delta_t(X)$, $\delta(X)$, and $\pi((c\mathcal{A})^X, t)$ for $X \in L(c\mathcal{A})$.

diagram. Edges appearing in the Hasse diagram of $L(c\mathcal{A})$ but not $L(\mathcal{A})$ are marked by dotted lines. Recall that $L_\infty = L(\mathcal{A}_\infty)$ is isomorphic to $L(c\mathcal{A})$ with the top element $T(c\mathcal{A})$ removed.

Table 5.3 gives the values of $\mu(X)$, $p(X)$, $\pi((\mathcal{A}^X, t)$, $\delta_t(X)$, $\delta(X)$, and $\pi(c\mathcal{A}^X, t)$ for $X \in L(c\mathcal{A})$. Theorem 2.11 applied to $c\mathcal{A}$ tells us that for any $H \in c\mathcal{A}$, the left hand side of Equation 3.7 will be $\pi(\mathcal{A}, t) = 9t^2 + 6t + 1$. Note that in our index notation, $H_{i_1 \dots i_p} \leq H_{j_1 \dots j_p}$ precisely when $\{i_1, \dots, i_p\} \subseteq \{j_1, \dots, j_p\}$. For example, if $H = H_1$, there are six terms in the sum on the left hand side of Equation 3.7, corresponding to H_1 , H_{135} , H_{146} , $H_{12\infty}$, and $H_{123456\infty}$. The left hand side is thus

$$\begin{aligned} \sum_{\substack{H \leq X, \\ X \in \bar{L}(c\mathcal{A})}} t^{r(X)-1} \pi((c\mathcal{A})^X, t) p(X) \\ &= t^0(2t^2 + 3t + 1) + 3t^1(t + 1) + 4t^2 \\ &= 9t^2 + 6t + 1, \end{aligned}$$

as expected.

Now let $\lambda = (\lambda_1, \dots, \lambda_6) \in \mathbb{C}^6$ be a weight system for \mathcal{A} . Also, for notational purposes, let $\lambda_{i_1 \dots i_p} = \lambda_{i_1} + \dots + \lambda_{i_p}$. \mathcal{A}_∞ is the arrangement obtained by adjoining hyperplane H_∞ to the image of \mathcal{A} in $\mathbb{C}P^2$. In this case, then, Varchenko's Beta function of Equation 3.2 is given by

$$\beta(\mathcal{A}, \lambda) = \frac{(\prod_{i=1}^4 \Gamma(\lambda_i + 1)) (\prod_{i=5}^6 \Gamma(\lambda_i + 1)^2) \Gamma(\lambda_{135} + 1) \Gamma(\lambda_{146} + 1) \Gamma(\lambda_{245} + 1) \Gamma(\lambda_{236} + 1)}{\Gamma(\lambda_{123456} + 1)^2 \Gamma(\lambda_{3456} + 1) \Gamma(\lambda_{1256} + 1)}.$$

One can see that for each i , $1 \leq i \leq 6$, the multiplicity of λ_i is the same in the numerator and the denominator. λ_1 and λ_5 , for example, appear with multiplicity 3 and 4, respectively, in both the numerator and denominator. Moreover, for each $H \in \mathcal{A}_\infty$,

$$\sum_{\substack{H \leq X, \\ X \in L_\infty}} \delta_t(X) = \frac{(9t^2 + 6t + 1) - 4t^2}{1 + t} = 5t + 1,$$

which is in agreement with Equation 3.9.

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