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L^2-boundedness of Marcinkiewicz integral with rough kernel

Yong DING

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Abstract. In this paper the author given the L^2-boundedness of the Marcinkiewicz integral with rough kernel on product domains when Ω is in $L(\log^+ L)^2(S^{n-1}\times S^{m-1})$.

Key words: Marcinkiewicz integral, rough kernel, product domains.

1. Introduction

It is well known that in [6] E.M. Stein introducted the Marcinkiewicz integral operators of higher dimension as the following

$$\mu_\Omega(f)(x)=\left(\int_0^\infty |F_t(x)|^2\frac{dt}{t^3}\right)^{1/2},$$

where

$$F_t(x)=\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}f(y)\,dy,$$

$\Omega\in L^1(S^{n-1})$ is a homogeneous of degree zero satisfying $\int_{S^{n-1}}\Omega(x')\,dx'=0$, and S^{n-1} denotes the unit sphere of $\mathbb{R}^n$. Stein proved that

Theorem A *Under the conditions above, if $\Omega\in Lip_\alpha(S^{n-1})$ $(0<\alpha\le 1)$, then*

$$\|\mu_\Omega(f)\|_p\le C\|f\|_p,\quad 1<p\le 2,$$

$$|\{x: x\in\mathbb{R}^n, \mu_\Omega(f)(x)>\lambda\}|\le\frac{C}{\lambda}\|f\|_1.$$

In [1], A. Benedek, A. Calderon and R. Panzone proved that if $\Omega\in C^1(S^{n-1})$, then $\mu_\Omega(f)$ is bounded operator on $L^p(\mathbb{R}^n)$ $(1<p<\infty)$. A. Torchinsky and S. Wang considered the weighted L^p-boundedness of

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$\mu_\Omega(f)$, they got that if $\Omega \in Lip_\alpha(S^{n-1})$ $(0 < \alpha \le 1)$ and for $1 < p < \infty$, $\mu_\Omega(f)$ is $L^p(\mathbb{R}^n)$ bounded operator, then for $\omega \in A_p$, the Muckenhoupt weight, $\mu_\Omega(f)$ is bounded on $L^p(\omega)$ $(1 < p < \infty)$. (see [7])

On the other hand, the L^p-boundedness for singular integral operators T on product domains $\mathbb{R}^n \times \mathbb{R}^m$, defined by

$$Tf(x,y) = p.v. \iint_{\mathbb{R}^n \times \mathbb{R}^m} \frac{\Omega(x-u, y-v)}{|x-u|^n |y-v|^m} f(u,v)\, dudv,$$

had been discussed for Ω in different function spaces on $S^{n-1} \times S^{m-1}$. For example, in [4], R. Fefferman proved that T is bounded operator on $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ if Ω satisfies some regularity conditions. In [2], J. Duoandikoetxea proved that the smoothness conditions assumed on Ω are unnecessary for the $L^p(\mathbb{R}^n \times \mathbb{R}^m)$-boundedness of operator T. Recentely, Y.S. Jiang and S.Z. Lu give the $L^2(\mathbb{R}^n \times \mathbb{R}^m)$-boundedness of T with rough kernel by using the method of block decomposition for functions [5].

In this paper we shall consider the boundedness of the Marcinkiewicz integral operators $\mu_\Omega(f)$ on product domains, defined by

$$\mu_\Omega(f)(x,y) = \left(\int_0^\infty \int_0^\infty |F_{t,s}(x,y)|^2 \frac{dtds}{t^3 s^3} \right)^{1/2},$$

where

$$F_{t,s}(x,y) = \iint_{\substack{|x-u|\le t \\ |y-v|\le s}} \frac{\Omega(x-u, y-v)}{|x-u|^{n-1} |y-v|^{m-1}} f(u,v)\, dudv,$$

$\Omega \in L^1(S^{n-1} \times S^{m-1})$ and satisfies the following conditions:

$$\Omega(tx, sy) = \Omega(x,y) \quad \text{for any } t,s > 0, \tag{1.1}$$

$$\int_{S^{n-1}} \Omega(x', y')\, dx' = 0 \quad \text{for any } y' \in S^{m-1}, \tag{1.2}$$

$$\int_{S^{m-1}} \Omega(x', y')\, dy' = 0 \quad \text{for any } x' \in S^{n-1}. \tag{1.3}$$

The main result of this paper is stated as follows.

Theorem 1 *Suppose that $\Omega \in L(\log^+ L)^2$ $(S^{n-1} \times S^{m-1})$ satisfying* (1.1)–(1.3). *Then $\mu_\Omega(f)$ is bounded on $L^2(\mathbb{R}^n \times \mathbb{R}^m)$ for $n \ge 2$ and $m \ge 2$.*

2. Proor of Theorem 1

Let us begin by proving a lemma.

Lemma 1 *Suppose that $\Omega(x', y') \in L^\infty(S^{n-1} \times S^{m-1})$, then for any $0 < \alpha < 1$ there is a C such that for any $j, k \in \mathbb{Z}$,*

$$\int_{2^j}^{2^{j+1}} \int_{2^k}^{2^{k+1}} \left| \iint_{S^{n-1}\times S^{m-1}} \Omega(u', v') e^{-i(ru'\cdot x + hv'\cdot y)}\, du'dv' \right| \frac{drdh}{rh} \\ \leq C\|\Omega\|_{L^\infty(S^{n-1}\times S^{m-1})} |2^j x|^{-\alpha/2} |2^k y|^{-\alpha/2}.$$

Proof. Obviously,

$$\begin{aligned}
&\int_{2^j}^{2^{j+1}} \int_{2^k}^{2^{k+1}} \left| \iint_{S^{n-1}\times S^{m-1}} \Omega(u', v') e^{-i(ru'\cdot x + hv'\cdot y)}\, du'dv' \right| \frac{drdh}{rh} \\
&\leq C\bigg(\int_{2^j}^{2^{j+1}} \int_{2^k}^{2^{k+1}} \bigg| \iint_{S^{n-1}\times S^{m-1}} \Omega(u', v') \\
&\qquad \times e^{-i(ru'\cdot x + hv'\cdot y)}\, du'dv' \bigg|^2 \frac{drdh}{rh} \bigg)^{1/2} \\
&= C\bigg(\int_{2^j}^{2^{j+1}} \int_{2^k}^{2^{k+1}} \iint_{S^{n-1}\times S^{m-1}} \iint_{S^{n-1}\times S^{m-1}} \\
&\qquad \Omega(u', v')\overline{\Omega(w', z')} e^{-i[r(u'-w')\cdot x + h(v'-z')\cdot y]}\, du'dv'dw'dz' \frac{drdh}{rh} \bigg)^{1/2} \\
&= C\bigg(\iint_{S^{n-1}\times S^{m-1}} \iint_{S^{n-1}\times S^{m-1}} \Omega(u', v')\overline{\Omega(w', z')} \\
&\qquad \int_{2^j}^{2^{j+1}} \int_{2^k}^{2^{k+1}} e^{-i[r(u'-w')\cdot x + h(v'-z')\cdot y]} \frac{drdh}{rh}\, du'dv'dw'dz' \bigg)^{1/2}.
\end{aligned}$$

By [3] we know that for any $0 < \alpha < 1$

$$\begin{aligned}
&\left| \int_{2^j}^{2^{j+1}} \int_{2^k}^{2^{k+1}} e^{-i[r(u'-w')\cdot x + h(v'-z')\cdot y]} \frac{drdh}{rh} \right| \\
&= \left| \int_{2^j}^{2^{j+1}} e^{-ir(u'-w')\cdot x} \frac{dr}{r} \right| \left| \int_{2^k}^{2^{k+1}} e^{-ih(v'-z')\cdot y} \frac{dh}{h} \right| \\
&\leq C \min\{1, |2^j x(u'-w')|^{-1}\} \min\{1, |2^k y(v'-z')|^{-1}\} \\
&\leq C|2^j x|^{-\alpha} |x(u'-w')|^{-\alpha} |2^k y|^{-\alpha} |y(v'-z')|^{-\alpha}.
\end{aligned}$$

Using

$$\iint_{S^{n-1}\times S^{n-1}} \frac{1}{|x(u'-w')|^{\alpha}}\, du'dw' < \infty$$

and

$$\iint_{S^{m-1}\times S^{m-1}} \frac{1}{|y(v'-z')|^{\alpha}}\, dv'dz' < \infty,$$

we get

$$\int_{2^j}^{2^{j+1}} \int_{2^k}^{2^{k+1}} \left| \iint_{S^{n-1}\times S^{m-1}} \Omega(u',v') e^{-i(ru'\cdot x + hv'\cdot y)}\, du'dv' \right| \frac{drdh}{rh}$$
$$\leq C\|\Omega\|_{L^\infty(S^{n-1}\times S^{m-1})} |2^j x|^{-\alpha/2} |2^k y|^{-\alpha/2}.$$

□

The proof of Lemma 1 is complete.

We now turn to the proof of Theorem 1. If we denote Fourier transform of f on product domains by $\hat{f}$, i.e.

$$\hat{f}(\xi,\eta) = \iint_{\mathbb{R}^n\times\mathbb{R}^m} e^{-i(x\cdot\xi + y\cdot\eta)} f(x,y)\, dxdy,$$

then we have

$$\begin{aligned}
\hat{F}_{t,s}(\xi,\eta) &= \iint_{\mathbb{R}^n\times\mathbb{R}^m} e^{-i(x\cdot\xi+y\cdot\eta)} F_{t,s}(x,y)\, dxdy \\
&= \iint_{\mathbb{R}^n\times\mathbb{R}^m} e^{-i(x\cdot\xi+y\cdot\eta)} \iint_{\substack{|x-u|\leq t \\ |y-v|\leq s}} \frac{\Omega(x-u,y-v)}{|x-u|^{n-1}|y-v|^{m-1}} \\
&\qquad \times f(u,v)\, dudv\, dxdy \\
&= \hat{f}(\xi,\eta)\cdot \iint_{\substack{|u|\leq t \\ |v|\leq s}} \frac{\Omega(u,v)}{|u|^{n-1}|v|^{m-1}} e^{-i(u\cdot\xi+v\cdot\eta)}\, dudv.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\iint_{\mathbb{R}^n\times\mathbb{R}^m} |\mu_\Omega(f)(x,y)|^2\, dxdy \\
&\quad = \iint_{\mathbb{R}^n\times\mathbb{R}^m} \int_0^\infty \int_0^\infty |F_{t,s}(x,y)|^2 \frac{dtds}{t^3s^3}\, dxdy \\
&\quad = \int_0^\infty \int_0^\infty \iint_{\mathbb{R}^n\times\mathbb{R}^m} |F_{t,s}(x,y)|^2\, dxdy \frac{dtds}{t^3s^3}
\end{aligned}$$

$$= \int_0^\infty \int_0^\infty \iint_{\mathbb{R}^n\times\mathbb{R}^m} |\hat{F}_{t,s}(\xi,\eta)|^2\, d\xi d\eta \frac{dtds}{t^3s^3}$$
$$= \iint_{\mathbb{R}^n\times\mathbb{R}^m} |\hat{f}(\xi,\eta)|^2 \Big(\int_0^\infty \int_0^\infty \Big| \iint_{\substack{|u|\le t\\ |v|\le s}} \frac{\Omega(u,v)}{|u|^{n-1}|v|^{m-1}}$$
$$\times\, e^{-i(u\cdot\xi+v\cdot\eta)}\, dudv \Big|^2 \frac{dtds}{t^3s^3}\Big)\, d\xi d\eta.$$

It is easy to see that

$$\Big(\int_0^\infty \int_0^\infty \Big| \iint_{\substack{|u|\le t\\ |v|\le s}} \frac{\Omega(u,v)}{|u|^{n-1}|v|^{m-1}} e^{-i(u\cdot\xi+v\cdot\eta)}\, dudv \Big|^2 \frac{dtds}{t^3s^3}\Big)^{1/2}$$
$$= \Big(\int_0^\infty \int_0^\infty \Big| \int_0^s \int_0^t \iint_{S^{n-1}\times S^{m-1}} \Omega(u',v')$$
$$\times\, e^{-i(ru'\cdot\xi+hv'\cdot\eta)}\, du'dv'drdh \Big|^2 \frac{dtds}{t^3s^3}\Big)^{1/2}$$
$$\le C \int_0^\infty \int_0^\infty \Big| \iint_{S^{n-1}\times S^{m-1}} \Omega(u',v') e^{-i(ru'\cdot\xi+hv'\cdot\eta)}\, du'dv' \Big| \frac{drdh}{rh},$$

therefore, by homogeneity we need only prove that there is a constant C such that

$$\sup_{\substack{x'\in S^{n-1}\\ y'\in S^{m-1}}} \int_0^\infty \int_0^\infty \Big| \iint_{S^{n-1}\times S^{m-1}} \Omega(u',v')$$
$$\times\, e^{-i(ru'\cdot x'+hv'\cdot y')}\, du'dv' \Big| \frac{drdh}{rh} \le C, \tag{2.1}$$

when $\Omega \in L(\log^+ L)^2(S^{n-1}\times S^{m-1})$.

We write

$$\int_0^\infty \int_0^\infty \Big| \iint_{S^{n-1}\times S^{m-1}} \Omega(u',v') e^{-i(ru'\cdot x'+hv'\cdot y')}\, du'dv' \Big| \frac{drdh}{rh}$$
$$= \Big(\int_0^2 \int_0^2 + \int_2^\infty \int_0^2 + \int_0^2 \int_2^\infty + \int_2^\infty \int_2^\infty \Big)$$
$$\Big| \iint_{S^{n-1}\times S^{m-1}} \Omega(u',v') e^{-i(ru'\cdot x'+hv'\cdot y')}\, du'dv' \Big| \frac{drdh}{rh}$$
$$= I + II + III + IV.$$

From the cancelation conditions (1.2) and (1.3), we see that for any $x' \in$

S^{n-1} and $y' \in S^{m-1}$, there is a C such that

$$\begin{aligned} I &= \int_0^2 \int_0^2 \left| \iint_{S^{n-1}\times S^{m-1}} \Omega(u',v') e^{-i(ru'\cdot x' + hv'\cdot y')}\, du'dv' \right| \frac{drdh}{rh} \\ &= \int_0^2 \int_0^2 \Bigg| \iint_{S^{n-1}\times S^{m-1}} \Omega(u',v')(e^{-iru'\cdot x'} - 1) \\ &\qquad \times (e^{-ihv'\cdot y'} - 1)\, du'dv' \Bigg| \frac{drdh}{rh} \le C. \end{aligned}$$

In order to give the estimation of II, III and IV, we need introduce some notations as following:

$$\begin{aligned} E_l &= \{(u',v') : (u',v') \in S^{n-1} \times S^{m-1}, 2^l < |\Omega(u',v')| \le 2^{l+1}\}, \\ &\qquad\qquad \text{for } \ l \ge 1 \\ E_0 &= \{(u',v') : (u',v') \in S^{n-1} \times S^{m-1}, |\Omega(u',v')| \le 2\}, \end{aligned}$$

$\Omega_l(u',v') = \Omega(u',v')\chi_{E_l}(u',v')$, where $\chi_{E_l}(u',v')$ is the characteristic function of E_l. Let us first consider II. In the estimation of II we assume that $0 < h < 2$ and $2^j < r \le 2^{j+1}$ for $j \ge 1$. Thus

$$\begin{aligned} II &= \int_2^\infty \int_0^2 \left| \iint_{S^{n-1}\times S^{m-1}} \Omega(u',v') e^{-i(ru'\cdot x' + hv'\cdot y')}\, du'dv' \right| \frac{drdh}{rh} \\ &= \sum_{j=1}^\infty \int_{2^j}^{2^{j+1}} \int_0^2 \Bigg| \iint_{S^{n-1}\times S^{m-1}} \Omega(u',v') \\ &\qquad \times e^{-i(ru'\cdot x' + hv'\cdot y')}\, du'dv' \Bigg| \frac{drdh}{rh} \\ &\le \sum_{l\ge 0} \sum_{j=1}^\infty \int_{2^j}^{2^{j+1}} \int_0^2 \Bigg| \iint_{S^{n-1}\times S^{m-1}} \Omega_l(u',v') \\ &\qquad \times e^{-i(ru'\cdot x' + hv'\cdot y')}\, du'dv' \Bigg| \frac{drdh}{rh} \\ &= \left(\sum_{l\ge 0} \sum_{1\le j\le 2l} + \sum_{l\ge 0} \sum_{j>2l} \right) \int_{2^j}^{2^{j+1}} \int_0^2 \Bigg| \iint_{S^{n-1}\times S^{m-1}} \Omega_l(u',v') \\ &\qquad \times e^{-i(ru'\cdot x' + hv'\cdot y')}\, du'dv' \Bigg| \frac{drdh}{rh} \\ &= II_1 + II_2 \end{aligned}$$

For II_1, we have by (1.2), (1.3)

$$\begin{aligned} II_1 &= \sum_{l\ge 0}\sum_{1\le j\le 2l}\int_{2^j}^{2^{j+1}}\int_0^2\left|\iint_{S^{n-1}\times S^{m-1}}\Omega_l(u',v')e^{-iru'\cdot x'}\right. \\ &\qquad \times\left.(e^{-ihv'\cdot y'}-1)\,du'dv'\right|\frac{drdh}{rh} \\ &\le \sum_{l\ge 0}\sum_{1\le j\le 2l}\int_{2^j}^{2^{j+1}}\int_0^2\iint_{S^{n-1}\times S^{m-1}}|\Omega_l(u',v')| \\ &\qquad \times|e^{-ihv'\cdot y'}-1|\,du'dv'\frac{dhdr}{hr} \\ &\le C\sum_{l\ge 0}\sum_{1\le j\le 2l}\log 2\cdot\|\Omega_l\|_{L^1(S^{n-1}\times S^{m-1})} \\ &\le C\sum_{l\ge 0} l\log 2\cdot 2^{l+1}|E_l|\le C\|\Omega\|_{L\log^+L(S^{n-1}\times S^{m-1})}, \end{aligned}$$

and for II_2

$$\begin{aligned} II_2 &= \sum_{l\ge 0}\sum_{j>2l}\int_{2^j}^{2^{j+1}}\int_0^2\left|\iint_{S^{n-1}\times S^{m-1}}\Omega_l(u',v')e^{-iru'\cdot x'}\right. \\ &\qquad \times\left.(e^{-ihv'\cdot y'}-1)\,du'dv'\right|\frac{dhdr}{hr} \\ &\le \sum_{l\ge 0}\sum_{j>2l}\int_0^2\int_{S^{m-1}}\int_{2^j}^{2^{j+1}}\left|\int_{S^{n-1}}\Omega_l(u',v')e^{-iru'\cdot x'}\,du'\right|\frac{dr}{r} \\ &\qquad \times|e^{-ihv'\cdot y'}-1|\,dv'\frac{dh}{h} \\ &\le C\sum_{l\ge 0}\sum_{j>2l}\int_{S^{m-1}}\|\Omega_l(\cdot,v')\|_{L^\infty(S^{n-1})}\,dv'2^{-\alpha j/2} \\ &\le C\sum_{l\ge 0}\sum_{j>2l}2^{l+1}\cdot|E_l|2^{-\alpha j/2} \\ &\le C\sum_{l\ge 0}2^{l+1}\cdot|E_l|2^{-\alpha 2l/2}\le C\sum_{l\ge 0}2^l\cdot|E_l| \\ &\le C\|\Omega\|_{L^1(S^{n-1}\times S^{m-1})}. \end{aligned}$$

Hence there is a C such that for any $(x',y')\in S^{n-1}\times S^{m-1}$ we have $II = II_1+II_2\le C$. The same proof works for III and we have $III\le C$. It remains to estimate IV. We denote

$$\begin{aligned}
IV &= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\int_{2^j}^{2^{j+1}}\int_{2^k}^{2^{k+1}}\bigg|\iint_{S^{n-1}\times S^{m-1}}\Omega(u',v') \\
&\qquad \times e^{-i(ru'\cdot x'+hv'\cdot y')}\,du'dv'\bigg|\frac{dhdr}{hr} \\
&\le \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l\ge 0}\int_{2^j}^{2^{j+1}}\int_{2^k}^{2^{k+1}}\bigg|\iint_{S^{n-1}\times S^{m-1}}\Omega_l(u',v') \\
&\qquad \times e^{-i(ru'\cdot x'+hv'\cdot y')}\,du'dv'\bigg|\frac{dhdr}{hr} \\
&= \left(\sum_{l=0}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}+\sum_{l\ge 1}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\right)\int_{2^j}^{2^{j+1}}\int_{2^k}^{2^{k+1}} \\
&\qquad \bigg|\iint_{S^{n-1}\times S^{m-1}}\Omega_l(u',v')e^{-i(ru'\cdot x'+hv'\cdot y')}\,du'dv'\bigg|\frac{dhdr}{hr} \\
&= IV_1 + IV_2.
\end{aligned}$$

By Lemma 1 we have

$$IV_1 \le C\|\Omega_0\|_{L^\infty(S^{n-1}\times S^{m-1})}\sum_{j=1}^{\infty}2^{-\alpha j/2}\sum_{k=1}^{\infty}2^{-\alpha k/2}\le C.$$

Now, we do the decomposition for IV_2. Choosing $s\in\mathbb{N}$ such that $s\alpha>1$, where α is as in Lemma 1.

$$\begin{aligned}
IV_2 &= \sum_{l\ge 1}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\int_{2^j}^{2^{j+1}}\int_{2^k}^{2^{k+1}}\bigg|\iint_{S^{n-1}\times S^{m-1}}\Omega_l(u',v') \\
&\qquad \times e^{-i(ru'\cdot x'+hv'\cdot y')}\,du'dv'\bigg|\frac{dhdr}{hr} \\
&= \Bigg(\sum_{l\ge 1}\sum_{1\le j\le sl}\sum_{1\le k\le sl}+\sum_{l\ge 1}\sum_{1\le j\le sl}\sum_{k>sl}+\sum_{l\ge 1}\sum_{j>sl}\sum_{1\le k\le sl}+\sum_{l\ge 1}\sum_{j>sl}\sum_{k>sl}\Bigg) \\
&\qquad \int_{2^j}^{2^{j+1}}\int_{2^k}^{2^{k+1}}\bigg|\iint_{S^{n-1}\times S^{m-1}}\Omega_l(u',v') \\
&\qquad \times e^{-i(ru'\cdot x'+hv'\cdot y')}\,du'dv'\bigg|\frac{dhdr}{hr} \\
&= A_1 + A_2 + A_3 + A_4.
\end{aligned}$$

Let us now estimate $A_i, (i = 1, 2, 3, 4)$ respectively. For A_1, we have

$$\begin{aligned}
A_1 &\le C \sum_{l\ge 1} \sum_{1\le j\le sl} \sum_{1\le k\le sl} \|\Omega_l\|_{L^1(S^{n-1}\times S^{m-1})} (\log 2)^2 \\
&\le C \sum_{l\ge 1} l^2 \cdot 2^{l+1} |E_l| \cdot (\log 2)^2 \\
&\le C \|\Omega\|_{L(\log^+ L)^2 (S^{n-1}\times S^{m-1})} \le C.
\end{aligned}$$

For A_2, we have

$$\begin{aligned}
A_2 &\le \sum_{l\ge 1} \sum_{1\le j\le sl} \sum_{k>sl} \int_{2^j}^{2^{j+1}} \int_{S^{n-1}} \int_{2^k}^{2^{k+1}} \left| \int_{S^{m-1}} \Omega_l(u', v') \right. \\
&\qquad \left. \times e^{-ihv'\cdot y'} \, dv' \right| \frac{dh}{h} \, du' \frac{dr}{r} \\
&\le C \sum_{l\ge 1} \sum_{1\le j\le sl} \sum_{k>sl} \log 2 \cdot \int_{S^{n-1}} \|\Omega_l(u', \cdot)\|_{L^\infty(S^{m-1})} \, du' 2^{-\alpha k/2} \\
&\le C \sum_{l\ge 1} \sum_{1\le j\le sl} \sum_{k>sl} \log 2 \cdot 2^{l+1} |E_l| \cdot 2^{-\alpha k/2} \\
&\le C \sum_{l\ge 1} \sum_{1\le j\le sl} \log 2 \cdot 2^{l+1} |E_l| \cdot 2^{-\alpha sl/2} \\
&\le C \sum_{l\ge 1} l \log 2 \cdot 2^{l+1} |E_l| \le C \|\Omega\|_{L \log^+ L (S^{n-1}\times S^{m-1})} \le C.
\end{aligned}$$

Similar considerations apply to A_3. Let us now estimate A_4.

$$\begin{aligned}
A_4 &= \sum_{l\ge 1} \sum_{j>sl} \sum_{k>sl} \int_{2^j}^{2^{j+1}} \int_{2^k}^{2^{k+1}} \left| \iint_{S^{n-1}\times S^{m-1}} \Omega_l(u', v') \right. \\
&\qquad \left. \times e^{-i(ru'\cdot x' + hv'\cdot y')} \, du' dv' \right| \frac{dh dr}{hr} \\
&\le C \sum_{l\ge 1} \sum_{j>sl} \sum_{k>sl} \|\Omega_l\|_{L^\infty(S^{n-1}\times S^{m-1})} 2^{-\alpha k/2} 2^{-\alpha j/2} \\
&\le C \sum_{l\ge 1} \sum_{j>sl} \sum_{k>sl} 2^{l+1} \cdot 2^{-\alpha k/2} 2^{-\alpha j/2} \\
&\le C \sum_{l\ge 1} 2^l \cdot 2^{-\alpha sl/2} 2^{-\alpha sl/2} \le C.
\end{aligned}$$

Thus, there is a C such that for any $(x', y') \in S^{n-1} \times S^{m-1}$

$$IV_2 = A_1 + A_2 + A_3 + A_4 \le C.$$

To sum up, from the estimations above for I, II, III and IV we see that (2.1) holds for $\Omega \in L(\log^+ L)^2(S^{n-1} \times S^{m-1})$ and Theorem 1 follows.

Remark.

We may consider the $L^2(S^{n-1} \times S^{m-1})$-boundedness of a class of the Marcinkiewicz integral operators $\mu_{\Omega,b}(f)$ on product domains with radial function $b(r,s)$ on $\mathbb{R}^+ \times \mathbb{R}^+$, defined by

$$\mu_{\Omega,b}(f)(x,y) = \left(\int_0^\infty \int_0^\infty |F_{t,s}(x,y)|^2 \frac{dtds}{t^3 s^3} \right)^{1/2},$$

where

$$F_{t,s}(x,y) = \iint_{\substack{|x-u| \le t \\ |y-v| \le s}} \frac{\Omega(x-u, y-v) b(|x-u|, |v-y|)}{|x-u|^{n-1} |y-v|^{m-1}} f(u,v)\, dudv.$$

We have the following result:

Theorem 2 *Suppose that $\Omega \in L(\log^+ L)^2(S^{n-1} \times S^{m-1})$ satisfying* (1.1)–(1.3) *and $b(r,s) \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^+)$. Then $\mu_{\Omega,b}(f)$ is bounded on $L^2(\mathbb{R}^n \times \mathbb{R}^m)$ for $n \ge 2$ and $m \ge 2$.*

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Department of Mathematics
Beijing Normal University
Beijing, 100875
P.R. of China