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Extension of submanifolds of $\mathbb{C}^n$ preserving the number of negative Levi eigenvalues

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Abstract. Given a totally real C^2-submanifold S of a complex manifold X, it is obvious that there exists a hypersurface M, in a neighborhood of any point of S, which contains S and which is the boundary of a strictly pseudoconvex domain. We prove here that if S is *generic*, then there exists a hypersurface M through S which has the same number of negative (or positive) Levi eigenvalues as S at a prescribed conormal. (Resp. at all common conormals when we assume in addition that the rank of the Levi-form L_S is constant.) As an application we show how to *lift* complex submanifolds from S to $\dot{T}_S^*X$, the conormal bundle to S in X, when L_S is semidefinite of constant rank (cf. Bedford-Fornaess [1] for the case of codim $S=1$). We point out that our method is not adequate to describe the behavior of the Levi form of M on points outside S. In particular it is still an open problem whether any submanifold S whose Levi form is positive semi-definite, is contained in a pseudoconvex hypersurface M.

Some of the results discussed here are also exposed in [9].

Key words: CR manifolds - real/complex symplectic structures.

1. Statement and Proof of the Main Result

Let X be a complex manifold of dimension n, S a real C^2-submanifold of X with $\mathrm{codim}_X S = l$, $\pi : T^*X \to X$ the cotangent bundle to X, $\pi : T_S^*X \to S$ the conormal bundle to S in X. For a point $p=(z,\zeta) \in \dot{T}_S^*X$ $(= T_S^*X \setminus \{0\})$, choose a real C^2-function r with $r|_S \equiv 0$ and $\partial r(z) = p$, and define the Levi form of S at p by

$$L_S(p) = \partial\bar{\partial} r(z)|_{T_z^{\mathbb{C}}S}, \tag{1}$$

where $T^{\mathbb{C}}S = TS \cap \sqrt{-1}TS$. Denote by $s_S^{+,-,0}(p)$ the numbers of respectively positive, negative, and null eigenvalues of $L_S(p)$.

Assume that S is *generic* in the sense that

$$(T_S^*X)_z \cap \sqrt{-1}(T_S^*X)_z = \{0\}. \tag{2}$$

Fix $p_o \in \dot{T}_S^*X$.

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Theorem 1 *We may find a hypersurface M (in an open neighborhood of $z_o \stackrel{\text{def.}}{=} \pi(p_o)$) such that*

$$\begin{cases} M \supset S \\ p_o \in \dot{T}^*_M X \\ s^-_M(p_o) = s^-_S(p_o). \end{cases} \tag{3}$$

(Similarly there exists M which satisfies (3) with s^- replaced by s^+.)

Proof. We take complex coordinates z in a neighborhood B of z_o in X, and identify in these coordinates $X \simeq T_z X \, \forall z \in B$. We take the canonically associated complex symplectic coordinates (z, ζ) in T^*X. The action of the canonical 1-form $\omega = \omega^{\mathbb{R}} + \sqrt{-1}\omega^{\mathbb{I}}$ is then defined by means of the Hermitian product of X and that of $\omega^{\mathbb{R}}$ through the Euclidean product of $X^{\mathbb{R}}$ the real underlying manifold to X. This provides an identification of $T^*_S X$ to $TS^\perp$, the Euclidean orthogonal to TS. We shall also denote by $\sigma = \sigma^{\mathbb{R}} + \sqrt{-1}\sigma^{\mathbb{I}} (= \mathrm{d}\omega)$ the canonical 2-form on T^*X. We define the *complex modulus* $\|\zeta\| = (\sum_{i=1}^n \zeta_i^2)^{\frac{1}{2}}$ where we choose the determination of the square root which is positive for real ζ. In particular $\|\zeta\|$ makes sense when $\sum_i \zeta_i^2 \notin \mathbb{R}^-$. This is the case of any $\zeta \in (T^*_S X)_z$, when z is close to z_o. (In fact, by (2) the coordinates can be chosen so that any $\zeta \in (T^*_S X)_{z_o}$ is real.) We write any $\tau \in (X \setminus S) \cap B$ as:

$$\tau = z - |\zeta| \frac{\zeta}{\|\zeta\|} \tag{4}$$

for an unique $(z; \zeta) \in \dot{T}^*_S X$ with $z \in B'$ and $|\zeta|$ small. In fact it is easy to check that the normals issued from different points of a C^2-manifold S cannot have non-trivial intersection in a neighborhood of S. And this is still true if we replace normal directions $\frac{\zeta}{|\zeta|}$ by $\frac{\zeta}{\|\zeta\|}$. By (4), $X \setminus S$ and $\dot{T}^*_S X$ are thus identified in neighborhoods of z_o and $(z_o; 0)$ respectively. This provides an *orthogonal projection* h and a *distance function* δ, defined locally by:

$$h : X \setminus S \to S, h(\tau) = z, \quad \delta : X \setminus S \to \dot{\mathbb{R}}^+, \delta(\tau) = \|\tau - z\| (= |\zeta|).$$

We have also to notice that $X \setminus S$ is foliated by the hypersurfaces of *fixed distance* to S:

$$\tilde{S}_t = \{\tau \in B; \delta(\tau) = t\}$$

$$= \left\{ \tau = z - t\frac{\zeta}{\|\zeta\|}; (z,\zeta) \in \dot{T}^*_S X \cap \pi^{-1}(B') \right\},$$

with B and B' neighborhoods of τ_o and z_o in X and S respectively.

We fix t and write also $\tilde{S}$ instead of $\tilde{S}_t$. We introduce a complex symplectic diffeomorphism $\chi = \chi_t$ of $\dot{T}^*X$ defined, for $\sum \zeta_i^2 \notin \mathbb{R}^-$, by:

$$\chi : (z;\zeta) \mapsto \left(z - t\frac{\zeta}{\|\zeta\|}; \zeta \right).$$

We remark that $\pi\chi_t(T^*_S X) = \tilde{S}$ and that $\chi(T^*_S X)$ has to be $\mathbb{R}$-Lagrangian (i.e. Lagrangian for $\sigma^{\mathbb{R}}$) because χ preserves Lagrangianity. It follows:

$$\chi_t(T^*_S X) = T^*_{\tilde{S}} X. \tag{5}$$

This implies in particular that

$$T_{h(\tau)} S \subset T_\tau \tilde{S}, \quad \forall \tau \in \tilde{S}, \tag{6}$$

under the identification, in coordinates, $X \simeq T_\tau X \simeq T_{h(\tau)} X$. $\tilde{S}$ being a hypersurface, we identify the conormals $q \in T^*_{\tilde{S}} X$ in a neighborhood of q_o to the base-points $\tau = \pi(q) \in \tilde{S}$. □

To carry on our proof we need to state now some Lemmas.

Lemma 2 *There exists* $R = R_t \subset \tilde{S}$ *with* $\dim R = \dim S$ *and such that*

(i) $T^{\mathbb{C}}_{\tau_o} R \supset \operatorname{Ker} L_{\tilde{S}}(\tau_o)$

(ii) $T_{\tau_o} R = \Phi_t(T_{z_o} S)$,

where Φ_t *is a linear transformation of* $\mathbb{C}^n$ *with* $\Phi_t - \mathrm{Id} = O(t)$.

Proof. We denote by $(r_1 = 0, \dots, r_l = 0)$ $(l = \operatorname{codim} S)$ a system of independent equations for S. We set $p_o = (z_o; \zeta_o)$, observe that we can assume $\zeta_o \in \mathbb{R}^n$, $|\zeta_o| = 1$ due to (2), and choose an equation $r = 0$ for S which satisfies $\partial r(z_o) = \zeta_o$. We write $\lambda_S(p_o) :\overset{\text{def.}}{=} T_{p_o} T^*_S X$, and observe that we have the parametric description:

$$\lambda_S(p_o) = \left\{ \left(u, \sum_j t_j \partial r_j(z_o) + \partial\partial r(z_o) u + \partial\bar{\partial} r(z_o) \bar{u} \right); u \in T^{\mathbb{C}}_{z_o} S,\ (t_j) \in \mathbb{R}^l \right\}.$$

It follows

$$\begin{aligned}\lambda_S(p_o)\cap\sqrt{-1}\lambda_S(p_o) &= \Big\{\Big(u,\sum_j t_j\partial r_j(z_o)+\partial\partial r(z_o)u+\partial\bar\partial r(z_o)\bar u\Big)\\ &= \Big(\sqrt{-1}w,\sqrt{-1}\sum_j s_j\partial r_j(z_o)\\ &\quad+\partial\partial r(z_o)\sqrt{-1}w-\partial\bar\partial r(z_o)\overline{\sqrt{-1}w}\Big)\\ &\qquad \text{for } u,\, w\in T^{\mathbb{C}}_{z_o}S,\ (t_j),\ (s_j)\in\mathbb{R}^l\Big\}.\end{aligned}$$

This implies $u=\sqrt{-1}w$ and moreover

$$\partial\bar\partial r(z_o)\bar u=-\frac{1}{2}\Big(\sum_j(t_j-\sqrt{-1}s_j)\partial r_j(z_o)\Big)\quad \text{(i.e. } u\in\operatorname{Ker}L_S(p_o)).$$

In particular $\sum_j t_j\partial r_j(z_o)=-2\Re\mathrm{e}\,\partial\bar\partial r(z_o)\bar u$. Also notice that

$$-2\Re\mathrm{e}\,\partial\bar\partial r(z_o)\bar u+\partial\bar\partial r(z_o)\bar u=-\overline{\partial\bar\partial r(z_o)\bar u}=-\bar\partial\partial r(z_o)u.$$

It follows

$$\begin{aligned}&\lambda_S(p_o)\cap\sqrt{-1}\lambda_S(p_o)\\ &\quad=\{(u,v);u\in\operatorname{Ker}L_S(p_o),\ v=-2\Re\mathrm{e}\,\partial\bar\partial r(z_o)\bar u\\ &\qquad\qquad+\partial\partial r(z_o)u+\partial\bar\partial r(z_o)\bar u)\}\\ &\quad=\{(u,v);u\in\operatorname{Ker}L_S(p_o),\ v=\partial\partial r(z_o)u-\bar\partial\partial r(z_o)u\}.\end{aligned}\tag{7}$$

In particular

$$\lambda_S(p_o)\cap\sqrt{-1}\lambda_S(p_o)\ \underset{\pi'}{\overset{\sim}{\to}}\ \operatorname{Ker}L_S(p_o),$$

is one-to-one. Clearly similar injectivity for π' and similar parametric description as (7) also holds for $\lambda_{\tilde S}(q_o)\cap\sqrt{-1}\lambda_{\tilde S}(q_o)$ ($q_o=\chi(p_o)$).

Let us define now a linear transformation on $\mathbb{C}^n$ by $\Phi_t: u\mapsto u+t(v(u)-\zeta_o\langle\zeta_o,v(u)\rangle)$ where $v(u)=\partial\partial r(z_o)u-\bar\partial\partial r(z_o)u$. Note that we have:

$$\begin{aligned}\pi'\chi_t'(p_o):(u,v)&\mapsto ut\frac{v(\sum_i\zeta_{oi}^2)-\zeta_o\langle\zeta_o,v\rangle}{(\sum_i\zeta_{oi}^2)^{\frac{3}{2}}}\\ &=ut(v-\zeta_o\langle\zeta_o,v\rangle).\end{aligned}$$

(Note here that $\sum_i\zeta_{oi}^2=1$.)

Thus with the notation $q_o = \chi(p_o)$, the diagram

$$\begin{array}{ccc} \lambda_S(p_o) \cap \sqrt{-1}\lambda_S(p_o) & \overset{\chi'}{\underset{\sim}{\longrightarrow}} & \lambda_{\tilde{S}}(q_o) \cap \sqrt{-1}\lambda_{\tilde{S}}(q_o) \\ \pi' \downarrow & & \downarrow \pi' \\ \operatorname{Ker} L_S(p_o) & \underset{\Phi_t}{\overset{\sim}{\longrightarrow}} & \operatorname{Ker} L_{\tilde{S}}(q_o), \end{array} \tag{8}$$

is commutative. We write $\tau_o = \pi(q_o)$, denote by g the projection $g : T_{\tau_o}X \to \tilde{S}$ along the normal at τ_o, and put $R = g(\Phi_t(T_{z_o}S))$. R satisfies all requirements of Lemma 2. □

When dealing with a hypersurface $\tilde{S}$ (and for a choice of an orientation $\pm q_o$), we write $L_{\tilde{S}}(\tau_o)$, $\tau_o = \pi(q_o)$ instead of $L_{\tilde{S}}(q_o)$. We point out that (8) shows that

$$\operatorname{rank} L_{\tilde{S}}(\tau_o) = \operatorname{rank} L_S(p_o) + (l-1). \tag{9}$$

We also point out that (i) and (ii) imply, for small t:

$$L_{\tilde{S}}(\tau_o)|_{T^{\mathbb{C}}_{\tau_o}R} \sim L_S(p_o), \tag{10}$$

where " $\sim$ " means equivalence in signature and rank. Let us identify $\frac{(T^*_S X)_{z_o}}{(T^*_{\tilde{S}}X)_{\tau_o}}$ to a totally real plane N orthogonal to $T_{\tau_o}R$ in $T_{\tau_o}\tilde{S}$ by the aid of the Euclidean structure of $X^{\mathbb{R}} = T_{\tau_o}X^{\mathbb{R}} = T_{z_o}X^{\mathbb{R}}$, and define $\tilde{N} = N \oplus \sqrt{-1}N$. Thus $\tilde{N}$ is the orthogonal complement of $T^{\mathbb{C}}_{\tau_o}R$ in $T^{\mathbb{C}}_{\tau_o}\tilde{S}$. We note that $\{t\frac{\zeta}{\|\zeta\|}; \zeta \in \mathbb{R}^l \simeq (T^*_S X)_{z_o}\}$ is the spherical surface in $\mathbb{R}^l$ of radius t (small), and N is (identified to) its tangent plane at $(-1, \dots, 0)$. It follows that the real Hessian $\operatorname{Hess}_{\tilde{S}}$ verifies

$$\operatorname{Hess}_{\tilde{S}}(\tau_o)(v,v) = -2t^{-1}|v|^2 \quad \forall v \in N.$$

Note also that $\operatorname{Hess}_{\tilde{S}}(\tau_o)(\sqrt{-1}v, \sqrt{-1}v) \le c|v|^2 \quad \forall v \in N$. This implies

$$\begin{aligned} L_{\tilde{S}}(\tau_o)(\bar{v}, v) &= \frac{1}{4}\left[\operatorname{Hess}_{\tilde{S}}(\tau_o)(v,v) + \operatorname{Hess}_{\tilde{S}}(\tau_o)(\sqrt{-1}v, \sqrt{-1}v)\right] \\ &\le \left(-\frac{t^{-1}}{2} + c\right)|v|^2 \le -\frac{t^{-1}}{3}|v|^2 \quad \forall v \in N. \end{aligned} \tag{11}$$

We recall that $\operatorname{Ker} L_{\tilde{S}}(\tau_o) \hookrightarrow T^{\mathbb{C}}_{\tau_o}R$. Thus we may find $\tilde{N}' \subset T^{\mathbb{C}}_{\tau_o}\tilde{S}$ transversal to $T^{\mathbb{C}}_{\tau_o}R$ and such that:

$$L_{\tilde{S}}(\tau_o)(\bar{u}, v) = 0 \quad \forall u \in T^{\mathbb{C}}_{\tau_o}R,\ v \in \tilde{N}'. \tag{12}$$

By choosing t small enough, we may suppose that (11) still holds with the new $\tilde{N}'$. It follows that

$$s^-_{\tilde{S}}(\tau_o) = s^-_S(p_o) + (l-1) \quad (\text{and } \ s^+_{\tilde{S}}(\tau_o) = s^+_S(p_o)).$$

We take now a hypersurface $\tilde{M}$ which intersect $\tilde{S}$ along R with order of contact 2 and with $\tilde{M}^+ \subset \tilde{S}^+$ (where $\tilde{M}^+$, $\tilde{S}^+$ are the closed half-spaces with boundary $\tilde{M}$, $\tilde{S}$ and inward conormal q). We note that this implies

$$\chi^{-1}(T^*_{\tilde{M}}X) = T^*_M X \quad \text{for a hypersurface } \ M \supset S,$$

due to the assumption on the order of contact of $\tilde{M}$ with $\tilde{S}$. We have clearly

$$L_{\tilde{M}}(\tau_o)|_{T^{\mathbb{C}}_{\tau_o}R} = L_{\tilde{S}}(\tau_o)|_{T^{\mathbb{C}}_{\tau_o}R}(\sim L_S(p_o) \ \text{ for } \ t \text{ small}). \tag{13}$$

Lemma 3 *We have*

$$L_{\tilde{M}}(\tau_o)(\bar{u}, v) = 0 \quad \forall u \in T^{\mathbb{C}}_{\tau_o}R,\ \ v \in \tilde{N}'. \tag{14}$$

Proof. We choose complex coordinates $z = (z_1, z', z'')$ such that $\tau_o = 0$, $q = \mathrm{d}y_1$, $T_{\tau_o}X = \mathbb{C}_{z_1} \times T^{\mathbb{C}}_{\tau_o}\tilde{S} = \mathbb{C}_{z_1} \times T^{\mathbb{C}}_{\tau_o}R \times \tilde{N} = \mathbb{C}_{z_1} \times \mathbb{C}^{n-l}_{z'} \times \mathbb{C}^{l-1}_{z''}$. We take equations $y_1 = h_1$ and $y_1 = h_2$ for $\tilde{M}$ and $\tilde{S}$ respectively, and set $h = h_1 - h_2$. We have

$$h|_R \equiv 0, \quad \partial h|_R \equiv 0. \tag{15}$$

It follows

$$\sum_j \bar{a}_j \bar{\partial}_{z_j} \partial h|_R \equiv 0 \ \text{ if } \ \Re\mathrm{e}\Big(\sum_j a_j \partial_{z_j}\Big) \in T^{\mathbb{C}}_{\tau_o}R.$$

In particular, since $T^{\mathbb{C}}_{\tau_o}R = \Re\mathrm{e}\,(\mathrm{Span}_{\mathbb{C}}(\partial_{z'}))$, then $\bar{\partial}_{z'}\partial h(\tau_o) = 0$. Thus $L_h(\bar{u}, v) = 0\ \forall u \in T^{\mathbb{C}}_{\tau_o}R$; in particular the property "$L_{h_i}(\bar{u}, v) = 0\ \forall u \in T^{\mathbb{C}}_{\tau_o}R$, $v \in \tilde{N}'$" holds for $i = 1$ iff it holds for $i = 2$. Thus (12) and (14) are equivalent. □

End of proof of Theorem 1

It is also clear that we can take $\tilde{M}$ such that (11) holds for $L_{\tilde{M}}$ and for

$\tilde{N}'$ (with a new c). Recalling also (13), we have for small t:

$$s^-_{\tilde{M}}(\tau_o) = s^-_{\tilde{S}}(p_o) + (l-1) \quad (\text{and} \;\; s^+_{\tilde{M}}(\tau_o) = s^+_{\tilde{S}}(p_o)).$$

We note now that, from $\lambda_{\tilde{M}}(\tau) \cap \sqrt{-1}\lambda_{\tilde{M}}(\tau) \underset{\chi'^{-1}}{\overset{\sim}{\to}} \lambda_M(p) \cap \sqrt{-1}\lambda_M(p)$, ($\tau = \pi\chi(p)$), we get, similarly to (9):

$$\operatorname{rank} L_M(p) = \operatorname{rank} L_{\tilde{M}}(\tau). \tag{16}$$

It follows:

$$s^-_M(p_o) = s^-_S(p_o) \quad \text{and} \quad s^+_M(p_o) = s^+_S(p_o) + (l-1). \tag{17}$$

Thus M satisfies all requirements in the statement of Theorem 1. □

Theorem 4 *Let* $\operatorname{rank} L_S(p) \equiv \text{const}\, \forall p$ *in* $\dot{T}^*_S X$ *close to* p_o*, and assume that* S *is of class* C^3*. Then there exists a germ of a hypersurface* M *at* z_o *such that*

$$s^-_M(p) \equiv s^-_S(p_o) \quad \forall p \in S \times_M T^*_M X. \tag{18}$$

Proof. We transform $T^*_S X \underset{\chi}{\overset{\sim}{\to}} T^*_{\tilde{S}} X$ ($\chi = \chi_t$, $\tilde{S} = \tilde{S}_t$). Since

$$\operatorname{Ker} L_{\tilde{S}}(\tau) \underset{\pi'}{\overset{\sim}{\leftarrow}} \lambda_{\tilde{S}}(q) \cap \sqrt{-1}\lambda_{\tilde{S}}(q) \underset{\chi'}{\overset{\sim}{\leftarrow}} \lambda_S(p) \cap \sqrt{-1}\lambda_S(p)$$
$$\underset{\pi'}{\overset{\sim}{\to}} \operatorname{Ker} L_S(p),$$

has constant rank, then it is integrable (= closed under Lie-brackets) according to [4]. (see also [8]). For this the assumption of C^3-regularity for S is required.

Thus each $\tilde{S} = \tilde{S}_t$ is foliated by the (complex) integral leaves of $\operatorname{Ker} L_{\tilde{S}}$. Since the hypersurfaces $\tilde{S}_t$ give in turn a t-parameter foliation of $X \setminus S$, then we get a foliation of $X \setminus S$ by complex leaves tangent to the bundle:

$$\mathcal{W}(\tau) \overset{\text{def.}}{:=} \operatorname{Ker} L_S(h(\tau); \zeta_\tau) \quad \text{with} \quad \frac{|\zeta_\tau|\zeta_\tau}{\|\zeta_\tau\|} = \tau - h(\tau).$$

Take a decomposition $TS = L \oplus \operatorname{Ker} L_S$ such that L_S is diagonal (with unitary eigenvalues) in $(L \cap \sqrt{-1}L)$. Define R to be the union of the integral leaves of $\mathcal{W}$ issued from $g(\Phi_{|\zeta_t|}L)$ ($g : TX \to \tilde{S}$). R is a germ of a submanifold of $\tilde{S}$ at τ_o which satisfies:

$$\begin{cases} T_\tau^{\mathbb{C}} R \supset \operatorname{Ker} L_{\tilde{S}}(\tau) & \forall \tau \in R, \\ T_\tau R = \Phi_t^\tau(T_{z_o}S) & \text{with } |\Phi_t^\tau - \mathrm{Id}| < \epsilon \text{ for } |(\tau, t)| < \delta_\epsilon. \end{cases} \tag{19}$$

We still have

$$L_{\tilde{S}}(\tau)|_{T_\tau^{\mathbb{C}} R} \sim L_S(p_o), \tag{20}$$

and, for a decomposition $T_\tau^{\mathbb{C}} \tilde{S} = T_\tau^{\mathbb{C}} R \oplus \tilde{N}'_\tau$:

$$L_{\tilde{S}}(\tau)(\bar{v}, v) \le -ct^{-1}|v|^2 \quad \forall v \in \tilde{N}'_\tau, \tag{21}$$

$$L_{\tilde{S}}(\tau)(\bar{u}, v) \le \epsilon |u||v| \quad \forall u \in T_\tau^{\mathbb{C}} R, \forall v \in \tilde{N}'_\tau. \tag{22}$$

From (20), (21), (22), and, essentially, by the first of (19), we get $s^-_{\tilde{S}}(q) = s^-_S(p_o) + (l-1)$, $\forall p$. We take a hypersurface $\tilde{M}$ which intersect $\tilde{S}$ along R with order of contact 2 and with $\tilde{M}^+ \subset \tilde{S}^+$. It is not restrictive to assume $\tilde{M}$ invariant under the flow of $\mathcal{W}$. For otherwise, if f is a projection along the $\mathcal{W}$-integral leaves, one replaces $\tilde{M}$ by $f^{-1}f\tilde{M}$. (Remark here that $R = f^{-1}fR$.) We have obviously:

$$L_{\tilde{M}}(\tau)|_{T_\tau^{\mathbb{C}} R} = L_{\tilde{S}}(\tau)|_{T_\tau^{\mathbb{C}} R}(\sim L_S(p_o)) \quad t \text{ small}, \tau \in R \text{ close to } \tau_o.$$

We also have

$$L_{\tilde{M}}(\tau)(\bar{u}, v) \le \epsilon |u||v| \quad \forall u \in T_\tau^{\mathbb{C}} R, \quad \forall v \in \tilde{N}'_\tau,$$

$$L_{\tilde{M}}(\tau)(\bar{v}, v) \le -ct^{-1}|v|^2 \quad \forall v \in \tilde{N}'_\tau$$

$$L_{\tilde{M}}(\tau)(\bar{u}, w) = 0 \quad \forall u \in \operatorname{Ker} L_{\tilde{S}}(\tau)(= \mathcal{W}(\tau)), \quad \forall w \in T_\tau^{\mathbb{C}} \tilde{M}, \quad \forall \tau \in R,$$

(because $\tilde{M}$ is invariant under the flow of $\mathcal{W}$). It follows:

$$s^-_{\tilde{M}}(q) = s^-_S(p_o) + (l-1) \quad \forall q \in R \times_{\tilde{M}} T^*_{\tilde{M}} X. \tag{23}$$

From (23) we get the conclusion as in Theorem 1. □

Corollary 5 *In the situation of Theorem* 1 (*resp.* 4), *we have*

$$\operatorname{Ker} L_M(p_o) = \operatorname{Ker} L_S(p_o)$$
$$(\text{resp. } \operatorname{Ker} L_M(p) = \operatorname{Ker} L_S(p)\, \forall p \in S \times_M T^*_M X).$$

Proof. It is an immediate consequence of the isomorphisms:

$$\operatorname{Ker} L_S(p) \underset{\Phi_t}{\overset{\sim}{\rightarrow}} \operatorname{Ker} L_{\tilde{S}}(\tau) = \operatorname{Ker} L_{\tilde{M}}(\tau) \underset{\Phi_t^{-1}}{\overset{\sim}{\rightarrow}} \operatorname{Ker} L_M(p).$$

$\square$

2. An application to complex curves in pseudoconvex manifolds

Let X be a complex manifold of dimension n. In [1] it is proved that any complex curve γ in a pseudoconvex hypersurface $S \subset X$ can be *lifted* to a complex curve in $\dot{T}^*_S X$. We extend here the above result to the case of $\operatorname{codim} S > 1$ or $\dim \gamma > 1$.

Theorem 6 *Let S be a generic submanifold of X of codimension l, p_o a point of $\dot{T}^*_S X$, $z_o = \pi(p_o)$, and suppose*

$$s^-_S(p) \equiv 0 \quad \textit{for any} \;\; p \in \dot{T}^*_S X \;\; \textit{close to} \; p_o. \tag{24}$$

We also suppose that there exists a hypersurface M with $M \supset S$, $T^*_M X \ni p_o$ and which satisfies:

$$\operatorname{Ker} L_S(p) \subset \operatorname{Ker} L_M(p) \quad \forall p \in S \times_M \dot{T}^*_M X, \; p \textit{ close to } p_o. \tag{25}$$

Let γ be a complex submanifold of S. Then there exists γ^*, complex submanifold of $\dot{T}^*_S X$, which contains p_o and such that $\pi(\gamma^*) = \gamma$.

Proof. Take an equation $r = 0$ for M with $\partial r(z_o) = p_o$. Then

$$L_r(z)(w, \bar{w}) \geq 0 \quad \forall w \in T^{\mathbb{C}}_z M, \forall z \in S.$$

Let $u \in \dot{T}^{\mathbb{C}}_z \gamma$; clearly $L_r(z)(u, \bar{u}) = 0$. Thus the above inequality implies:

$$L_r(z)(w, \bar{u}) = 0 \quad \forall z \in \gamma, \; \forall w \in T^{\mathbb{C}}_z M. \tag{26}$$

Let $\chi = \chi_{-t}$ be the complex symplectic transformation $\chi : (z;\zeta) \mapsto \left(z + t\frac{\zeta}{(\sum_i \zeta_i^2)^{\frac{1}{2}}}; \zeta\right)$. Thus for the hypersurface $\tilde{S} = \tilde{S}_{-t}$ (different from $\tilde{S} = \tilde{S}_{+t}$ of §1), we have $\chi(\dot{T}^*_S X) = \dot{T}^*_{\tilde{S}} X$. We remark that for $p \in \dot{T}^*_S X$ and with $q = \chi(p) \in \dot{T}^*_{\tilde{S}} X$, we have $\operatorname{rank} L_{\tilde{S}}(q) = \operatorname{rank} L_S(p) + (l-1)$. We also remark that, for t small: $s^+_{\tilde{S}}(q) = s^+_S(p) + (l-1)$. In particular we have:

$$s^-_{\tilde{S}}(q) \equiv 0 \quad \forall q \in \dot{T}^*_{\tilde{S}} X. \tag{27}$$

Let us define $\tilde{\gamma} = \left\{z + t\frac{\partial r(z)}{((\sum_i(\partial_{z_i}r(z))^2)^{\frac{1}{2}})}; \, z \in \gamma\right\}$. We claim that $\tilde{\gamma}$ is a complex manifold in $\tilde{S}$. In fact let us take coordinates $z = x + \sqrt{-1}y \in \mathbb{C}^n$ such that $\gamma = \{0\} \times \cdots \times \{0\} \times \mathbb{C}^d_{z''}$ where $d = \dim_{\mathbb{C}} \gamma$ and $z'' = (z_{n-d+1}, \ldots, z_n)$. What we need to prove is that:

$$\partial_{\bar{z}_h}\left(\frac{\partial r(z)}{(\sum_i(\partial_{z_i}r(z))^2)^{\frac{1}{2}}}\right)|_{\{0\}\times\cdots\times\mathbb{C}^d_{z''}} = 0, \quad \forall h \geq n-d+1, \tag{28}$$

or equivalently:

$$\partial_{\bar{z}_h}\partial_{z_j}r\Big(\sum_i(\partial_{z_i}r)^2\Big) - \partial_{z_j}r\sum_i(\partial_{\bar{z}_h}\partial_{z_i}r)(\partial_{z_i}r) = 0 \qquad \forall h \geq n-d+1, \, \forall j. \tag{29}$$

Let (e_i) be an orthonormal system in $\mathbb{C}^n$, and let $w_i^j = \partial_{z_i}re_j - \partial_{z_j}re_i$. Thus for any fixed j, the set of vectors w_i^j, $i = 1, \ldots, n$, $i \neq j$, is a basis for $T_z^{\mathbb{C}}M$. We may also assume that $u = e_h$. Then the term on the left side of (29) is equal to

$$\begin{aligned}\sum_i &\Big((\partial_{\bar{z}_h}\partial_{z_j}r)(\partial_{z_i}r)^2 - (\partial_{\bar{z}_h}\partial_{z_i}r)(\partial_{z_j}r)(\partial_{z_i}r)\Big) \\ &= \sum_i \Big((\partial_{\bar{z}_h}\partial_{z_j}r)(\partial_{z_i}r) - (\partial_{\bar{z}_h}\partial_{z_i}r)(\partial_{z_j}r)\Big)(\partial_{z_i}r) \\ &= \sum_i \Big(\partial\bar{\partial}r(w_i^j, \bar{u})\Big)(\partial_{z_i}r) = 0 \ \forall j,\end{aligned} \tag{30}$$

due to (26). It follows that $\tilde{\gamma}$ is a complex manifold in the pseudoconvex hypersurface $\tilde{S}$. Thus [1] applies (with suitable modifications because possibly $\dim_{\mathbb{C}} \gamma > 1$), and entails the existence of a complex manifold $\tilde{\gamma}^* \subset \dot{T}^*_{\tilde{S}}X$, such that $\pi(\tilde{\gamma}^*) = \tilde{\gamma}$. Finally if we define $\gamma^* \overset{\text{def.}}{:=} \chi^{-1}(\tilde{\gamma}^*)$, then γ^* is a complex manifold in $\dot{T}^*_S X$ which verifies $\pi(\gamma^*) = \gamma$. □

Remark 7 Let $s_S^-(p) \equiv 0 \, \forall p \in \dot{T}^*_S X$ at p_o. One should wonder whether there exists a pseudoconvex hypersurface M which contains S. But it is not clear if this is true. For this reason we apply [1] not directly to M but to $\tilde{S}$ (with $\dot{T}^*_{\tilde{S}}X = \chi(\dot{T}^*_S X)$). In this respect the crucial point is that $\tilde{\gamma}$ is still a complex manifold in $\tilde{S}$.

Example 8 Let us consider in $\mathbb{C}^3$ with coordinates $z = x + \sqrt{-1}y$:

$$S = \{z; x_3 = 0,\, x_1 = 0\},\;\; p = \mathrm{d}x_1,\;\; \gamma = \{0\} \times \mathbb{C}_{z_2} \times \{0\}.$$

For $M = \{z;\, x_1 = 0\}$, clearly γ can be *lifted* to a complex curve $\gamma^* \subset S \times_M \dot{T}^*_M X$ in (trivial) accordance with Theorem 6. But not any M has this property. For instance if we take $M = \{z;\, x_1 = x_2 x_3\}$, then L_M is non-degenerate and therefore $\dot{T}^*_M X$ contains no complex γ^* because otherwise $T\gamma^* \subset \dot{T}^*_M X \cap \sqrt{-1}\dot{T}^*_M X (\simeq \operatorname{Ker} L_M) = 0$ which is a contradiction.

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