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Morita-Mumford classes on finite cyclic subgroups of the mapping class group of closed surfaces

Takeshi UEMURA

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Abstract. Let G be a finite cyclic subgroup of the mapping class group of order m. We prove the Morita-Mumford classes restricted to G admit a certain kind of periodicity whose period is given by the Euler function $\phi(m)$. Using this periodicity theorem, we compute the Morita-Mumford classes on arbitrary finite cyclic subgroups of the automorphism group of Klein's quartic curve.

Key words: Morita-Mumford class, mapping class group, Klein curve.

Introduction

Let Σ_g be a closed oriented surface of genus $g \geq 2$, and M_g the mapping class group of Σ_g, which is the group of isotopy classes of orientation preserving diffeomorphisms of Σ_g. The cohomological study of M_g has been developed rapidly and has yielded many interesting results. The Morita-Mumford classes, defined by Morita [Mo1] and Mumford [Mu] independently, are a series of cohomology classes of M_g, whose zeroth term is equal to the Euler number $2-2g$ of Σ_g. Many mathematicians, including Harer [H2] [H3], Miller [Mi], and Morita [Mo1] [Mo2] [Mo3] [Mo4], have pointed out the importance of these classes for the study of the stable cohomology ring of M_g. Moreover, recently it is revealed by Akita that the Morita-Mumford classes play an important role in the study of the η-invariant of mapping tori of periodic mapping classes (see [Ak]). We are convinced that the Morita-Mumford classes contribute largely to the unstable cohomological study of M_g in the future.

The Morita-Mumford classes of surface bundles are defined as follows. Let $\pi : E \to B$ be an oriented fiber bundle whose fiber is Σ_g. (We call such a bundle a "surface bundle") The relative tangent bundle $T_{E/B}$ is the oriented real 2-dimensional vector bundle over E consisting of all the tangent vectors along the fibers. Take its Euler class $e := e(T_{E/B}) \in H^2(E; \boldsymbol{Z})$, then $e^{n+1} \in H^{2(n+1)}(E; \boldsymbol{Z})$. Let $\pi_! : H^n(E; \boldsymbol{Z}) \to H^{n-2}(B; \boldsymbol{Z})$ be the

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Gysin homomorphism, which is also called the "integral along the fibers", derived from the Serre spectral sequence of the surface bundle. Then the n-th Morita-Mumford class e_n is defined as follows:

$$e_n = e_n(E) := \pi_!(e^{n+1}) \in H^{2n}(B; \boldsymbol{Z}).$$

It is equal to the pull-back of $e_n \in H^{2n}(M_g; \boldsymbol{Z})$ by the holonomy homomorphism of $\pi_1(B)$ into M_g. Especially if $n = 0$, then e_0 is equal to the Euler number $2 - 2g$ of Σ_g.

The main purpose of this paper is to compute the Morita-Mumford classes on arbitrary finite cyclic subgroups of the automorphism group of the Klein curve. The Klein curve is defined by the equation

$$X^3Y + Y^3Z + Z^3X = 0$$

in the complex projective plane $\boldsymbol{CP^2}$, and it has been studied by many people, including Baker [Ba], Matsuura [Ma], Morifuji [Mf2], Prapavessi [P] and others. As is known, its genus is 3, and its automorphism group is isomorphic to the projective special linear group $PSL(2,7)$.

We will use a general formula for the Morita-Mumford classes (Theorem 2.1) to prove the main result in Section 3. Let C be a compact Riemann surface of genus g and G a finite cyclic group of order m. Suppose G acts on C in a faithful and holomorphic way. Consider the homotopy quotient $\pi : C_G \to B_G$ of this action, which is a surface bundle with fiber C. Let $\zeta = \exp(2\pi\sqrt{-1}/m)$, and $u_0 \in H^2(G; \boldsymbol{Z})$ the Euler class associated with the complex 1-dimensional G-module R given by multiplication by ζ. It is equal to the Euler class of the complex line bundle R_G over the classifying space B_G. Then the Morita-Mumford classes admit a certain kind of periodicity, whose period is $\phi(m)$, the number of integers between 1 and m relatively prime to m. Then

Theorem 2.1 *$e_{n+\phi(m)}(C_G) = e_n(C_G){u_0}^{\phi(m)} \in H^{2(n+\phi(m))}(G; \boldsymbol{Z})$ for $n \geq 0$.*

Theorem 2.1 is discussed in Section 2. In [Ak], Akita notices it for the case where m is a prime. In view of the affirmative solution of the Nielsen realization problem by Kerckhoff [Ke], any finite subgroup of M_g is realized as a holomorphic automorphism group of a suitable Riemann surface. Hence the periodicity theorem (Theorem 2.1) also holds for any cyclic subgroup of M_g. The main result of this paper is the following.

Theorem 3.1 *Let C be the Klein curve and G a finite cyclic group. Suppose G acts on C in a faithful and holomorphic way. Let $\zeta = \exp(2\pi\sqrt{-1}/7)$, and $\omega = \exp(2\pi\sqrt{-1}/3)$. Then the Morita-Mumford classes of this action are given as follows:*

(1) *If $G \cong \boldsymbol{Z}/7$, then*

$$e_n(C_G) = \begin{cases} 3{u_0}^n, & \text{if } n \text{ is a multiple of } 3, \\ 0, & \text{otherwise,} \end{cases}$$

in $H^{2n}(G;\boldsymbol{Z}) \cong \boldsymbol{Z}/7$, where $u_0 \in H^2(G;\boldsymbol{Z})$ denotes the Euler class associated with the complex 1-dimensional G-module given by multiplication by ζ.

(2) *If $G \cong \boldsymbol{Z}/3$, then*

$$e_n(C_G) = \begin{cases} 2{v_0}^n, & \text{if } n \text{ is even}, \\ 0, & \text{if } n \text{ is odd}, \end{cases}$$

in $H^{2n}(G;\boldsymbol{Z}) \cong \boldsymbol{Z}/3$, where $v_0 \in H^2(G;\boldsymbol{Z})$ denotes the Euler class associated with the complex 1-dimensional G-module given by multiplication by ω.

(3) *If $G \cong \boldsymbol{Z}/2$ or $\boldsymbol{Z}/4$, then $e_n(C_G) = 0$ for $n \geq 0$ in $H^{2n}(G;\boldsymbol{Z})$.*

Theorem 3.1 implies that there exist two kinds of finite cyclic subgroups of M_3. One satisfies $e_1 = 0$ and $e_2 \neq 0$, the other $e_1 = e_2 = 0$ and $e_3 \neq 0$. In Section 4, we construct an action of a finite cyclic group on a closed oriented surface satisfying $e_1 = e_2 = \cdots = e_{n-1} = 0$ and $e_n \neq 0$ when n (≥ 4) is an even number or a multiple of 3. Finally in Section 5, we consider the case where C is a hyperelliptic curve, and give two actions of finite cyclic groups. Especially if the genus of C is one, one of them satisfies $e_{\text{odd}} \neq 0$ and $e_{\text{even}} = 0$.

1. Preliminaries

In this section, we recall a fixed-point formula of the Morita-Mumford classes on finite groups ([KU]). In [KU], we studied the Morita-Mumford classes on finite subgroups of M_g in the following situation. Let G be a finite group and C a compact Riemann surface of genus $g \geq 0$. Suppose G acts on C in a faithful and holomorphic way. Consider the universal principal G-bundle $E_G \to B_G$. Then it induces the homotopy quotient (which is also called "the Borel construction") $\pi : C_G \to B_G$ of this action. The space C_G

is the quotient of $E_G \times C$ by the diagonal action of G. The map π induced by the first projection provides an oriented fiber bundle with fiber C

$$C \to C_G \xrightarrow{\pi} B_G.$$

Its Morita-Mumford class $e_n(C_G) \in H^{2n}(B_G; \boldsymbol{Z}) = H^{2n}(G; \boldsymbol{Z})$ is equal to the restriction of e_n to the subgroup G.

Denote the isotropy group at a point $p \in C$ by G_p. The singular set

$$S := \{p \in C \mid G_p \neq \{1\}\}$$

is a G-stable finite subset of C, since the action is faithful and holomorphic. Let $\xi_p = (E_{G_p} \times T_pC)/G_p$ be the oriented real 2-dimensional vector bundle over B_{G_p} associated with the action of G_p on the tangent space T_pC and $e(\xi_p) \in H^2(B_{G_p}; \boldsymbol{Z}) = H^2(G_p; \boldsymbol{Z})$ its Euler class. Since the transfer map $\mathrm{cor}_{G_p}^G : H^*(G_p; \boldsymbol{Z}) \to H^*(G; \boldsymbol{Z})$ is invariant under conjugation, the cohomology class $\mathrm{cor}_{G_p}^G(e(\xi_p)^n) \in H^{2n}(G; \boldsymbol{Z})$ is constant on each G-orbit (see for example [Br].) Then we obtain an explicit formula for the Morita-Mumford classes $e_n(C_G)$ in terms of fixed-point data.

Theorem 1.1 (Kawazumi-Uemura) *In the situation stated above we have*

$$e_n(C_G) = \sum_{p \in S/G} \mathrm{cor}_{G_p}^G(e(\xi_p)^n) \in H^{2n}(B_G; \boldsymbol{Z}) = H^{2n}(G; \boldsymbol{Z})$$

for any $n \geq 1$.

It should be noted that this fixed-point formula is deduced from a general formula of Morita-Mumford classes for fiberwise branched coverings of surface bundles by Miller [Mi] and Morita [Mo1]. The right-hand side depends only on the isotropy groups and their actions on the tangent spaces at the fixed-points.

2. A periodicity theorem for the Morita-Mumford classes

Let C be a compact Riemann surface of genus g. Suppose a finite cyclic group G of order m acts on C in a faithful and holomorphic way. Let $\zeta = \exp(2\pi\sqrt{-1}/m)$, and choose a generator γ of G. Then we consider the complex 1-dimensional G-module R where the action of γ is given by the multiplication by ζ, and define $u_0 \in H^2(G; \boldsymbol{Z})$ by the Euler class associated with R. Throughout this paper, we will call u_0 simply "the Euler class given by multiplication by ζ". Then the Morita-Mumford classes admit a certain kind of periodicity, whose period is $\phi(m)$, the number of integers between 1 and m relatively prime to m. In other words, $\phi(m)$ is the Euler function of m. Then we obtain the following result.

Theorem 2.1 *$e_{n+\phi(m)}(C_G) = e_n(C_G){u_0}^{\phi(m)} \in H^{2(n+\phi(m))}(G; \boldsymbol{Z})$ for $n \geq 0$.*

Proof. Let $S = \coprod_{i=1}^{l} G\cdot p_i$ be the G-stable decomposition of the singular set and m_i the order of $G\cdot p_i$, so that $\frac{m}{m_i} = |G_{p_i}|$. Let $\zeta_i = \exp(2\pi\sqrt{-1}/\frac{m}{m_i})$. Then the action γ^{m_i} on the tangent space $T_{p_i}C$ is equal to the multiplication by $\zeta_i^{k_i}$ for some integer k_i relatively prime to m. From Theorem 1.1, when $n \geq 1$, the Morita-Mumford classes of this action is given as follows:

$$e_n(C_G) = \left(\sum_{i=1}^{l} m_i {k_i}^n\right) {u_0}^n.$$

As is well-known, $k_i^{\phi(\frac{m}{m_i})} \equiv 1 \pmod{\frac{m}{m_i}}$. Since $\phi(\frac{m}{m_i})$ divides $\phi(m)$, this congruence implies $m_i k_i^{\phi(m)} \equiv m_i \pmod{m}$. Therefore we obtain

$$\begin{aligned} e_{n+\phi(m)}(C_G) &= \left(\sum_{i=1}^{l} m_i {k_i}^{n+\phi(m)}\right) {u_0}^{n+\phi(m)} \\ &= \left(\sum_{i=1}^{l} m_i {k_i}^{n}\right) {u_0}^{n} {u_0}^{\phi(m)} = e_n(C_G){u_0}^{\phi(m)} \end{aligned}$$

in $H^{2(n+\phi(m))}(G; \boldsymbol{Z}) \cong \boldsymbol{Z}/m$. In the case where $n = 0$ we have $\sum_{i=1}^{l} m_i \equiv 2 - 2g = e_0(C_G) \pmod{m}$ from the classical Riemann-Hurwitz formula. Hence we obtain

$$e_{\phi(m)}(C_G) = (2-2g)u_0^{\phi(m)} = e_0(C_G)u_0^{\phi(m)}$$

similarly. This concludes the proof. □

Corollary 2.1 *$e_{s\phi(m)}(C_G) = (2-2g){u_0}^{s\phi(m)} = e_0(C_G)u_0^{s\phi(m)}$ for any integer $s \geq 1$.*

If m = 2, 3, 4 and 6, then $\phi(m) \leq 2$. Using Theorem 2.1 and Corollary 2.1, we deduce the following corollaries.

Corollary 2.2 *If $G \cong \boldsymbol{Z}/2$, then $e_n(C_G) = 0$ for $n \geq 0$.*

Corollary 2.3 *If $G \cong \boldsymbol{Z}/3$, $\boldsymbol{Z}/4$ or $\boldsymbol{Z}/6$, then*

$$e_n(C_G) = \begin{cases} (2-2g){u_0}^n, & \text{if } n \text{ is even,} \\ e_1 u_0^{n-1}, & \text{if } n \text{ is odd.} \end{cases}$$

3. An application to the Klein curve

Let C be the complex algebraic curve defined by the equation

$$X^3Y + Y^3Z + Z^3X = 0 \tag{1}$$

in the complex projective plane $\boldsymbol{CP}^2$. The curve C is of genus 3, and called the Klein curve. It is known that the automorphism group $\mathrm{Aut}(C)$ is isomorphic to the projective special linear group $PSL(2,7)$ which has order 168. Moreover $\mathrm{Aut}(C)$ has the presentation

$$PSL(2,7) = \langle s,t \mid s^2 = t^3 = (st)^7 = [s,t]^4 = 1\rangle,$$

where $[s,t] = sts^{-1}t^{-1}$ denotes the commutator of s and t. We may regard it as a subgroup of M_3.

The purpose of this section is to compute the Morita-Mumford classes on arbitrary cyclic subgroups of $PSL(2,7)$ as an application of Theorem 1.1 and Theorem 2.1. The conjugacy classes of $PSL(2,7)$ are as follows (see [I]):

Conjugacy class	1	2	3	4	7_1	7_2
Number of elements	1	21	56	42	24	24

Table 1. Conjugacy classes of $PSL(2,7)$

In Table 1, each conjugacy class is denoted by the order of its elements, and 7_1 and 7_2 mean the different classes. This Table 1 indicates that any two cyclic subgroups of $PSL(2,7)$ are conjugate to each other if they have the same order, and each of them is isomorphic to $\boldsymbol{Z}/2$, $\boldsymbol{Z}/3$, $\boldsymbol{Z}/4$ or $\boldsymbol{Z}/7$.

The main result in this paper is the following.

Theorem 3.1 *Let C be the Klein curve and G a finite cyclic group. Suppose G acts on C in a faithful and holomorphic way. Let $\zeta = \exp(2\pi\sqrt{-1}/7)$, and $\omega = \exp(2\pi\sqrt{-1}/3)$. Then the Morita-Mumford classes of this action are given as follows:*

(1) *If* $G \cong \boldsymbol{Z}/7$, *then*

$$e_n(C_G) = \begin{cases} 3{u_0}^n, & \text{if } n \text{ is a multiple of } 3, \\ 0, & \text{otherwise,} \end{cases}$$

in $H^{2n}(G;\boldsymbol{Z}) \cong \boldsymbol{Z}/7$, *where* u_0 *denotes the Euler class given by multiplication by* ζ.

(2) *If* $G \cong \boldsymbol{Z}/3$, *then*

$$e_n(C_G) = \begin{cases} 2{v_0}^n, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd,} \end{cases}$$

in $H^{2n}(G;\boldsymbol{Z}) \cong \boldsymbol{Z}/3$, *where* v_0 *denotes the Euler class given by multiplication by* ω.

(3) *If* $G \cong \boldsymbol{Z}/2$, *then* $e_n(C_G) = 0$ *for* $n \geq 0$ *in* $H^{2n}(G;\boldsymbol{Z}) \cong \boldsymbol{Z}/2$.

(4) *If* $G \cong \boldsymbol{Z}/4$, *then* $e_n(C_G) = 0$ *for* $n \geq 0$ *in* $H^{2n}(G;\boldsymbol{Z}) \cong \boldsymbol{Z}/4$.

Proof. We recall that the genus of the Klein curve is 3. We see from [KU] that $e_1 = 0$, since $PSL(2,7)$ is a perfect group. Hence (2), (3) and (4) follow from Corollary 2.2 and 2.3 immediately.

In order to prove (1), we define an automorphism γ of C as follows: (see for example [AR], [Kl])

$$\gamma(X,Y,Z) := (\zeta X, \zeta^4 Y, \zeta^2 Z),$$

where $\zeta = \exp(2\pi\sqrt{-1}/7)$. It induces an element γ of order 7 of the automorphism group $PSL(2,7)$. We put $G = \langle\gamma\rangle < PSL(2,7)$. Since any cyclic subgroups of $PSL(2,7)$ of order 7 is conjugate to G, it suffices to compute $e_n(C_G)$.

On the open subset $\{Z \neq 0\}$, substituting $x := X/Z$ and $y := Y/Z$ into (1), we obtain the following function of two variables:

$$f := x^3 y + y^3 + x.$$

Then $\gamma(x) = \zeta^{-1}x$ and $\gamma(y) = \zeta^2 y$. We can easily see that $[0:0:1]$ is the unique fixed point of γ on $\{Z \neq 0\}$. By the implicit function theorem, the variable y can serve as a coordinate at $(x,y) = (0,0)$ since $f_x(0,0) \neq 0 = f_y(0,0)$. Let $u_0 \in H^2(G;\boldsymbol{Z})$ be the Euler class given by multiplication by ζ. Then we can see that the contribution at $[0:0:1]$ is $(2u_0)^n$.

In a similar way, on $\{X \neq 0\}$, $[1:0:0]$ is the unique fixed point and its contribution is u_0^n, and on $\{Y \neq 0\}$, $[0:1:0]$ is the unique fixed point and its contribution is $(-3u_0)^n$. Therefore we obtain

$$\begin{aligned} e_n(C_G) &= (2u_0)^n + u_0^n + (-3u_0)^n \\ &= \{2^n + 1 + 2^{2n}\}u_0^n \end{aligned}$$

in $H^{2n}(G; \boldsymbol{Z}) \cong \boldsymbol{Z}/7$. This concludes the proof. □

Remark 3.1. As is known, we have another action γ_0 of order 7 such that

$$\gamma_0(X, Y, Z) := (\zeta X, \zeta^2 Y, \zeta^4 Z)$$

(see for example [Ba].) If we compute the Morita-Mumford classes using this action, we obtain the following:

$$e_n(C_G) = \begin{cases} -3u_0{}^n, & \text{if } n \text{ is a multiple of 3,} \\ 0, & \text{otherwise,} \end{cases}$$

in $H^{2n}(G; \boldsymbol{Z}) \cong \boldsymbol{Z}/7$.

Remark 3.2. The cyclic actions on the Klein curve C are explicitly given by [Kl], [P], and [Ba]. We can also compute the Morita-Mumford classes on $\boldsymbol{Z}/3$ by using the action τ of order 3 given by

$$\tau(X, Y, Z) := (Y, Z, X) \text{ (cyclic permutation.)}$$

In fact, the fixed points of τ are $[1 : \omega : \omega^2]$ and $[1 : \omega^2 : \omega]$, so using $e_1 = 0$ (recall that $PSL(2, 7)$ is perfect), we obtain the same result as in Theorem 3.1.

4. Some actions of cyclic groups on surfaces

Theorem 3.1 implies the existence of a finite cyclic subgroup of M_3 satisfying $e_1 = 0$, $e_2 \neq 0$, and $e_1 = e_2 = 0$, $e_3 \neq 0$. So we consider the following problem.

Problem Construct a finite cyclic subgroup of M_g satisfying $e_1 = e_2 = \cdots = e_{n-1} = 0$ and $e_n \neq 0$ for each $n \geq 4$.

In this section, we will give two affirmative partial answers to this problem.

Theorem 4.1 *For an arbitrary integer $m \geq 0$, there exists an action of a finite cyclic group G on a closed oriented surface C satisfying $e_1(C_G) = e_2(C_G) = \cdots = e_{2m-1}(C_G) = 0$ and $e_{2m}(C_G) \neq 0$.*

Theorem 4.2 *For an arbitrary integer $m \geq 0$, there exists an action of a finite cyclic group G on a closed oriented surface C satisfying $e_1(C_G) = e_2(C_G) = \cdots = e_{3m-1}(C_G) = 0$ and $e_{3m}(C_G) \neq 0$.*

Proof of Theorem 4.1. By Dirichlet's Theorem, there exists a prime p satisfying $p = 2ml + 1$ for some integer $l \geq 1$. Let k be a primitive root of p, so that $k^{p-1} \equiv 1 \pmod p$ and $k_0 := k^l$. We consider the following situation. At first, let S_i^2 be the 2-sphere of radius $a > 0$ inside $\boldsymbol{R}^3$ defined by the following equation:

$$S_i^2 = \{(x, y, z) \in \boldsymbol{R}^3 \mid x^2 + y^2 + \{z + 3(i-1)a\}^2 = a^2\}$$

for $1 \leq i \leq m$. Secondly, take $2p$ points

$$\begin{aligned} p_{i_+}^j &= \left(\frac{\sqrt{3}}{2} a \cos\left(\frac{2j\pi}{p}\right), \frac{\sqrt{3}}{2} a \sin\left(\frac{2j\pi}{p}\right), \left(-3i + \frac{7}{2}\right) a\right) \\ p_{i_-}^j &= \left(\frac{\sqrt{3}}{2} a \cos\left(\frac{2j\pi}{p}\right), \frac{\sqrt{3}}{2} a \sin\left(\frac{2j\pi}{p}\right), \left(-3i + \frac{5}{2}\right) a\right) \end{aligned}$$

on each S_i^2 $(0 \leq j \leq p-1)$. Take sufficiently small open discs $U_{i_\pm}^j$ centered at $p_{i_\pm}^j$ respectively, and connect $U_{i_-}^j$ and $U_{(i+1)_+}^{k_0 j}$ with a tube for each i, j. Then we obtain a closed oriented surface C of genus $(p-1)(m-1)$. We define an action of the cyclic group $G = \boldsymbol{Z}/p$ on C as follows. Rotate S_i^2 by $2k_0^{i-1}\pi/p$ about the z-axis. From the construction, these actions extend to the action of $G = \boldsymbol{Z}/p$ on the whole surface C. Let $u_0 \in H^2(G; \boldsymbol{Z})$ be the Euler class given by multiplication by $\zeta = \exp(2\pi\sqrt{-1}/p)$. Then the isotropy group of each singular point is G, namely, this action is semi-free. The fixed points on ${S_i}^2$ are $(0, 0, (-3i + 3 \pm 1)a)$. Considering the contribution of each fixed point, the n-th Morita-Mumford class of this action is

$$\begin{aligned} e_n(C_G) &= {u_0}^n + (-u_0)^n + (k_0 u_0)^n + (-k_0 u_0)^n + \cdots \\ &\qquad + ({k_0}^{m-1} u_0)^n + (-{k_0}^{m-1} u_0)^n \\ &= \{1 + (-1)^n + {k_0}^n + (-k_0)^n + \cdots \\ &\qquad + {k_0}^{(m-1)n} + (-k_0)^{(m-1)n}\} u_0^n \end{aligned}$$

in $H^{2n}(G; \boldsymbol{Z}) \cong \boldsymbol{Z}/p$. It is obvious that $e_n(C_G) = 0$ when n is an odd number. If $n = 2t$ $(1 \leq t \leq m-1)$, then

$$e_{2t}(C_G) = 2(1 + {k_0}^{2t} + {k_0}^{4t} + \cdots + {k_0}^{2(m-1)t}){u_0}^{2t}$$

$$\begin{aligned} &= 2 \cdot \frac{{k_0}^{2mt} - 1}{{k_0}^{2t} - 1} {u_0}^{2t} \\ &= 0, \end{aligned}$$

since ${k_0}^{2mt} = k^{2mlt} = 1$, and ${k_0}^{2t} = k^{2lt} \neq 0$. If $n = 2m$, then

$$\begin{aligned} e_{2m}(C_G) &= 2(1 + {k_0}^{2m} + {k_0}^{4m} + \cdots + {k_0}^{2m(m-1)}){u_0}^{2m} \\ &= 2(1 + k^{2ml} + k^{4ml} + \cdots + k^{2ml(m-1)}){u_0}^{2m} \\ &= 2 \cdot 1 \cdot m{u_0}^{2m} \\ &\neq 0 \end{aligned}$$

in $\boldsymbol{Z}/p$ since $m < p - 1 = 2ml$. This concludes the proof. □

Consider the case where $m = 1$ in the proof of Theorem 4.1. Then the genus of C is zero. Therefore C is isomorphic to the complex projective line $\boldsymbol{P}^1$. Any action of a finite cyclic group on $\boldsymbol{P}^1$ is conjugate to the rotation as above. We can regard C as the unit sphere S^2 in $\boldsymbol{R}^3$ by a suitable diffeomorphism. So we can define the action of $G = \boldsymbol{Z}/p$ on C, which is the rotation of C by $2a\pi/p$ about the z-axis for some integer $1 \leq a \leq [\frac{p}{2}]$. Here $[\frac{p}{2}]$ denotes the largest integer less than or equal to $\frac{p}{2}$. Therefore we obtain the following.

Proposition 4.1 *Let C be the Riemann sphere $\boldsymbol{P}^1$. Suppose $G = \boldsymbol{Z}/p$ acts on C as above. Let $u_0 \in H^2(G; \boldsymbol{Z})$ be the Euler class given by multiplication by $\zeta = \exp(2\pi\sqrt{-1}/m)$. Then*

$$e_n(C_G) = \begin{cases} 2a{u_0}^n, & \text{if } n \text{ is even}, \\ 0, & \text{if } n \text{ is odd}. \end{cases}$$

Proof of Theorem 4.2. By Dirichlet's Theorem, there exists a prime p satisfying $p = 3ml + 1$ for some integer $l \geq 1$. Let k be a primitive root of p, and $k_0 := k^l$, and a (≥ 2) the smallest integer satisfying $p \mid 1 + a + a^2$. Define the complex algebraic curve C_0 by

$$X^{a+1}Y + Y^{a+1}Z + Z^{a+1}X = 0$$

in $\boldsymbol{CP}^2$. It is not difficult to see that C_0 is a non-singular curve, and its genus is $a(a+1)/2$ by *Plücker's formula*. Prepare m copies C_i $(1 \leq i \leq m)$ of the curve C_0. Similarly in the proof of Theorem 3.1, we define an automorphism γ_i on each C_i as follows:

$$\gamma_i(X,Y,Z) := (\zeta^{k_0^{i-1}}X, \zeta^{k_0^{i-1}a^2}Y, \zeta^{k_0^{i-1}a}Z),$$

where $\zeta = \exp(2\pi\sqrt{-1}/p)$. Note that $\gamma_i = \gamma_1^{k_0^{i-1}}$.

Each γ_i induces an action of the cyclic group $G = \boldsymbol{Z}/p$ on C_i. We can easily see that the singular set $S_i \subset C_i$ of G is

$$S_i = \{[1:0:0],[0:1:0],[0:0:1]\}.$$

Choose two points $p_i, q_i \in C_i - S_i$ such that $G\cdot p_i \cap G\cdot q_i = \emptyset$. Define $p_i^j := \gamma_i^j(p_i)$ and $q_i^j := \gamma_i^j(q_i)$ for $0 \le j \le p-1$. Note that the action of G on $C_i - S_i$ is free. Take sufficiently small open discs $U_{i,j}$ and $V_{i,j}$ in C_i centered at p_i^j and q_i^j, respectively. Connect $V_{i,j}$ and $U_{i+1,j}$ with a tube for each i,j $(1 \le i \le m-1)$. Then we obtain a closed oriented surface C of genus $a(a+1)(p-1)(m-1)/2$. From this construction, the automorphisms γ_i's extend to the action of $G = \boldsymbol{Z}/p$ on the whole surface C.

Let $u_0 \in H^2(G;\boldsymbol{Z})$ be the Euler class given by multiplication by $\zeta = \exp(2\pi\sqrt{-1}/p)$. Clearly this action is semi-free, and we can compute the contribution of each fixed point similarly in the proof of Theorem 3.1. Therefore the n-th Morita-Mumford class of the action on C_i is

$$\begin{aligned} e_n((C_i)_G) &= [\{{k_0}^{i-1}(a-1)\}^n + \{{k_0}^{i-1}(1-a^2)\}^n \\ &\qquad + \{{k_0}^{i-1}(a^2-a)\}^n]{u_0}^n, \end{aligned}$$

and that of the action on the whole surface C is

$$\begin{aligned} &e_n(C_G) \\ &= \sum_{i=1}^{m} e_n((C_i)_G) \\ &= (1+{k_0}^n+\cdots+{k_0}^{(m-1)n})\{(a-1)^n+(1-a^2)^n+(a^2-a)^n\}{u_0}^n \end{aligned}$$

in $H^{2n}(G;\boldsymbol{Z}) \cong \boldsymbol{Z}/p$. It is easy to check that $e_n(C_G) = 0$ when n is not a multiple of 3. If $n = 3t$ $(1 \le t \le m-1)$, then

$$(1+{k_0}^{3t}+\cdots+{k_0}^{3t(m-1)}) = \frac{{k_0}^{3mt}-1}{{k_0}^{3t}-1} = \frac{k^{3lmt}-1}{k^{3lt}-1} = 0,$$

since ${k_0}^{3mt} = k^{3mlt} = 1$, and ${k_0}^{3t} = k^{3lt} \neq 0$. Therefore $e_{3t}(C_G) = 0 \in \boldsymbol{Z}/p$.

If $n = 3m$, then

$$\begin{aligned}(1 + {k_0}^{3m} + \cdots + {k_0}^{3m(m-1)}) &= 1 + k^{3ml} + \cdots + k^{3m(m-1)l} \\ &= 1 \cdot m \neq 0.\end{aligned}$$

Therefore it is easy to see that $e_{3m}(C_G) = 3m(a-1)^{3m}{u_0}^{3m} \neq 0$ in $\boldsymbol{Z}/p$. This concludes the proof. $\square$

5. Hyperelliptic curves

In this section, we consider the case where C is a hyperelliptic curve, and give two actions of finite cyclic groups.

Example 5.1. Consider two complex plane curves

$$w^2 = z(1 - z^{2g}), \qquad {w_1}^2 = z_1({z_1}^{2g} - 1)$$

for $g \geq 1$. Glueing them each other by the map $z_1 = z^{-1}$ and $w_1 = z^{-g-1}w$, we obtain a hyperelliptic curve C of genus g. Let $\zeta = \exp(2\pi\sqrt{-1}/4g)$, consider the action

$$\gamma : (z, w) \longmapsto (\zeta^{2k} z, \zeta^k w) \quad (k = 1, 2, \ldots, 4g-1).$$

Then it gives an automorphism of C of order $4g$. Its singular set S is

$$S = \{(0,0), \infty, (\zeta^{2j}, 0);\ j = 0, 1, \ldots, 2g-1\},$$

where ∞ denotes the point at infinity: $(z_1, w_1) = (0, 0)$. This action is not semi-free since the isotropy groups of $(0,0)$ and ∞ are $\langle\gamma\rangle$, but that of $(\zeta^{2j}, 0)$ is $\langle\gamma^{2g}\rangle$. Here $\langle\gamma\rangle$ (resp. $\langle\gamma^{2g}\rangle$) denotes the automorphism group of C generated by γ (resp. γ^{2g}). Let $u_0 \in H^2(\langle\gamma\rangle; \boldsymbol{Z})$ (resp. $v_0 \in H^2(\langle\gamma^{2g}\rangle; \boldsymbol{Z})$ be the Euler class given by multiplication by ζ (resp. ζ^{2g}). Then u_0^n (resp. v_0^n) generates the group $H^{2n}(\langle\gamma\rangle; \boldsymbol{Z}) \cong \boldsymbol{Z}/4g$ (resp. $H^{2n}(\langle\gamma^{2g}\rangle; \boldsymbol{Z}) \cong \boldsymbol{Z}/2$) for each n.

Then Theorem 1.1 implies

$$e_n(C_{\langle\gamma\rangle}) = {u_0}^n + \{-(2g+1)\}^n {u_0}^n + \mathrm{cor}_{\langle\gamma^{2g}\rangle}^{\langle\gamma\rangle} {v_0}^n \in H^{2n}(\langle\gamma\rangle; \boldsymbol{Z}).$$

From well-known properties of the transfer map, we can easily see that $\mathrm{cor}_{\langle\gamma^{2g}\rangle}^{\langle\gamma\rangle} {v_0}^n = [\langle\gamma\rangle : \langle\gamma^{2g}\rangle]{u_0}^n = 2g{u_0}^n$ (see for example [Br]). Therefore we obtain

$$e_n(C_{\langle\gamma\rangle}) = \begin{cases} (2+2g){u_0}^n, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd,} \end{cases}$$

in $H^{2n}(\langle\gamma\rangle; \boldsymbol{Z}) \cong \boldsymbol{Z}/4g$. Especially if $g = 1$, then $2 + 2g \equiv 0 \pmod 4$. So $e_n(C_{\langle\gamma\rangle}) = 0$ for any $n \geq 0$.

Example 5.2. Consider two complex plane curves

$$w^2 = z(1 - z^{2g+1}), \qquad {w_1}^2 = {z_1}^{2g+1} - 1$$

for $g \geq 1$. Glueing them each other by the map $z_1 = z^{-1}$ and $w_1 = z^{-g-1}w$, we obtain a hyperelliptic curve C of genus g. Let $\zeta = \exp(2\pi\sqrt{-1}/(4g+2))$, consider the action

$$\gamma : (z, w) \longmapsto (\zeta^{2k}z, \zeta^k w) \quad (k = 1, 2, \ldots, 4g+1).$$

Then it gives an automorphism of C of order $4g + 2$. Its singular set S is

$$S = \{(0,0), \infty_-, \infty_+, (\zeta^{2j}, 0)\ ;\ j = 0, 1, \ldots, 2g\},$$

where ∞_- and ∞_+ denote the points at infinity: $(z_1, w_1) = (0, \pm\sqrt{-1})$. This action is not semi-free since the isotropy group of $(0,0)$ is $\langle\gamma\rangle$, but those of ∞_- and ∞_+ are $\langle\gamma^2\rangle$, and that of $(\zeta^{2j}, 0)$ is $\langle\gamma^{2g+1}\rangle$. Take the Euler classes $u_0 \in H^2(\langle\gamma\rangle; \boldsymbol{Z})$, $m_0 \in H^2(\langle\gamma^2\rangle; \boldsymbol{Z})$, and $v_0 \in H^2(\langle\gamma^{2g+1}\rangle; \boldsymbol{Z})$ similarly in Example 5.1.

Then Theorem 1.1 implies

$$e_n(C_{\langle\gamma\rangle}) = {u_0}^n + \mathrm{cor}^{\langle\gamma\rangle}_{\langle\gamma^2\rangle}(-m_0)^n + \mathrm{cor}^{\langle\gamma\rangle}_{\langle\gamma^{2g+1}\rangle}\, {v_0}^n \in H^{2n}(\langle\gamma\rangle; \boldsymbol{Z}).$$

Note that the actions at the points at infinity are $z_1 \mapsto \zeta^{-2k}z_1$ and $w_1 \mapsto \zeta^{-(2g+1)k}w_1 = (-1)^k w_1$, so the contribution at each point is $(-m_0)^n$. Similarly in Example 5.1, we can easily see that $\mathrm{cor}^{\langle\gamma\rangle}_{\langle\gamma^2\rangle}(-m_0)^n = [\langle\gamma\rangle : \langle\gamma^2\rangle]{u_0}^n = 2(-1)^n {u_0}^n$, and $\mathrm{cor}^{\langle\gamma\rangle}_{\langle\gamma^{2g+1}\rangle}\, {v_0}^n = [\langle\gamma\rangle : \langle\gamma^{2g+1}\rangle]{u_0}^n = (2g+1){u_0}^n$. Therefore we obtain

$$e_n(C_{\langle\gamma\rangle}) = \begin{cases} 2(2+g){u_0}^n, & \text{if } n \text{ is even,} \\ 2g{u_0}^n, & \text{if } n \text{ is odd,} \end{cases}$$

in $H^{2n}(\langle\gamma\rangle; \boldsymbol{Z}) \cong \boldsymbol{Z}/(4g+2)$. Especially if $g = 1$, then $2(2+1) \equiv 0 \pmod 6$.

So we obtain

$$e_n(C_{\langle\gamma\rangle}) = \begin{cases} 0, & \text{if } n \text{ is even,} \\ 2{u_0}^n, & \text{if } n \text{ is odd,} \end{cases}$$

in $H^{2n}(\langle\gamma\rangle; \boldsymbol{Z}) \cong \boldsymbol{Z}/6$. This example shows that $e_{\text{odd}} \neq 0$ and $e_{\text{even}} = 0$, and differs from the others described above.

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Hokurei High School
Shin-ei, Kiyota-ku
Sapporo 004-0839, Japan
E-mail: someday@netfarm.ne.jp