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Abel-Tauber theorems for Hankel and Fourier transforms and a problem of Boas

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Abstract. We prove Abel-Tauber theorems for Hankel and Fourier transforms. For example, let f be a locally integrable function on $[0, \infty)$ which is eventually decreasing to zero at infinity. Let $\rho = 3, 5, 7, \ldots$ and ℓ be slowly varying at infinity. We characterize the asymptotic behavior $f(t) \sim \ell(t)t^{-\rho}$ as $t \to \infty$ in terms of the Fourier cosine transform of f. Similar results for sine and Hankel transforms are also obtained. As an application, we give an answer to a problem of R.P. Boas on Fourier series.

Key words: Abel-Tauber theorems, Hankel transforms, Fourier transforms, Fourier series, Π-variation.

1. Introduction and results

As a prototype, we use Fourier cosine transforms to explain our problem. Let f be a locally integrable, eventually decreasing function on $[0, \infty)$ which tends to zero at infinity, and let F_c be its Fourier cosine transform. Let $\rho > 0$ and ℓ be slowly varying at infinity (see below). We are concerned with Abel-Tauber theorems which characterize the asymptotic behavior $f(t) \sim \ell(t)t^{-\rho}$ as $t \to \infty$ in terms of F_c. It turns out that the values $1, 3, 5, \ldots$ of ρ are exceptional. For $\rho \neq 1, 3, 5, \ldots$, one can obtain the desired Abel-Tauber theorems using regular variation — or Karamata theory. See Bingham-Goldie-Teugels [BGT, Ch. 4], where references to earlier work by Hardy and Rogosinski, Aljančić, Bojanić and Tomić, Vuilleumier, Zygmund and others are given. However the same theorems do not hold for $\rho = 1, 3, 5, \ldots$. These exceptional values are related to the power series expansion of the kernel $\cos x$ (see Soni-Soni [SS]).

In [I1], one of the authors showed that one could use Π-variation — or de Haan theory in the terminology of [BGT] — to obtain the desired Abel-Tauber theorem for cosine transforms when $\rho = 1$. For theorems of the same type, we refer to [I1] (cosine series and integrals), [I2] (sine series and integrals), [I3] (Fourier-Stieltjes coefficients), and Bingham-Inoue [BI]

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(Hankel transforms).

In this paper, we consider the remaining exceptional values, e.g., $\rho = 3, 5, \ldots$ for cosine transforms. In fact, as in [BI], we consider those for Hankel transforms from the beginning; the results for cosine and sine transforms follow as special cases. As an application, we give an answer to a problem of R.P. Boas on Fourier series.

We write R_0 for the class of slowly varying functions at infinity, that is, the class of positive measurable ℓ, defined on some neighbourhood of infinity, satisfying

$$\ell(\lambda x)/\ell(x) \to 1 \qquad (x \to \infty) \qquad \forall \lambda > 0.$$

For $\ell \in R_0$, the class Π_ℓ is the class of measurable f satisfying

$$\{f(\lambda x) - f(x)\}/\ell(x) \to c \log \lambda \qquad (x \to \infty) \qquad \forall \lambda > 0$$

for some constant c, called the *ℓ-index* of f. See [BGT] for background.

Let $\nu \geq -1/2$, $t^{\nu+\frac{1}{2}}h(t) \in L^1_{\mathrm{loc}}[0,\infty)$, and h be ultimately decreasing to zero at infinity. We consider the *Hankel Transform*

$$H_\nu(x) := \int_0^{\infty-} h(t)(xt)^{1/2} J_\nu(xt)dt \qquad (0 < x < \infty), \tag{1.1}$$

where $\int_0^{\infty-}$ denotes an improper integral $\lim_{M\to\infty} \int_0^M$ and J_ν is the Bessel function

$$J_\nu(x) = \sum_{j=0}^{\infty} c_{\nu,j} x^{\nu+2j} \qquad (0 \leq x < \infty)$$

with

$$c_{\nu,j} := \frac{(-1)^j}{2^{\nu+2j} \cdot j! \cdot \Gamma(\nu + j + 1)} \quad (\nu \geq -1/2,\ j = 0, 1, \ldots). \tag{1.2}$$

Since the improper integral on the right of (1.1) converges uniformly on each (a, ∞) with $a > 0$, H_ν is finite and continuous on $(0, \infty)$.

For $n \in \mathbb{N}$ and $x \in (0, \infty)$, we define $\bar{H}_{\nu,n}$ by

$$\begin{aligned}&\bar{H}_{\nu,n}(x)\\ &:= x^{\nu+\frac{1}{2}+2n}\left\{H_\nu(1/x) - \sum_{j=0}^{n-1} c_{\nu,j}\int_0^\infty t^{\nu+\frac{1}{2}+2j}h(t)dt \cdot x^{-\nu-\frac{1}{2}-2j}\right\}\end{aligned} \tag{1.3}$$

if $\int_0^\infty t^{\nu-\frac{3}{2}+2n}h(t)dt < \infty$.

Theorem 1 *Let $\ell \in R_0$ and $n \in \mathbb{N}$. Let $\nu \geq -1/2$, $t^{\nu+\frac{1}{2}}h(t) \in L^1_{\text{loc}}[0,\infty)$, and h be ultimately decreasing to zero at infinity, with Hankel transform H_ν. Then*

$$h(t) \sim t^{-\nu-\frac{3}{2}-2n}\ell(t) \qquad (t \to \infty) \tag{1.4}$$

if and only if

$$\int_0^\infty t^{\nu-\frac{3}{2}+2n}h(t)dt < \infty \quad \text{and} \quad \bar{H}_{\nu,n} \in \Pi_\ell \ \text{with } \ell\text{-index } c_{\nu,n}. \tag{1.5}$$

Note that Theorem 1 includes results for Fourier cosine and sine transforms, as

$$x^{1/2}J_{-1/2}(x) = \sqrt{\frac{2}{\pi}}\cos x, \qquad x^{1/2}J_{1/2}(x) = \sqrt{\frac{2}{\pi}}\sin x.$$

For $x \in (0,\infty)$, we define $\bar{H}_\nu$ by

$$\bar{H}_\nu(x) := x^{\nu+\frac{1}{2}}H_\nu(1/x). \tag{1.6}$$

We will prove Theorem 1 by reducing the problem to the following known result (which corresponds to the case $n = 0$ of (1.4)):

Theorem A ([BI], extending [I1], [I2]) *Let ν, h, H_ν and ℓ be as in Theorem 1. Then*

$$h(t) \sim t^{-\nu-\frac{3}{2}}\ell(t) \qquad (t \to \infty) \tag{1.7}$$

if and only if

$$\bar{H}_\nu \in \Pi_\ell \ \text{with } \ell\text{-index } c_{\nu,0}. \tag{1.8}$$

The cosine case $\nu = -\frac{1}{2}$ of Theorem A is due to [I1], the sine case $\nu = \frac{1}{2}$ to [I2], and the general case $\nu \geq -\frac{1}{2}$ to Bingham-Inoue [BI].

The theorems above treat the boundary cases to the following known Abel-Tauber theorem for Hankel transforms:

Theorem B ([RS], [SS], extending [P], [B]) *Let ν, h, H_ν and ℓ be as in Theorem 1.*

(1) *For* $0<\rho<\nu+\frac{3}{2}$,

$$h(t)\sim t^{-\rho}\ell(t) \qquad (t\to\infty) \tag{1.9}$$

if and only if

$$H_\nu(x)\sim x^{\rho-1}\ell(1/x)\cdot 2^{\frac{1}{2}-\rho}\frac{\Gamma(\frac{3}{4}+\frac{\nu}{2}-\frac{\rho}{2})}{\Gamma(\frac{1}{4}+\frac{\nu}{2}+\frac{\rho}{2})} \quad (x\to 0+). \tag{1.10}$$

(2) *Let* $n\in\mathbb{N}$ *and* $\nu-\frac{1}{2}+2n<\rho<\nu+\frac{3}{2}+2n$. *Then* (1.9) *holds if and only if* $\int_0^\infty t^{\nu-\frac{3}{2}+2n}h(t)dt<\infty$ *and*

$$H_\nu(x)-\sum_{j=0}^{n-1}c_{\nu,j}\int_0^\infty t^{\nu+\frac{1}{2}+2j}h(t)dt\cdot x^{\nu+\frac{1}{2}+2j}$$
$$\sim x^{\rho-1}\ell(1/x)\cdot 2^{\frac{1}{2}-\rho}\frac{\Gamma(\frac{3}{4}+\frac{\nu}{2}-\frac{\rho}{2})}{\Gamma(\frac{1}{4}+\frac{\nu}{2}+\frac{\rho}{2})} \quad (x\to 0+). \tag{1.11}$$

The part (1) of Theorem B is due to Pitman [P], Bingham [B], and Ridenhour-Soni [RS], while the part (2) to Soni-Soni [SS].

We focus on Fourier (cosine and sine) transforms. Let $f\in L^1_{\rm loc}[0,\infty)$ and f be ultimately decreasing to zero at infinity. We write $F_{\rm c}$ for the *Fourier cosine transform* of f:

$$F_{\rm c}(x)=\int_0^{\infty-}f(t)\cos(xt)dt \qquad (0<x<\infty). \tag{1.12}$$

Similarly, let $g(t)t\in L^1_{\rm loc}[0,\infty)$, and g be ultimately decreasing to zero at infinity. We write $G_{\rm s}$ for the *Fourier sine transform* of g:

$$G_{\rm s}(x)=\int_0^{\infty-}g(t)\sin(xt)dt \qquad (0<x<\infty). \tag{1.13}$$

Now, at least formally,

$$F_{\rm c}^{(2j)}(0)=(-1)^j\int_0^\infty t^{2j}f(t)dt,$$
$$G_{\rm s}^{(2j+1)}(0)=(-1)^j\int_0^\infty t^{2j+1}g(t)dt.$$

So for $n \in \mathbb{N}$ we define $\bar{F}_{c,n}$ by

$$\bar{F}_{c,n}(x) := x^{2n}\left\{F_c(1/x) - \sum_{j=0}^{n-1} \frac{F_c^{(2j)}(0)}{(2j)!}x^{-2j}\right\} \quad (0 < x < \infty) \tag{1.14}$$

if $F_c \in C^{2n-2}([0,\infty))$. Similarly, for $n \in \mathbb{N}$, we define $\bar{G}_{s,n}$ by

$$\bar{G}_{s,n}(x) := x^{2n+1}\left\{G_s(1/x) - \sum_{j=0}^{n-1} \frac{G_s^{(2j+1)}(0)}{(2j+1)!}x^{-2j-1}\right\} \quad (0 < x < \infty) \tag{1.15}$$

if $G_s \in C^{2n-1}([0,\infty))$. Here as usual, $C^m([0,\infty))$ is the class of functions which are of $C^m(I)$-class for some open neighbourhood I of $[0,\infty)$.

Theorem 2 *Let $\ell \in R_0$ and $n \in \mathbb{N}$. Let $f \in L^1_{\text{loc}}[0,\infty)$ and f be ultimately decreasing to zero at infinity, with Fourier cosine transform F_c. Then*

$$f(t) \sim t^{-2n-1}\ell(t) \qquad (t \to \infty) \tag{1.16}$$

if and only if

$$F_c \in C^{2n-2}([0,\infty)) \;\; \textit{and} \;\; \bar{F}_{c,n} \in \Pi_\ell \;\; \textit{with } \ell\textit{-index } \frac{(-1)^n}{(2n)!}. \tag{1.17}$$

Theorem 3 *Let $\ell \in R_0$ and $n \in \mathbb{N}$. Let $g(t)t \in L^1_{\text{loc}}[0,\infty)$ and g be ultimately decreasing to zero at infinity, with Fourier sine transform G_s. Then*

$$g(t) \sim t^{-2n-2}\ell(t) \qquad (t \to \infty) \tag{1.18}$$

if and only if

$$G_s \in C^{2n-1}([0,\infty)) \;\; \textit{and} \;\; \bar{G}_{s,n} \in \Pi_\ell \;\; \textit{with } \ell\textit{-index } \frac{(-1)^n}{(2n+1)!}. \tag{1.19}$$

Remark. In Theorem 2, $F_c \in C^{2n-2}([0,\infty))$ implies that the limit $F_c(0+)$ exists and that F_c, with $F_c(0) := F_c(0+)$, is in $C^{2n-2}([0,\infty))$; similarly for the meaning of $G_s \in C^{2n-1}([0,\infty))$ in Theorem 3.

We will prove Theorems 2 and 3 using Theorem 1.

We give an application of Theorem 3 to probability theory. Let X be a real random variable defined on a probability space $(\Omega, \mathcal{F}, P)$. The *tail-sum* of X is the function T defined by

$$T(x) := P(X \leq -x) + P(X > x) \qquad (0 \leq x < \infty).$$

Note that T is finite and decreases to zero at infinity. Now

$$\{1 - U(\xi)\}/\xi = \int_0^{\infty-} T(x)\sin(x\xi)dx \qquad (0 < \xi < \infty),$$

where U is the real part of the characteristic function of X:

$$U(\xi) := E[\cos(\xi X)] \qquad (\xi \in \mathbb{R})$$

(see [BGT, p. 336]). By Theorem 3, the asymptotic behavior

$$T(x) \sim x^{-2n-2}\ell(x) \qquad (x \to \infty)$$

with $n \in \mathbb{N}$ and $\ell \in R_0$ is characterized in terms of U.

We can apply Theorems 1 and A to Question 7.19 of Boas [Bo]. For $f \in L^1[0, \pi]$, we define its *Fourier cosine coefficients* a_n by

$$\begin{aligned} a_n &:= \frac{2}{\pi}\int_0^{\pi} f(t)\cos(nt)dt \quad (n = 1, 2, \ldots), \\ &:= \frac{1}{\pi}\int_0^{\pi} f(t)dt \quad (n = 0). \end{aligned} \tag{1.20}$$

Similarly, for $g \in L^1[0, \pi]$, we define its *Fourier sine coefficients* b_n by

$$b_n := \frac{2}{\pi}\int_0^{\pi} g(t)\sin(nt)dt \qquad (n = 1, 2, \ldots). \tag{1.21}$$

Theorem 4 *Let $f \in L^1[0, \pi]$ with Fourier cosine coefficients (a_k). We assume that $a_k \geq 0$ for all $k \geq 0$. Let $n \in \mathbb{N}$ and $\ell \in R_0$. Then*

$$\sum_{k=m}^{\infty} a_k \sim \frac{\ell(m)}{m^{2n}} \cdot \frac{1}{2n} \qquad (m \to \infty) \tag{1.22}$$

if and only if

$$f \in C^{2n-2}([0, \pi]) \quad \textit{and} \quad \bar{f}_n \in \Pi_\ell \ \textit{with } \ell\textit{-index } \frac{(-1)^n}{(2n)!}, \tag{1.23}$$

where

$$\bar{f}_n(x) := x^{2n}\left\{f(1/x) - \sum_{j=0}^{n-1} \frac{f^{(2j)}(0)}{(2j)!} x^{-2j}\right\} \quad (1/\pi \le x < \infty). \tag{1.24}$$

Corollary *In Theorem* 4, *we further assume that* (a_k) *is decreasing. Then* (1.23) *is equivalent to*

$$a_m \sim \frac{\ell(m)}{m^{2n+1}} \qquad (m \to \infty). \tag{1.25}$$

Theorem 5 *Let* $g \in L^1[0,\pi]$ *with Fourier sine coefficients* (b_k). *We assume that* $b_k \ge 0$ *for all* $k \ge 1$. *Let* $n \in \mathbb{N}$ *and* $\ell \in R_0$. *Then*

$$\sum_{k=m}^{\infty} b_k \sim \frac{\ell(m)}{m^{2n+1}} \cdot \frac{1}{2n+1} \qquad (m \to \infty) \tag{1.26}$$

if and only if

$$g \in C^{2n-1}([0,\pi]) \quad \text{and} \quad \bar{g}_n \in \Pi_\ell \text{ with } \ell\text{-index } \frac{(-1)^n}{(2n+1)!}, \tag{1.27}$$

where

$$\bar{g}_n(x) := x^{2n+1}\left\{g(1/x) - \sum_{j=0}^{n-1} \frac{g^{(2j+1)}(0)}{(2j+1)!} x^{-2j-1}\right\} \quad (1/\pi \le x < \infty). \tag{1.28}$$

Corollary *In Theorem* 5, *we further assume that* (b_k) *is decreasing. Then* (1.27) *is equivalent to*

$$b_m \sim \frac{\ell(m)}{m^{2n+2}} \qquad (m \to \infty). \tag{1.29}$$

Remark. We understand that $L^1[0,\pi]$ consists of equivalence classes with respect to the equivalence relation $f_1 \sim f_2 \Leftrightarrow f_1 = f_2$ a.e. So, e.g., in (1.23), $f \in C^{2n-2}([0,\pi])$ implies that there exists a function in $C^{2n-2}([0,\pi])$ which lies in the equivalence class of f and that we identify the function with f. In particular, if $\sum_{k=0}^{\infty} |a_k| < \infty$, then by [Z, Ch. III, Theorem 3.9] (Theorem of Lebesgue on Cesàro summability) $f \in C([0,\pi])$ and we may assume that

$f(x) = \sum_{k=0}^{\infty} a_k \cos(kx)$ for $0 \le x \le \pi$. Similarly, if $\sum_{k=1}^{\infty} |b_k| < \infty$, then $g \in C([0,\pi])$ and we may assume that $g(x) = \sum_{k=1}^{\infty} b_k \sin(kx)$ for $0 \le x \le \pi$.

For (1.26) with $n = 0$, we have the following:

Theorem 6 *Let g, (b_k) and ℓ be as in Theorem 5. We write $\bar{g}(x) := xg(1/x)$ for $x \ge 1/\pi$. Then*

$$\sum_{k=m}^{\infty} b_k \sim \frac{\ell(m)}{m} \qquad (m \to \infty) \tag{1.30}$$

if and only if

$$g \in C([0,\pi]) \quad \text{and} \quad \bar{g} \in \Pi_\ell \text{ with } \ell\text{-index } 1. \tag{1.31}$$

See also [I2, Theorem 1.2].

Theorems 4, 5 and 6 treat the boundary cases to the following known results due to Yong [Y]:

Theorem C ([Y]) *Let f, (a_k) and ℓ be as in Theorem 4. Let $n \in \mathbb{N}$ and $2n-1 < \rho < 2n+1$. Then*

$$\sum_{k=m}^{\infty} a_k \sim \frac{\ell(m)}{m^{\rho-1}} \cdot \frac{1}{\rho-1} \qquad (m \to \infty) \tag{1.32}$$

if and only if $f \in C^{2n-2}([0,\pi])$ and

$$f(x) - \sum_{j=0}^{n-1} \frac{f^{(2j)}(0)}{(2j)!} x^{2j} \sim \frac{\pi}{2\Gamma(\rho)\cos(\rho\pi/2)} x^{\rho-1} \ell(1/x) \quad (x \to 0+). \tag{1.33}$$

Theorem D ([Y]) *Let g, (b_k) and ℓ be as in Theorem 5. Let $n \in \mathbb{N}$ and $2n < \rho < 2n+2$. Then*

$$\sum_{k=m}^{\infty} b_k \sim \frac{\ell(m)}{m^{\rho-1}} \cdot \frac{1}{\rho-1} \qquad (m \to \infty) \tag{1.34}$$

if and only if $g \in C^{2n-1}([0,\pi])$ and

$$g(x) - \sum_{j=0}^{n-1} \frac{g^{(2j+1)}(0)}{(2j+1)!} x^{2j+1} \sim \frac{\pi}{2\Gamma(\rho)\sin(\rho\pi/2)} x^{\rho-1} \ell(1/x) \quad (x \to 0+). \tag{1.35}$$

Theorems 4, 5 and 6, together with Theorems C and D, give an answer to [Bo, Question 7.19].

2. Proof of Theorem 1

We note that (1.4) implies

$$\int_0^\infty t^{\nu-\frac{3}{2}+2n}h(t)dt < \infty. \tag{2.1}$$

So, when proving the equivalence of (1.4) and (1.5), we may assume (2.1).

We define $h_0, \ldots, h_{n-1}$ by

$$\begin{aligned} h_0(t) &:= \int_t^\infty h(s)s^{\nu+\frac{1}{2}}ds \qquad (0 \le t < \infty), \\ h_j(t) &:= \int_t^\infty h_{j-1}(s)sds \qquad (0 \le t < \infty,\ j = 1, \ldots, n-1). \end{aligned}$$

Since h is eventually non-negative, h_j are all eventually decreasing. By Fubini's theorem,

$$\begin{aligned} h_j(0) &= \int_0^\infty dt_j t_j \int_{t_j}^\infty dt_{j-1}t_{j-1} \cdots \int_{t_1}^\infty h(t_0)t_0^{\nu+\frac{1}{2}}dt_0 \\ &= \int_0^\infty dt_0 h(t_0)t_0^{\nu+\frac{1}{2}} \int_0^{t_0} dt_1 t_1 \cdots \int_0^{t_{j-1}} t_j dt_j \\ &= \frac{1}{2^j j!}\int_0^\infty h(t)t^{\nu+\frac{1}{2}+2j}dt. \end{aligned}$$

Since

$$\begin{aligned} &x^{-\mu}J_\mu(x) = O(1) \qquad (x \to \infty), \\ &\frac{d}{dx}\{x^{-\mu}J_\mu(x)\} = -x^{-\mu}J_{\mu+1}(x), \\ &x^{-\mu}J_\mu(x) \to c_{\mu,0} \qquad (x \to 0+) \end{aligned}$$

for any $\mu \ge -1/2$ (see Watson [W], pages 199 and 45), we obtain, by integration by parts,

$$\begin{aligned} &H_\nu(x) \\ &= x^{\nu+\frac{1}{2}}\int_0^{\infty-} h(t)t^{\nu+\frac{1}{2}}\{(tx)^{-\nu}J_\nu(tx)\}dt \\ &= x^{\nu+\frac{1}{2}}h_0(0)c_{\nu,0} - x^{\nu+\frac{1}{2}+2}\int_0^{\infty-} h_0(t)t\{(tx)^{-\nu-1}J_{\nu+1}(tx)\}dt = \cdots \end{aligned}$$

$$= \sum_{j=0}^{n-1} (-1)^j x^{\nu+\frac{1}{2}+2j} h_j(0) c_{\nu+j,0}$$
$$+ (-1)^n x^{\nu+\frac{1}{2}+2n} \int_0^{\infty-} h_{n-1}(t) t\{(tx)^{-\nu-n} J_{\nu+n}(tx)\} dt$$
$$= \sum_{j=0}^{n-1} (-1)^j x^{\nu+\frac{1}{2}+2j} h_j(0) c_{\nu+j,0}$$
$$+ (-1)^n x^n \int_0^{\infty-} g(t)(tx)^{1/2} J_{\nu+n}(tx) dt,$$

where

$$g(t) := t^{-\nu+\frac{1}{2}-n} h_{n-1}(t) \qquad (0 < t < \infty).$$

Since

$$(-1)^j h_j(0) c_{\nu+j,0} = c_{\nu,j} \int_0^{\infty} t^{\nu+\frac{1}{2}+2j} h(t) dt,$$

we have

$$\bar{H}_{\nu,n}(x) = (-1)^n x^{(\nu+n)+\frac{1}{2}} \int_0^{\infty-} g(t)(t/x)^{1/2} J_{\nu+n}(t/x) dt. \tag{2.2}$$

Now $t^{(\nu+n)+\frac{1}{2}} g(t) \in L^1_{\text{loc}}[0,\infty)$ and g is eventually decreasing to zero, whence by Theorem A (with ν replaced by $\nu+n$) (1.5) is equivalent to

$$g(t) \sim t^{-\nu-n-\frac{3}{2}} \ell(t) \cdot \frac{(-1)^n c_{\nu,n}}{c_{\nu+n,0}} = t^{-\nu-n-\frac{3}{2}} \ell(t) \cdot \frac{1}{2^n n!} \qquad (t \to \infty)$$

or

$$h_{n-1}(t) \sim t^{-2} \ell(t) \cdot \frac{1}{2^n n!} \qquad (t \to \infty). \tag{2.3}$$

Since h_j is eventually decreasing, $\log\{h_j(t)t\}$ is slowly increasing, whence by the Monotone Density Theorem (see [BGT, §1.7]) (2.3) is equivalent to

$$h_0(t) = \int_t^{\infty} s^{\nu+\frac{1}{2}} h(s) ds \sim t^{-2n} \ell(t) \cdot \frac{1}{2n} \qquad (t \to \infty). \tag{2.4}$$

By assumption, h is eventually decreasing, whence $\log\{h(t)t^{\nu+\frac{1}{2}}\}$ is slowly increasing. Again by the Monotone Density Theorem, (2.4) is equivalent to (1.4). This completes the proof. □

3. Proofs of Theorems 2 and 3

Lemma 3.1 *Let $n \in \mathbb{N}$, and let $f \in L^1_{\text{loc}}[0,\infty)$, f be ultimately decreasing to zero at infinity, with Fourier cosine transform F_c. If $F_c \in C^{2n-2}([0,\infty))$ and*

$$F_c(x) - \sum_{j=0}^{n-1} \frac{F_c^{(2j)}(0)}{(2j)!} x^{2j} = O(x^{2n-2}) \qquad (x \to 0+), \tag{3.1}$$

then $\int_0^\infty t^{2n-2} f(t)dt < \infty$. In particular,

$$F_c^{(2j)}(x) = (-1)^j \int_0^\infty t^{2j} f(t) \cos(xt) dt$$
$$(0 \le x < \infty,\ j = 0, \ldots, n-1).$$

Remark. For the meaning of $F_c \in C^{2n-2}([0,\infty))$, see the remark just after Theorem 3.

Proof. We first show that we lose no generality by supposing that f is finite and decreasing on $[0,\infty)$.

Choose X so large that f is finite and decreasing on $[X,\infty)$. Set

$$\tilde{f}(t) := f(X) \quad (0 \le t < X), \qquad := f(t) \quad (X \le t),$$

and let $\tilde{F}_c$ be its Fourier cosine transform:

$$\tilde{F}_c(x) := \int_0^{\infty-} \tilde{f}(t) \cos(xt) dt \qquad (0 < x < \infty).$$

Set $D(x) := F_c(x) - \tilde{F}_c(x)$. Then

$$D(x) = \int_0^X \{f(t) - f(X)\} \cos(xt) dt \qquad (0 < x < \infty),$$

and so D can be extended to a function in $C^\infty([0,\infty))$. Moreover,

$$\begin{aligned} D(x) &- \sum_{j=0}^{n-1} \frac{D^{(2j)}(0)}{(2j)!} x^{2j} \\ &= \int_0^X \{f(t) - f(X)\} \left\{ \cos(tx) - \sum_{j=0}^{n-1} \frac{(-1)^j}{(2j)!} (xt)^{2j} \right\} dt \\ &= O(x^{2n}) \qquad (x \to 0+). \end{aligned}$$

So for F_c to be in $C^{2n-2}([0,\infty))$ and satisfy (3.1) it is necessary and sufficient that $\tilde{F}_c$ has the same properties. Thus we may replace f by $\tilde{f}$ — that is, we may assume that f is finite and decreasing on $[0,\infty)$.

Since $F_c(x) \to F_c(0)$ as $x \to 0+$, we have $\int_0^\infty f(t)dt < \infty$ by [SS, Theorem 20] (with $k(t) = \cos t$). In particular,

$$F_c(x) = \int_0^\infty f(t)\cos(xt)dt \qquad (0 \le x < \infty). \tag{3.2}$$

If $n \ge 2$, then we proceed to the next step. We follow the idea of the proofs of Chan[C, Theorems 1–10]. By (3.1), $F_c(x) - F_c(0) = O(x^2)$ as $x \to 0+$ or, by (3.2),

$$\int_0^\infty f(t)\{1-\cos(tx)\}\,dt = O(x^2) \qquad (x \to 0+).$$

Since the integrand is non-negative,

$$\int_0^{1/x} f(t)\{1-\cos(xt)\}\,dt = O(x^2) \qquad (x \to 0+).$$

By [C, Lemma 3] (or directly), $1-\cos(tx) \ge (tx)^2/4$ for $0 \le tx \le 1$, whence

$$x^2\int_0^{1/x} t^2 f(t)dt = O(x^2) \qquad (x \to 0+)$$

or

$$\int_0^{1/x} t^2 f(t)dt = O(1) \qquad (x \to 0+).$$

Thus $\int_0^\infty t^2 f(t)dt < \infty$ and so

$$F_c^{(2)}(x) = (-1)\int_0^\infty t^2 f(t)\cos(xt)dt \qquad (0 \le x < \infty). \tag{3.3}$$

If $n \ge 3$, then we proceed to the next step. By (3.1),

$$F_c(x) - \left\{F_c(0) + \frac{F_c^{(2)}(0)}{2!}x^2\right\} = O(x^4) \qquad (x \to 0+)$$

or, by (3.3),

$$\int_0^\infty f(t)\left[\cos(tx) - \left\{1 - \frac{(xt)^2}{2!}\right\}\right]dt = O(x^4) \qquad (x \to 0+).$$

Since

$$\cos u - \left(1 - \frac{u^2}{2!}\right) \geq 0 \quad (0 \leq u < \infty), \qquad \geq \frac{u^4}{2 \cdot 4!} \quad (0 \leq u \leq 1)$$

(see [C, Lemma 3]), we have, as $x \to 0+$,

$$\frac{x^4}{2 \cdot 4!} \int_0^{1/x} t^4 f(t)dt \leq \int_0^{1/x} f(t)\left[\cos(tx) - \left\{1 - \frac{(xt)^2}{2!}\right\}\right]dt$$
$$= O(x^4),$$

whence $\int_0^\infty t^4 f(t)dt < \infty$. Therefore

$$F_{\mathrm{c}}^{(4)}(x) = (-1)^2 \int_0^\infty t^4 f(t)\cos(xt)dt \qquad (0 \leq x < \infty).$$

If $n \geq 4$, then in a similar way we obtain inductively

$$\int_0^\infty t^{2j} f(t)dt < \infty,$$
$$F_{\mathrm{c}}^{(2j)}(x) = (-1)^j \int_0^\infty t^{2j} f(t)\cos(xt)dt \qquad (0 \leq x < \infty)$$

for $j = 3, \ldots, n-1$. This completes the proof. □

Proof of Theorem 2. If (1.16) holds, then $\int_0^\infty t^{2n-2} f(t)dt < \infty$, and so $F_{\mathrm{c}} \in C^{2n-2}([0,\infty))$ and

$$F_{\mathrm{c}}^{(2j)}(0) = (-1)^j \int_0^\infty t^{2j} f(t)dt \qquad (j = 0, \ldots, n-1). \tag{3.4}$$

Therefore, by Theorem 1 with $\nu = -1/2$, (1.17) follows.

Conversely, if F_{c} satisfies (1.17), then by [BGT, Theorem 3.7.4] we obtain (3.1), whence, by Lemma 3.1, $\int_0^\infty t^{2n-2} f(t)dt < \infty$ as well as (3.4). Therefore, by Theorem 1 with $\nu = -1/2$, (1.16) follows. □

Lemma 3.2 *Let $n \in \mathbb{N}$, and let $g(t)t \in L^1_{\mathrm{loc}}[0,\infty)$, g be ultimately decreasing to zero at infinity, with Fourier sine transform G_{s}. If $G_{\mathrm{s}} \in C^{2n-1}([0,\infty))$ and*

$$G_{\mathrm{s}}(x) - \sum_{j=0}^{n-1} \frac{G_{\mathrm{s}}^{(2j+1)}(0)}{(2j+1)!} x^{2j+1} = O(x^{2n-1}) \qquad (x \to 0+), \tag{3.5}$$

then $\int_0^\infty t^{2n-1}g(t)dt < \infty$. *In particular,*

$$G_{\mathrm{s}}^{(2j+1)}(x) = (-1)^j \int_0^\infty t^{2j+1}g(t)\cos(xt)dt \qquad (0 \le x < \infty,\ j = 0, \ldots, n-1).$$

The proof of Lemma 3.2 is quite similar to that of Lemma 3.1; we use [SS, Theorem 20] with $k(t) = \sin t$ as well as [C, Lemma 2] instead of [C, Lemma 3]. We omit the details.

The proof of Theorem 3 is also quite similar to that of Theorem 2; we use Theorem 1 with $\nu = 1/2$ as well as Lemma 3.2 instead of Lemma 3.1. The details are omitted.

4. Proofs of Theorems 4, 5 and 6

Lemma 4.1 *Let* $n \in \mathbb{N}$, *and let* $g \in L^1[0,\pi]$ *with Fourier sine coefficients* (b_k). *We assume* $b_k \ge 0$ *for all* $k \ge 1$. *If* $g \in C^{2n-1}([0,\pi])$ *and*

$$g(x) - \sum_{j=0}^{n-1} \frac{g^{(2j+1)}(0)}{(2j+1)!}x^{2j+1} = O(x^{2n-1}) \qquad (x \to 0+), \tag{4.1}$$

then $\sum_{k=1}^\infty k^{2n-1}b_k < \infty$. *In particular,*

$$g^{(2j+1)}(x) = (-1)^j \sum_{k=1}^\infty k^{2j+1}b_k \cos(kx) \qquad (0 \le x \le \pi,\ j = 0, 1, \ldots, n-1).$$

Proof. Since g' is bounded on $[0,\pi]$, $g \in$ Lip 1 (in the sense of [Bo, pp. 46–47]). By [Bo, Theorem 7.28], we have $\sum_{k=1}^\infty kb_k < \infty$. Therefore,

$$g(x) = \sum_{k=1}^\infty b_k \sin(kx), \quad g'(x) = \sum_{k=1}^\infty kb_k \cos(kx) \quad (0 \le x \le \pi). \tag{4.2}$$

If $n \ge 2$, then we proceed to the next step. As in the proof of Lemma 3.1, we follow the idea of the proofs of [C, Theorems 1–10]. By (4.1), $g(x) - xg^{(1)}(0) = O(x^3)$ as $x \to 0+$, or by (4.2),

$$\sum_{k=1}^\infty b_k \{kx - \sin(kx)\} = O(x^3) \qquad (x \to 0+).$$

Since

$$u - \sin u \geq 0 \quad (0 \leq u < \infty), \qquad u - \sin u \geq \frac{u^3}{2 \cdot 3!} \quad (0 \leq u \leq 1)$$

(see [C, Lemma 2]), we have

$$\frac{m^{-3}}{2 \cdot 3!} \sum_{k=1}^{m} k^3 b_k \leq \sum_{k=1}^{m} b_k \{(k/m) - \sin(k/m)\} = O(m^{-3}) \quad (m \to \infty),$$

whence $\sum_{k=1}^{\infty} k^3 b_k < \infty$ and

$$g^{(3)}(x) = (-1) \sum_{k=1}^{\infty} k^3 b_k \cos(kx) \qquad (0 \leq x \leq \pi).$$

If $n \geq 3$, then in a similar way we can show inductively

$$\sum_{k=1}^{\infty} k^{2j+1} b_k < \infty,$$

$$g^{(2j+1)}(x) = (-1)^j \sum_{k=1}^{\infty} k^{2j+1} b_k \cos(kx) \quad (0 \leq x \leq \pi)$$

for $j = 2, \ldots, n-1$. This completes the proof. □

Proof of Theorem 5. By [BGT, Theorem 3.7.4], (1.27) implies (4.1), whence by Lemma 4.1,

$$\sum_{k=1}^{\infty} k^{2n-1} b_k < \infty. \tag{4.3}$$

On the other hand, by partial summation, (1.26) also implies (4.3). Therefore, when proving the equivalence of (1.26) and (1.27), we may assume (4.3), whence $g \in C^{2n-1}([0, \pi])$, and

$$g^{(2j+1)}(x) = (-1)^j \sum_{k=1}^{\infty} k^{2j+1} b_k \cos(kx)$$
$$(0 \leq x \leq \pi, \ j = 0, \ldots, n-1).$$

Following [SS, pp. 620–621], we define a function h by

$$h(t) := \begin{cases} \sum_{k=1}^{\infty} b_k & (t < 1), \\ \sum_{k=n+1}^{\infty} b_k & (n \le t < n+1,\ n = 1, 2, \ldots). \end{cases} \tag{4.4}$$

Then

$$\begin{aligned} g(x) &= -\int_{[0,\infty)} \sin(xt)dh(t) = x\int_0^{\infty-} h(t)\cos(xt)dt \\ &= \sqrt{\frac{\pi}{2}} x H_{-1/2}(x) \qquad (0 \le x \le \pi) \end{aligned} \tag{4.5}$$

(recall $H_{-1/2}$ from (1.1)). On the other hand, for $j = 0, 1, \ldots, n-1$,

$$\begin{aligned} \int_0^{\infty} t^{2j} h(t)dt &= \frac{1}{2j+1} \sum_{n=0}^{\infty} \left(\sum_{k=n+1}^{\infty} b_k \right) \{(n+1)^{2j+1} - n^{2j+1}\} \\ &= \frac{1}{2j+1} \sum_{k=1}^{\infty} b_k \sum_{n=0}^{k-1} \{(n+1)^{2j+1} - n^{2j+1}\} \\ &= \frac{1}{2j+1} \sum_{k=1}^{\infty} k^{2j+1} b_k = \frac{(-1)^j}{2j+1} g^{(2j+1)}(0). \end{aligned} \tag{4.6}$$

In particular, $\int_0^{\infty} t^{2n-2} h(t)dt < \infty$. Recall $\bar{H}_{-1/2,n}$ from (1.3). By (4.5) and (4.6),

$$\bar{H}_{-1/2,n}(x) = \sqrt{\frac{2}{\pi}} x^{2n+1} \left\{ g(1/x) - \sum_{j=0}^{n-1} \frac{g^{(2j+1)}(0)}{(2j+1)!} x^{-2j-1} \right\}. \tag{4.7}$$

Now (1.26) is equivalent to

$$h(t) \sim \frac{\ell(t)}{t^{2n+1}} \cdot \frac{1}{2n+1} \qquad (t \to \infty),$$

which by Theorem 1 is equivalent to

$$\bar{H}_{-1/2,n} \in \Pi_\ell \text{ with } \ell\text{-index } \frac{c_{-1/2,n}}{2n+1} = \sqrt{\frac{2}{\pi}} \frac{(-1)^n}{(2n+1)!} \tag{4.8}$$

or to (1.27) by (4.7). This completes the proof. □

Proof of Theorem 6. By [BGT, Theorem 3.7.4], (1.31) implies $g(t)/t \in L^1[0,\pi]$. Therefore, since

$$\sum_{k=1}^{n} \sin kx = \frac{\sin\{\frac{1}{2}x(n+1)\}\sin(\frac{1}{2}xn)}{\sin(\frac{1}{2}x)}$$

and $|\sin(\frac{1}{2}x)| \geq x/\pi$ for $(0 \leq x \leq \pi)$, we obtain

$$\begin{aligned}\sum_{k=1}^{n} b_k &= \left|\frac{2}{\pi}\int_0^\pi g(t)\sum_{k=1}^{n}\sin(kt)dt\right| \\ &= \left|\frac{2}{\pi}\int_0^\pi g(t)\frac{\sin\{\frac{1}{2}t(n+1)\}\sin(\frac{1}{2}tn)}{\sin(\frac{1}{2}t)}dt\right| \\ &\leq 2\int_0^\pi \frac{|g(t)|}{t}dt < \infty.\end{aligned}$$

Thus

$$\sum_{k=1}^{\infty} b_k < \infty. \tag{4.9}$$

On the other hand, (1.30) also implies (4.9). So, when proving the equivalence of (1.30) and (1.31), we may assume (4.9), whence $g \in C([0,\pi])$ and

$$g(x) = \sum_{k=1}^{\infty} b_k \sin kx \qquad (0 \leq x \leq \pi).$$

We define h by (4.4). Then (1.30) is equivalent to

$$h(t) \sim \frac{\ell(t)}{t} \quad (t \to \infty). \tag{4.10}$$

On the other hand, by (4.5) and Theorem A with $\nu = -1/2$, (4.10) is equivalent to (1.31). This completes the proof. □

Lemma 4.2 *Let $n \in \mathbb{N}$, and let $f \in L^1[0,\pi]$ with Fourier cosine coefficients (a_k). We assume $a_k \geq 0$ for all $k \geq 0$. If $f \in C^{2n-2}([0,\pi])$ and*

$$f(x) - \sum_{j=0}^{n-1} \frac{f^{(2j)}(0)}{(2j)!}x^{2j} = O(x^{2n-2}) \quad (x \to 0+), \tag{4.11}$$

then $\sum_{k=0}^{\infty} k^{2n-2} a_k < \infty$. *In particular,*

$$f^{(2j)}(x) = (-1)^j \sum_{k=0}^{\infty} k^{2j} a_k \cos(kx) \qquad (0 \le x \le \pi,\ j = 0, 1, \ldots, n-1).$$

Proof. Since $f(x)$ approaches $f(0)$ as $x \to 0+$, we have $\sum_{k=0}^{\infty} a_k < \infty$ by [Bo, Theorem 7.26]. The rest of the proof is similar to that of Lemma 4.1 (see also the proof of Lemma 3.1), whence we omit the details. □

Proof of Theorem 4. By [BGT, Theorem 3.7.4], (1.23) implies (4.11), whence by Lemma 4.2

$$\sum_{k=0}^{\infty} k^{2n-2} a_k < \infty. \tag{4.12}$$

On the other hand, by partial summation, (1.22) also implies (4.12). Therefore, when proving the equivalence of (1.22) and (1.23), we may assume (4.12), whence $f \in C^{2n-2}([0, \pi])$ and

$$f^{(2j)}(x) = (-1)^j \sum_{k=0}^{\infty} k^{2j} a_k \cos(kx) \qquad (0 \le x \le \pi,\ j = 0, 1, \ldots, n-1).$$

Following [SS, p. 623], we define a function h by

$$h(t) := \begin{cases} \sum_{k=0}^{\infty} a_k & (t < 0), \\ \sum_{k=n+1}^{\infty} a_k & (n \le t < n+1,\ n = 0, 1, \ldots). \end{cases} \tag{4.13}$$

Then (1.22) is equivalent to

$$h(t) \sim \frac{\ell(t)}{t^{2n}} \cdot \frac{1}{2n} \qquad (t \to \infty). \tag{4.14}$$

Recall $H_{1/2}$ from (1.1). Since

$$f(x) = -\int_{[0,\infty)} \cos(xt) dh(t) = f(0) - x \int_0^{\infty-} h(t) \sin(xt) dt$$

for $0 \le x \le \pi$, we obtain

$$f(0) - f(x) = \sqrt{\frac{\pi}{2}} x H_{1/2}(x) \qquad (0 \le x \le \pi). \tag{4.15}$$

First we assume $n = 1$. Then, by Theorem A with $\nu = 1/2$, (4.14) holds if and only if

$$\bar{H}_{1/2}(x) \in \Pi_\ell \text{ with } \ell\text{-index } \frac{c_{1/2,0}}{2} = \frac{1}{2}\sqrt{\frac{2}{\pi}}$$

(recall $\bar{H}_{1/2}$ from (1.6)), which by (4.15) is equivalent to (1.23) with $n = 1$.

Next we assume $n \ge 2$. For $j = 0, 1, \ldots, n-2$,

$$\int_0^\infty t^{2j+1} h(t)dt = \frac{1}{2j+2} \sum_{k=0}^\infty k^{2j+2} a_k = \frac{(-1)^{j+1}}{2j+2} f^{(2j+2)}(0). \tag{4.16}$$

In particular, $\int_0^\infty t^{2n-3} h(t)dt < \infty$. Recall $\bar{H}_{1/2,n-1}$ from (1.3). By (4.15) and (4.16),

$$\bar{H}_{1/2,n-1}(x) = -\sqrt{\frac{2}{\pi}} x^{2n} \left\{ f(1/x) - \sum_{j=0}^{n-1} \frac{f^{(2j)}(0)}{(2j)!} x^{-2j} \right\}. \tag{4.17}$$

By Theorem 1 with $\nu = 1/2$, (4.14) is equivalent to

$$\bar{H}_{1/2,n-1} \in \Pi_\ell \text{ with } \ell\text{-index } \frac{c_{1/2,n-1}}{2n} = \sqrt{\frac{2}{\pi}} \frac{(-1)^{n-1}}{(2n)!},$$

which by (4.17) is equivalent to (1.23). This completes the proof. □

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