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A class of univalent functions II

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Abstract. In this paper we consider certain properties of the class of functions $f(z) = z + a_2z^2 + \cdots$ which are analytic in the unit disc and satisfy the condition

$$\left|f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu} - 1\right| < \lambda, \quad 0 < \mu < 1, \quad 0 < \lambda \leq 1 \quad [3].$$

Key words: univalent, starlike.

1. Introduction and preliminaries

Let H denote the class of functions analytic in the unit disc $U = \{z : |z| < 1\}$ and let $A \subset H$ be the class of normalized analytic functions f in U such that $f(0) = f'(0) - 1 = 0$. Let

$$S^*(\beta) = \left\{f \in A : \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \beta,\ 0 \leq \beta < 1,\ z \in U\right\}$$

denote the class of *starlike functions of order* β. We put $S^* \equiv S^*(0)$ (the class of *starlike functions*). It is well-known that these classes belong to the class of univalent functions in U (see, for example [2]). Also, it is known that the class

$$B_1(\mu) = \left\{f \in A : \mathrm{Re}\left\{f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right\} > 0,\ \mu > 0,\ z \in U\right\} \quad (1)$$

is the class of univalent functions in U ([1]).

In the paper [3] the author considered the class of functions $f \in A$ defined by the condition

$$\left|f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu} - 1\right| < \lambda, \quad (2)$$

where $0 < \mu < 1$, $0 < \lambda \leq 1$, $z \in U$, i.e. for $-1 < \mu < 0$ in (1). In the same

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paper it is proved that for

$$0 < \lambda \le \frac{1-\mu}{\sqrt{(1-\mu)^2+\mu^2}}, \quad 0 < \mu < 1, \tag{3}$$

in (2) we have that $f \in S^*$. The problems of starlikeness of order β and convexity was considered in [3] and [5].

We note that for the limit cases $\mu = 0$, $\lambda = 1$ and $\mu = 1$, $\lambda = 1$, we obtain the classes defined by the conditions

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 \quad \text{and} \quad \left|f'(z)\left(\frac{z}{f(z)}\right)^2 - 1\right| < 1,$$

respectively. The first class is the subclass of S^*, the second one is the subclass of univalent functions in U ([6], [4]).

In this paper by using another approach we will give some results concerning to the class of functions defined by the condition (2).

2. Results and consequences

We start with the result which is similar to the appropriate result in [4].

Theorem 1 *Let $f \in A$ satisfy the condition (2) with $0 < \mu < 1$. Then we have the representation*

$$\left(\frac{z}{f(z)}\right)^{\mu} = 1 - \mu\lambda z^{\mu}\int_0^z \frac{\omega(t)}{t^{\mu+1}}dt, \tag{4}$$

or, equivalently,

$$\left(\frac{z}{f(z)}\right)^{\mu} = 1 - \mu\lambda\int_0^1 \frac{\omega(tz)}{t^{\mu+1}}dt, \tag{4'}$$

where

$$\omega \in H, \quad \omega(0) = 0, \quad |\omega(z)| < 1, \quad z \in U. \tag{5}$$

Proof. From (2) we have $\left(\frac{z}{f(z)}\right)^{\mu+1} f'(z) = 1 + \lambda\omega(z)$, where ω satisfies the condition (5). We can write the last relation in the form $\left(\frac{1}{f^{\mu}(z)} - \frac{1}{z^{\mu}}\right)' = -\mu\lambda\frac{\omega(z)}{z^{\mu+1}}$. Since $\left(\frac{1}{f^{\mu}(z)} - \frac{1}{z^{\mu}}\right)\Big|_{z=0} = 0$, then by integration from the previous relation we get

$$\frac{1}{f^{\mu}(z)} - \frac{1}{z^{\mu}} = -\mu\lambda \int_0^z \frac{\omega(t)}{t^{\mu+1}} dt,$$

and from here the form (4) and (4′). □

Corollary 1 *If $f \in A$ satisfies the condition* (2) *with*

$$0 < \lambda \le \min\left\{1, \frac{1-\mu}{\mu}\right\} = \begin{cases} 1, & 0 < \mu \le \frac{1}{2} \\ \frac{1-\mu}{\mu}, & \frac{1}{2} \le \mu < 1 \end{cases}, \tag{6}$$

then

$$\operatorname{Re}\left\{\left(\frac{z}{f(z)}\right)^{\mu}\right\} > 0, \quad z \in U.$$

Proof. Since, by Schwartz's lemma $|\omega(tz)| \le t|z|$, $z \in U$, then by (4′) we have

$$\begin{aligned}\operatorname{Re}\left\{\left(\frac{z}{f(z)}\right)^{\mu}\right\} &\ge 1 - \mu\lambda \int_0^1 \frac{|\omega(tz)|}{t^{\mu+1}} dt \\ &\ge 1 - \frac{\mu\lambda}{1-\mu}|z| > 1 - \frac{\mu\lambda}{1-\mu} \ge 0,\end{aligned}$$

for λ satisfies (6).

We note that if $f \in A$ satisfies (2) with condition (6), then we have the representation

$$f(z) = \frac{z}{\left(1 - \mu\lambda \int_0^1 \frac{\omega(tz)}{t^{\mu+1}} dt\right)^{\frac{1}{\mu}}} \tag{7}$$

(where we take the principal value), and so

$$f(z) = \frac{z}{(1 + b_1 z + b_2 z^2 + \cdots)^{\frac{1}{\mu}}}.$$

From (4′) or from (7) we easily derive the following □

Corollary 2 *If $f \in A$ satisfies the condition* (2) *with condition* (6), *then*

$$\frac{|z|}{\left(1+\frac{\mu\lambda}{1-\mu}|z|\right)^{\frac{1}{\mu}}} \leq |f(z)| \leq \frac{|z|}{\left(1-\frac{\mu\lambda}{1-\mu}|z|\right)^{\frac{1}{\mu}}}, \quad z \in U. \tag{8}$$

These results are sharp as the function $f(z) = \frac{z}{\left(1-\frac{\mu\lambda}{1-\mu}z\right)^{\frac{1}{\mu}}}$ *shows.*

Remark 1. If in (8) we put that $\mu \to 0$ then we have

$$|z|e^{-\lambda|z|} \leq |f(z)| \leq |z|e^{\lambda|z|}, \quad z \in U,$$

for the functions $f \in A$ with $\left|\frac{zf'(z)}{f(z)} - 1\right| < \lambda$, $0 < \lambda \leq 1$, which is true.

Theorem 2 *If $f \in A$ satisfies the condition* (2) *with condition* (6), *then*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\lambda|z|}{1-\mu-\mu\lambda|z|}, \quad z \in U.$$

Proof. From (4′) by using logarithmic differentiation we obtain

$$\frac{zf'(z)}{f(z)} = \frac{1+\lambda\omega(z)}{1-\mu\lambda\int_0^1 \frac{\omega(tz)}{t^{\mu+1}}dt},$$

and from here

$$\left|\frac{zf'(z)}{f(z)} - 1\right| = \left|\frac{\lambda\omega(z)+\mu\lambda\int_0^1 \frac{\omega(tz)}{t^{\mu+1}}dt}{1-\mu\lambda\int_0^1 \frac{\omega(tz)}{t^{\mu+1}}dt}\right| \leq \frac{\lambda|\omega(z)|+\mu\lambda\int_0^1 \frac{|\omega(tz)|}{t^{\mu+1}}dt}{1-\mu\lambda\int_0^1 \frac{|\omega(tz)|}{t^{\mu+1}}dt}$$

$$\leq \frac{\lambda|z|+\frac{\mu\lambda}{1-\mu}|z|}{1-\frac{\mu\lambda}{1-\mu}|z|} = \frac{\lambda|z|}{1-\mu-\mu\lambda|z|}.$$

□

Corollary 3 *If $f \in A$ satisfies the condition* (2) *with* $0 < \lambda \leq \frac{1-\mu}{1+\mu}$, $0 < \mu < 1$, *then f is starlike function and*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1, \quad z \in U.$$

Proof. For given λ, from the previous theorem, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le \frac{\lambda|z|}{1-\mu-\mu\lambda|z|} < \frac{\lambda}{1-\mu-\mu\lambda} \le 1, \quad z \in U.$$

□

Theorem 3 *Let $f \in A$ satisfy the condition* (2) *with $\frac{1}{2} \le \mu < 1$. Then* $\mathrm{Re}\{f'(z)\} > 0$, $z \in U$, *for $0 < \lambda \le \lambda_0$, where λ_0 is the smallest positive root of the equation*

$$a^2\lambda^2(3 - 4a^2\lambda^2)^2 + \lambda^2 - 1 = 0, \quad a = \frac{\mu}{1-\mu}. \tag{9}$$

Proof. From (2) we have $\left(\frac{z}{f(z)}\right)^{\mu} \prec 1 + \lambda_1 z$, $\lambda_1 = \frac{\lambda\mu}{1-\mu} = \lambda a$ (see [3]), and from here $\left|\arg\left(\frac{z}{f(z)}\right)^{\mu}\right| < \arctan\frac{\lambda_1}{\sqrt{1-\lambda_1^2}}$. Also, from (2) we obtain $\left(\frac{z}{f(z)}\right)^{\mu+1} f'(z) = 1 + \lambda\omega(z)$, where ω satisfies the condition (5). From there we can express $f'(z) = \left(\frac{f(z)}{z}\right)^{\mu+1}(1 + \lambda\omega(z))$ and

$$\begin{aligned}
|\arg f'(z)| &\le \frac{\mu+1}{\mu}\left|\arg\left(\frac{f(z)}{z}\right)^{\mu}\right| + |\arg(1+\lambda\omega(z))| \\
&< 3\arctan\frac{\lambda_1}{\sqrt{1-\lambda_1^2}} + \arctan\frac{\lambda}{\sqrt{1-\lambda^2}} \\
&= \arctan\frac{\lambda_1(3-4\lambda_1^2)}{(1-4\lambda_1^2)\sqrt{1-\lambda_1^2}} + \arctan\frac{\lambda}{\sqrt{1-\lambda^2}} \\
&= \arctan\frac{\dfrac{\lambda_1(3-4\lambda_1^2)}{(1-4\lambda_1^2)\sqrt{1-\lambda_1^2}} + \dfrac{\lambda}{\sqrt{1-\lambda^2}}}{1 - \dfrac{\lambda_1(3-4\lambda_1^2)}{(1-4\lambda_1^2)\sqrt{1-\lambda_1^2}}\dfrac{\lambda}{\sqrt{1-\lambda^2}}} \le \frac{\pi}{2}, \quad 0 < \lambda_1 < \frac{1}{2},
\end{aligned}$$

If $1 - \frac{\lambda_1(3-4\lambda_1^2)}{(1-4\lambda_1^2)\sqrt{1-\lambda_1^2}}\frac{\lambda}{\sqrt{1-\lambda^2}} \ge 0$, which is equivalent to (9).

From (9) we have that the condition $0 < \lambda_1 < \frac{1}{2}$ is really satisfied.

□

For $\mu = \frac{1}{2}$ in the previous theorem we obtain

Corollary 4 *Let $f \in A$ satisfy the condition*

$$\left| f'(z) \left(\frac{z}{f(z)} \right)^{\frac{3}{2}} - 1 \right| < \lambda, \quad z \in U,$$

where $0 < \lambda \leq \lambda_0$ and $\lambda_0 = 0.3827\ldots$ is the smallest positive root of the equation $\lambda^2(3-4\lambda^2)^2 + \lambda^2 - 1 = 0$. Then $\mathrm{Re}\{f'(z)\} > 0$, $z \in U$.

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