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Complete affine flows with nilpotent holonomy group

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Abstract. S. Matsumoto proved that every complete affine flow on a 3-dimensional closed manifold is virtually algebraic, that is, a lift of the flow by a finite covering map is isomorphic to a flow which is constructed by an algebraic way. In this paper, we shall show that, in the general dimension, a complete affine flow on a closed manifold with additional conditions is virtually algebraic.

Key words: foliation, transverse structure, homogeneous flow.

1. Introduction and Preliminaries

When a flow is given a transverse geometric structure as a one-dimensional foliation, we can understand more detailed behavior of the flow. For instance, flows with transverse hyperbolic structures are classified by Epstein [4]. The flows with a transverse similarity structure are also classified in dimension 3 and classified in the general dimension with additional conditions (Ghys [6], Nishimori [12], Asuke [1]).

Turning to flows with a general transverse affine structure, we have a classification of the complete affine flows in dimension 3 given by Matsumoto [9]. However, in the general dimension, almost nothing is known so far. The purpose of this paper is a detailed study of flows with transverse affine structure.

Let G be a Lie group acting on a q-dimensional manifold X by left. A (G,X)*-foliation* $(M,\mathcal{F})$ of codimension q is a foliation which has a foliation chart $\{(U_\alpha, f_\alpha), g_{\alpha\beta}\}$, where

(1) $f_\alpha : U_\alpha \to X$ is a submersion which defines $\mathcal{F}|_{U_\alpha}$,

(2) $g_{\alpha\beta} \in G$ and $f_\alpha(x) = g_{\alpha\beta} f_\beta(x)$ for $x \in U_\alpha \cap U_\beta$.

The (G,X)-foliation is developable. It means that there exists a submersion $\mathcal{D} : \tilde{M} \to X$ and a homomorphism $h : \pi_1(M) \to G$ such that

(a) $\mathcal{D}(\gamma(x)) = h(\gamma)\mathcal{D}(x)$ for any $x \in \tilde{M}$ and $\gamma \in \pi_1(M)$,

(b) each leaf of $\pi^*\mathcal{F}$ is a connected component of the inverse image of a point by the submersion $\mathcal{D}$,

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where $\pi : \tilde{M} \to M$ is the universal covering map. We call $\mathcal{D}$ a *developing map* and h a *holonomy homomorphism* of $\mathcal{F}$. The image Γ of h is called *holonomy group* of $\mathcal{F}$. A (G, X)-foliation is said to be *complete* if its developing map is a fiber bundle projection.

In the case X is also G, a (G, G)-foliation is called *Lie G-foliation.* It is known that a Lie G-foliation is always complete. In the case X is a homogeneous space of G, a (G, X)-foliation is called *transversely homogeneous foliation.* The following are typical examples of (G, X)-flows or one-dimensional (G, X)-foliations.

Example 1.1 Let H be a simply connected Lie group and H_0 a closed one-dimensional subgroup of H. Suppose H has a uniform lattice Δ. Let $\theta : H \to H/H_0$ be the natural projection and $\iota : \Delta \to H$ the inclusion. Since θ is a fiber bundle projection to H/H_0, the pair of maps $(\iota, \theta) : (\Delta, H) \to (H, H/H_0)$ defines a complete transversely homogeneous flow on $\Delta\backslash H$.

Example 1.2 In the example above, if H_0 is also normal, then H/H_0 is a Lie group. Hence the pair of maps $(\theta \circ \iota, \theta) : (\Delta, H) \to (H/H_0, H/H_0)$ defines a Lie H/H_0-flow on $\Delta\backslash H$.

One way to classify the (G, X)-flows is to find such flows as in the above examples which are conjugate to the given (G, X)-flow.

We denote $\mathrm{Aff}(n)$ the group of affine transformations of $\boldsymbol{R}^n$. We say an $(\mathrm{Aff}(n), \boldsymbol{R}^n)$-flow simply an *affine flow*. An affine flow is *algebraic* if it is conjugate to one of the flow in Example 1.1 and 1.2. An affine flow is *virtually algebraic* if it is finitely covered by an algebraic affine flow.

Theorem 1.3 (S. Matsumoto [9]) *Every complete affine flow on a closed 3-manifold is virtually algebraic.*

We are interested in a natural question whether Theorem 1.3 holds in the general dimension or not. To investigate the complete affine flows in the general dimension, we can not apply the proof of Theorem 1.3. Indeed Matsumoto's proof depends on the 3-manifold theories which are well developed. For example, he uses the classification of the 3-dimensional simply connected solvable Lie groups.

On the other hand, we have to find a way to construct a suitable simply connected Lie group N and a homomorphism $\theta : N \to \mathrm{Aff}(n)$ which define a given complete affine flow. We shall use "syndetic hulls" for a linear group.

Our main result is the following:

Theorem 1.4 *A complete affine flow is virtually algebraic if its holonomy homomorphism is injective and its holonomy group is nilpotent.*

Throughout this paper, the underlying manifold is assumed to have a solvable fundamental group.

This paper is organized as follows. In §2, we shall recall manifolds which have a solvable fundamental group. In §3, we shall introduce the notion of syndetic hulls for a linear group. The contents of this section are mostly algebraic. In §4, we shall argue about the syndetic hulls for the holonomy group of a complete affine flow. In §5, we shall forget the transverse geometric structure of flows for a moment and turn our attention to flows which are tangent to the fibers of a fiber bundle structure of the manifold. In §6, we shall take up a classification theorem of nilpotent Lie G-flow given by E. Ghys. In §7, we shall state when a transversely homogeneous flow has a cross section in terms of the holonomy group and syndetic hulls. In §8, we shall prove our main theorem.

2. Polycyclic groups

In this section, we shall recall manifolds with solvable fundamental group.

Definition 2.1 A group G is called *polycyclic* if G admit a sequence $G = G_0 \supset G_1 \supset \cdots \supset G_k = \{e\}$ of subgroups such that each G_i is normal in G_{i-1} and G_{i-1}/G_i is cyclic. A group G is called *strongly polycyclic* if each G_{i-1}/G_i is infinite cyclic.

By the definition, a polycyclic group is solvable and every subgroup of polycyclic group is again polycyclic and finitely generated. It is known that any polycyclic group has a finite index strongly polycyclic subgroup.

Theorem 2.2 (R. Bieri [2]) *Let Π be a solvable group. If there exists a closed manifold M which is a $K(\Pi, 1)$-space, then Π is polycyclic.*

Let Π and M be as in Theorem 2.2. It is known that every strongly polycyclic group has a finite index normal subgroup which is isomorphic to a uniform lattice of some simply connected solvable Lie group. Let Π' be a finite index strongly polycyclic subgroup of Π isomorphic to a uniform lattice of a simply connected solvable Lie group G. Then, there exists a $K(\Pi',1)$-space N which covers M by the finitely many sheets and is homotopy equivalent to G/Π'. This, together with the fact that $\mathrm{rank}(\Pi')$ is equal to $\dim G$, implies the following.

Corollary 2.3 *Let Π be a solvable group. If M is a closed manifold which is a $K(\Pi,1)$-space, then* $\mathrm{rank}\,\Pi = \dim M$.

3. Syndetic hulls of a linear group

In this section, we are going to recall syndetic hulls of a linear group. The contents of this section are mostly algebraic.

An algebraic hull of a subgroup G of $GL(m;\boldsymbol{R})$ is a Lie group containing G. A syndetic hull is also a Lie group containing G. However, it does not always exist and is not always unique even if it exists. But it could be smaller than the algebraic hull.

Before defining a syndetic hull, we shall recall some notions of the algebraic groups.

Definition 3.1 We shall say that G is *$\boldsymbol{R}$-algebraic* if there exists an algebraic group $\boldsymbol{G}$ of $GL(m;\boldsymbol{C})$ such that $G = \boldsymbol{G} \cap GL(m;\boldsymbol{R})$.

We shall say $\boldsymbol{R}$-algebraic simply algebraic in the following. In general, an algebraic group $G \subset GL(m;\boldsymbol{R})$ is closed in Lie group topology of $GL(m;\boldsymbol{R})$. So, G is a Lie subgroup of $GL(m;\boldsymbol{R})$.

Definition 3.2 The *Algebraic hull* $A(G)$ of G is the closure of G in Zariski topology of $GL(m;\boldsymbol{R})$.

The algebraic hull is the smallest algebraic group containing G.

Definition 3.3 An element $g \in GL(m;\boldsymbol{R})$ is called *unipotent* if $(g-I)^m = 0$, where I denotes the unit matrix. A subgroup G of $GL(m;\boldsymbol{R})$ is called unipotent if it consists only of unipotent elements.

Note that a unipotent subgroup is nilpotent. A unipotent algebraic group is connected. Conversely, a connected unipotent group is algebraic. We refer the reader to the books [3] and [13].

Definition 3.4 Let G be a subgroup of a solvable Lie group H. We denote $\bar{G}$ the closure of G and $\bar{G}^0$ the identity component of $\bar{G}$. The *rank* in H of

G is defined by the equation

$$\mathrm{rank}_H(G) = \mathrm{rank}(\bar{G}/\bar{G}^0) + \dim(\bar{G}^0),$$

where $\mathrm{rank}(\bar{G}/\bar{G}^0)$ is the rank of the solvable group $\bar{G}/\bar{G}^0$ as an abstract group.

Now we can state a theorem, according to which a syndetic hull is defined.

Theorem 3.5 (Fried-Goldman [5]) *Let $G \subset GL(m;\boldsymbol{R})$ be a solvable group. There exists a closed solvable subgroup $H \subset GL(m;\boldsymbol{R})$ satisfying the following conditions*:
(1) *H has finitely many components and each component intersects G,*
(2) *G and H has the same algebraic hull,*
(3) *(Syndetic condition) there exists a compact set K of H such that $H = K \cdot G$,*
(4) $\dim H \leq \mathrm{rank}_{GL} G$.
Moreover, if all eigenvalues of the elements of G are real, then H is uniquely determined.

Such a group H is called a *syndetic hull* of G. Roughly speaking, a syndetic hull for G is a Lie group spanned by G, as a vector space is spanned by its bases. To see this, we shall sketch the proof of the theorem.

Sketch of the proof of Theorem 3.5. Since $A(G)$ is algebraic, it has finitely many connected components. Thus, it is enough to prove the case where $A(G)$ is connected.

Step 1. Suppose G is abelian. Then $A(G)$ is a connected abelian Lie group. If $A(G)$ is a vector group, then the $\boldsymbol{R}$-span H of G is a unique syndetic hull. Suppose $A(G)$ is isomorphic to $\boldsymbol{R}^q/\Lambda$, where Λ is a discrete subgroup of $\boldsymbol{R}^q$. Let $\pi : \boldsymbol{R}^q \to \boldsymbol{R}^q/\Lambda$ be the natural projection. Since $\bar{G}/\bar{G}^0$ is finitely generated abelian group, we can see that there exists a subgroup G' such that $\pi(G') = G$ and $\mathrm{rank}_{\boldsymbol{R}^q} G' = \mathrm{rank}_{GL} G$. Set H' be the $\boldsymbol{R}$-span of G'. Then $H = \pi(H')$ is a syndetic hull for G.

Step 2. If G is unipotent, then $A(G)$ is the unique syndetic hull. Since G is solvable, $[G, G]$ is a unipotent subgroup. Let $p : A(G) \to A(G)/[A(G), A(G)]$ be the natural projection. Since $A(G)/[A(G), A(G)]$ is abelian, there exists a syndetic hull Q of $p(G)$ by Step 1. This, together with the fact that $A([G,G]) = [A(G), A(G)]$, implies that $H = p^{-1}(Q)$ is a syndetic hull for G. □

Note that if G is unipotent then its syndetic hull is unique and equal to the algebraic hull of G.

Definition 3.6 (Jordan decomposition) An element $g \in G \subset GL(m; \boldsymbol{R})$ has a unique decomposition $g = g_u g_s = g_s g_u$, where $g_u, g_s \in A(G)$, g_u is unipotent and g_s is semisimple. We call g_u the *unipotent* part of g.

Now we shall again recall some notions of the algebraic group theory. Let $G \subset GL(m; \boldsymbol{R})$ be an algebraic group. Then G has a maximum connected normal solvable subgroup N. We call N the *radical* of G. Let U be the set of the unipotent elements of N. Then U is an algebraic group and G decomposes as a semi-direct product $U \cdot R$. We call U the *unipotent radical* and R the *maximal reductive group* of G. Note that R is abelian and isomorphic to a product of copies of $\boldsymbol{R}^{\times}$ and $SO(2)$ if G is connected solvable.

Proposition 3.7 (Fried-Goldman [5]) *Let G be a connected solvable linear group. Then the set of the unipotent parts of the elements of G coincides with the unipotent radical of $A(G)$.*

4. Action of holonomy group

In this section, we are going to show that a complete affine flow with solvable fundamental group is actually a Lie G-flow or transversely homogeneous flow with respect to a Lie group smaller than the affine transformation group.

Let (M^{n+1}, φ) be a complete affine flow. The completeness gives some restriction of the action of the holonomy group as follows.

Lemma 4.1 *There does not exist a subspace of $\boldsymbol{R}^n$ homeomorphic to $\boldsymbol{R}^m$ for $m < n$ invariant under the action of the holonomy group.*

Proof. Suppose there exists a subspace L of $\boldsymbol{R}^n$ homeomorphic to $\boldsymbol{R}^m$ and $\Gamma(L) = L$, where $m < n$. Then $\mathcal{D}^{-1}(L)$ is invariant under the action of $\pi_1(M)$. Since $\mathcal{D} : \tilde{M} \to \boldsymbol{R}^n$ is a trivial $\boldsymbol{R}$-bundle, $\mathcal{D}^{-1}(L)$ is isomorphic to $\boldsymbol{R}^{m+1}$. Hence $N = \mathcal{D}^{-1}(L)/\pi_1(M)$ is a $K(\pi_1(M), 1)$-space. Since M is a $K(\pi_1(M), 1)$-space, M and N are homotopy equivalent. Thus, we have

$$H_{n+1}(N; \boldsymbol{Z}) = H_{n+1}(M; \boldsymbol{Z}) \cong \boldsymbol{Z}.$$

However $H_{n+1}(N; \boldsymbol{Z})$ is trivial since $\dim N = m+1 < n+1$. This contradicts the fact that M and N are homotopy equivalent. □

Applying Fried-Goldman's theorem to the holonomy group Γ of a complete affine flow (M^{n+1}, φ), we have a syndetic hull H of Γ contained in $\mathrm{Aff}(n)$ since $\mathrm{Aff}(n)$ is an algebraic group.

Lemma 4.2 *Every syndetic hull of the holonomy group of a complete affine flow (M^{n+1}, φ) acts on $\boldsymbol{R}^n$ transitively.*

Proof. Let H be a syndetic hull of the holonomy group Γ and $A(H) = U \cdot R$ the algebraic hull of H, where U is the unipotent radical and R is a maximal reductive subgroup of $A(H)$. We may assume H is connected.

Since R is reductive, there exists a fixed point $x_0 \in \boldsymbol{R}^n$. We may assume x_0 is the origin of $\boldsymbol{R}^n$. Then the isotropy group of U at x_0 is $U \cap GL(n; \boldsymbol{R})$, where $GL(n; \boldsymbol{R})$ is considered as a subgroup of $GL(n+1; \boldsymbol{R})$ in usual way. Hence the isotropy group of U at x_0 is unipotent algebraic and thus connected. Since U is diffeomorphic to some Euclidean space, the orbit $U(x_0)$ of x_0 is diffeomorphic to $\boldsymbol{R}^k$. Since U is a normal subgroup of $A(H)$, the orbit $U(x_0)$ is invariant under the action of $A(H)$. Hence $U(x_0)$ is invariant under the action of Γ. Then Lemma 4.1 implies that $U(x_0) = \boldsymbol{R}^n$.

By Proposition 3.7, for any $u \in U$ there exists $h \in H$ and $r \in R$ such that $h = ur$. Hence we have $H(x_0) = U(x_0)$. Thus H acts on $\boldsymbol{R}^n$ transitively. □

This implies that (M, φ) is actually a Lie G-flow or a transversely homogeneous flow with respect to a Lie group smaller than $\mathrm{Aff}(n)$.

Corollary 4.3 *If H is simply connected, then* $\dim H$ *is equal to either* $\dim M$ *or* $\dim M - 1$.

Proof. Let $p : H \to [H, H]\backslash H$ be the natural projection. Since H is simply connected solvable and $[H, H]$ is unipotent, $[H, H]\backslash H$ is a simply connected abelian group. Hence $[H, H]\backslash H$ is isomorphic to $\boldsymbol{R}^q$ for some q.

By the condition (3) of Theorem 3.5, Γ is a cocompact subgroup of H. Thus $p(\Gamma)$ is a cocompact subgroup of $[H, H]\backslash H$. Since $[H, H]\backslash H$ is isomorphic to $\boldsymbol{R}^q$, we have $\operatorname{rank} p(\Gamma) \geq q$. Since $[H, H]$ is connected unipotent, $\operatorname{rank} \Gamma \cap [H, H] \geq \dim[H, H]$. Thus we have

$$\dim H = \dim[H, H] + \dim([H, H]\backslash H)$$

$$\begin{aligned} &\leq \operatorname{rank}\Gamma \cap [H,H] + \operatorname{rank} p(\Gamma) \\ &= \operatorname{rank}\Gamma = \dim M. \end{aligned}$$

On the other hand, since H acts on $\boldsymbol{R}^n$ transitively, $\dim H \geq n$. Thus $\dim H$ is equal to either $\dim M$ or $\dim M - 1$. □

5. Flows on fiber bundles

In this section, we shall consider flows which are tangent to the fibers of a fiber bundle. The aim of this section is to show a condition which implies that these two flows are diffeomorphic.

In preceding sections, we considered flows with a transverse geometric structure. However, in this section, we shall forget the transverse geometric structures and turn our attention to their topological situation.

Theorem 5.1 *Let φ and ψ be nonsingular flows on closed manifolds M and N which fibers over closed manifolds. Suppose the dimension of M and N are the same and the dimension of the fibers are the same and ≥ 2. Assume that φ (resp. ψ) is tangent to the fibers of M (resp. N) and each fiber of M (resp. N) is a minimal set of φ (resp. ψ). If there exists a smooth map $f : M \to N$ such that*
(1) *f is transverse to ψ and $f^*\psi = \varphi$ as foliations,*
(2) *$f(x) \neq f(y)$ if $y \notin O_\varphi(x)$*
Then, there exists a diffeomorphism $f_0 : M \to N$ sending φ to ψ.

Let $\varphi : M \times \boldsymbol{R} \to M$ be a non-singular flow on a manifold M. We denote $\varphi((x,t))$ by $\varphi_t(x)$. An orbit $\varphi(\{x\} \times \boldsymbol{R})$ through a point $x \in M$ is denoted by $\mathcal{O}_\varphi(x)$. A part of the orbit $\varphi(\{x\} \times J)$ is denoted by $\mathcal{O}_\varphi(x; J)$, where J is a subset of $\boldsymbol{R}$.

Definition 5.2 A subset W of M is said to be *invariant* under φ if $\varphi(W \times \boldsymbol{R}) \subset W$. A closed invariant subset W is said to be *minimal* if there does not exist a proper closed invariant subset of W. A flow φ is said to be *minimal* if M is the minimal set.

Note that if W is a minimal set then each orbit of φ through a point of W is dense in W.

Let (N, ψ) be a non-singular flow on a manifold and $f : M \to N$ a smooth map.

Definition 5.3 The map f is said to be transverse to ψ if $T_{f(x)}N = f_*(T_xM) \oplus T_{f(x)}X_\psi$ for each point x of M, where T_yX_ψ is the tangent space to the orbit of ψ through y. In this case, f induces a one-dimensional foliation $f^*\psi$ on M.

Let (M,φ), (N,ψ) and $f : M \to N$ as in Theorem 5.1. Note that, since the dimension of the fibers are ≥ 2, the flows φ and ψ have no closed orbits. The conditions of f in the theorem and the compactness of M and N implies that f is surjective and induces a diffeomorphism among the base spaces.

We define a function $u : M \times \boldsymbol{R} \to \boldsymbol{R}$ by $f(\varphi(x;t)) = \psi(f(x); u(x,t))$. The map f not being a diffeomorphism means that u is not a monotone function if we fix a point of M. So, our task is to exchange u for a function $\nu : M \to \boldsymbol{R}$ which is monotone on each orbit of φ. To do this, we use an averaging technique. In short, for any $x \in M$, the point $f(\varphi(x;t))$ moves as t increase toward the positive (negative) direction on $\mathcal{O}_\psi(f(x))$ in the average.

The function u is differentiable and satisfies

$$u(x, t+s) = u(\varphi(x;s), t) + u(x,s). \tag{5.1}$$

This implies the following.

Lemma 5.4 *The differential of the function $u : M \times \boldsymbol{R} \to \boldsymbol{R}$ is bounded.*

Proof. By the equation (5.1), we have $\frac{\partial}{\partial t}u(x,t) = \frac{\partial}{\partial t}u(\varphi(x;t),0)$. Thus the compactness of M implies that there exists a constant C such that

$$\left|\frac{\partial}{\partial t}u(x,t)\right| < C \quad \text{for all} \quad x \in M \quad \text{and} \quad t \in \boldsymbol{R}. \tag{5.2}$$

□

We denote the bundle projection $\xi : M \to Q$ and F the fiber of the bundle. Let $\{U_1, \dots, U_k\}$ be an open covering of Q such that the bundle $\xi : M \to Q$ is trivial on each U_j and $\{V_1, \dots, V_k\}$ a refinement of $\{U_1, \dots, U_k\}$ such that $\operatorname{Cl} V_j \subset U_j$. We identify $M|_{U_i}$ by $U_i \times F$. We may assume that f sends open transverse sections $U_i \times B_i$ of φ diffeomorphically to open transverse sections $f(U_i \times B_i)$ of ψ. In this setting, we have the following.

Lemma 5.5 *There exists a constant $T > 0$ such that $\mathcal{O}_\varphi(x; (0,T))$ intersects $\{\xi(x)\} \times B_i$ if $x \in \operatorname{Cl} V_i$.*

Proof. Define a function $r_i : U_i \times F \to R$ for $i = 1, \dots, k$ by

$$r_i(x) = \inf\{t > 0 \mid \varphi(x;t) \in U_i \times B_i\}.$$

By the assumption that each fiber F_q over $q \in Q$ is a minimal set of φ, the orbit through x intersects $\{\xi(x)\} \times B_i \subset U_i \times B_i$ at positive times if $\xi(x) \in U_i$. So, each r_i is well-defined and upper semi-continuous. Hence each r_i has an upper bound T_i on $\mathrm{Cl}\, V_i \times F$. Since $\{V_1, \cdots V_k\}$ is an open cover of Q, we get the lemma by taking $T = \max\{T_i \mid i = 1, \dots, k\} + 1$. □

The following is a key lemma to prove Theorem 5.1.

Lemma 5.6 *There exists $T_0 > 0$ such that $u(x, T_0) > 0$ for any $x \in M$.*

Proof. For any $z \in f(V_i \times B_i)$, we define $s(z)$ as the number of times that $\mathcal{O}_\psi(z; (-2CT - \delta, 2CT + \delta))$ intersects $f(V_i \times B_i)$. Here C and T are the constants in (5.2) and Lemma 5.5 respectively and δ is a small positive number. Since $f(V_i \times B_i)$ is transverse to ψ, the function $s : f(V_i \times B_i) \to \boldsymbol{Z}$ is bounded. Fix $q \in V_i \subset Q$. Let m_q be the maximum value of the function s on $f(\{q\} \times B_i)$ and $z_q \in f(\{q\} \times B_i)$ a point such that $s(z_q) = m_q$. Since m_q is maximum on $f(\{q\} \times B_i)$, there exists a neighborhood O_q^i of $z_q \in f(V_i \times B)$ such that $s(z) \leq m_q + 2$ for any $z \in \mathrm{Cl}\, O_q^i$.

Let $x_q \in \{q\} \times B_i$ be the point such that $f(x_q) = z_q$. For any $y \in (f|_{V_i \times B_i})^{-1}(O_q^i) \subset V_i \times B_i$, we define

$$T(y) := \{t \in \boldsymbol{R} \mid \varphi(y;t) \in V_i \times B_i, |u(y,t)| < 2CT + \delta\}.$$

We claim that $\#T(y)$ is finite. In fact, if the equality $u(y,t) = u(y,t')$ holds for t and t' in $T(y)$, then $f(\varphi(y;t))$ and $f(\varphi(y;t'))$ is the same point of $f(V_i \times B_i)$. Since $f|_{V_i \times B_i} : V_i \times B_i \to f(V_i \times B_i)$ is a diffeomorphism, we have $\varphi(y;t) = \varphi(y;t')$. Since φ has no closed orbits, we have $t = t'$. Hence if t and t' are distinct values in $T(y)$, then $u(y,t)$ and $u(y,t')$ are distinct values. Thus we have $\#T(y) \leq s(f(y))$.

Choose a small neighborhood $A_q^i = E_q^i \times D_q^i$ of $x_q \in (f|_{V_i \times B_i})^{-1}(O_q^i)$ such that the sets $T(y) \subset \boldsymbol{R}$ for $y \in \mathrm{Cl}(A_q^i)$ have same upper bounds. In other words, there exist $\tau_q^i > 0$ such that $t < \tau_q^i$ for any $t \in T(y)$ and $y \in \mathrm{Cl}(A_q^i)$. Then, we have

$$|u(y,t)| \geq 2CT + \delta > 2CT, \quad \text{if } \varphi(y;t) \in V_i \times B_i \text{ for } t > \tau_q^i \text{ and } y \in \mathrm{Cl}(A_q^i). \tag{5.3}$$

Let $\{W_1, \ldots, W_k\}$ be a refinement of $\{V_1, \ldots, V_l\}$ such that $\operatorname{Cl} W_j \subset V_j$. Since $\operatorname{Cl} W_i$ is compact, $\operatorname{Cl} W_i$ is covered by finite number of E_q^i, namely $\{E_{q_1}^i, \ldots, E_{q_{k_i}}^i\}$. Take $\tau_0 > \max\{\tau_{q_j}^i \mid i = 1, \ldots, k, j = 1, \ldots, k_i\}$. Then by (5.3) we have

$$|u(x,t)| \geq 2CT + \delta > 2CT \quad \text{for any} \quad x \in A_{q_j}^i \quad \text{and for any} \quad t > \tau_0. \tag{5.4}$$

Now we shall show that, for any x in some $A_{q_k}^i$, either of the following holds,

(1) $u(x,t) > CT$ for $t > \tau_0$,

(2) $u(x,t) < -CT$ for $t > \tau_0$.

Let $\tau_0 < t_1 < \cdots$ be the maximal sequence such that $\varphi(x, t_j) \in U_i \times B_i$ for $j > 0$. Since $x \in A_{q_k}^i$ and $t_j > \tau_0$, we have $u(x, t_j) > 2CT$ or $u(x, t_j) < -2CT$ for each j by (5.4). Since $t_{j+1} - t_j < T$, Lemma 5.4 implies $|u(x, t_{j+1}) - u(x, t_j)| < CT$. Thus, we have either $u(x, t_j) > 2CT$ for all j or $u(x, t_j) < -2CT$ for all j.

For any $t > \tau_0$, there exists j such that $t_j \leq t \leq t_{j+1}$. Thus we have either $t - t_j \leq T/2$ or $t_{j+1} - t \leq T/2$. Hence we have either $|u(x,t) - u(x,t_j)| \leq CT/2$ or $|u(x,t_{j+1}) - u(x,t)| < CT/2$ by Lemma 5.4. So, we have either $u(x,t) > CT$ for any $t > \tau_0$ or $u(x,t) < -CT$ for any $t > \tau_0$. Changing the orientation of the parameterization if necessary, we have $u(x,t) > CT$ for $t > \tau_0$.

Finally, we shall show that there exists $T_0 > 0$ such that $u(x, T_0) > 0$ for any $x \in M$. As we define T, there exists $S > 0$ such that $\mathcal{O}_\varphi(x; (0,S))$ meets $\{\xi(x)\} \times D_{q_k}^i$ if $\xi(x) \in E_{q_k}^i$ for any $x \in M$. Let $\beta = \sup\{-u(M \times [0,S]), 0\}$. Take an integer m and a positive number T_0 such that

$$mCT > \beta \tag{5.5}$$

$$T_0 > m(\tau_0 + S). \tag{5.6}$$

Fix $x \in M$. Since $\{E_{q_j}^i\}$ is a covering of Q, there exists $E_{q_l}^i$ such that $\xi(x) \in E_{q_l}^i$. We define a sequence $v_1 < v_2 < \cdots < v_m$ as follows. Let v_1 be the first arrival time at $\{\xi(x)\} \times D_{q_l}^i$ of the orbit starting at x. If v_j is defined, let v_{j+1} be the the first arrival time at $\{\xi(x)\} \times D_{q_l}^i$ of the orbit starting at x after $v_j + \tau_0$. Hence we have $\varphi(x; v_j) \in \{\xi(x)\} \times D_{q_l}^i$ for all $1 \leq j \leq m-1$ and $v_j + \tau_0 \leq v_{j+1} \leq v_j + \tau_0 + S$ for all $j > 1$. Thus, we have

$$v_m \leq v_{m-1} + \tau_0 + S$$

$$\begin{aligned}&\le v_1+(m-1)(\tau_0+S)\\&< S+(m-1)(\tau_0+S)\\&< T_0-\tau_0.\end{aligned}$$

Since $\varphi(x;v_j)\in\{\xi(x)\}\times D^i_{qk}$ for all j and $u(x,t)>CT$ for $t>\tau_0$, we have

$$u(\varphi(x;v_j),v_{j+1}-v_j)>CT \quad\text{and}\quad u(\varphi(x;v_m),T_0-v_m)>CT. \tag{5.7}$$

By the equation (5.1), (5.7) and the setting (4), (5), we have

$$\begin{aligned}u(x,T_0) &= u(x,v_1)+\sum_{j=1}^{m-1}u(\varphi(x;v_j),v_{j+1}-v_j)\\&\quad+u(\varphi(x;v_m),T_0-v_m)\\&> -\beta+mCT\\&> 0.\end{aligned}$$

□

Proof of Theorem 5.1. Define a function $\nu:M\to\boldsymbol{R}$ by

$$\nu(x)=\frac{1}{T_0}\int_0^{T_0}u(x,t)dt$$

and a map $f_0:M\to N$ by $f_0(x)=\psi(f(x);\nu(x))$. On each orbit of φ, we have

$$\begin{aligned}f_0(\varphi(x;s)) &= \psi(f(\varphi(x;s));\nu(\varphi(x;s)))\\&= \psi(f(x);\nu(\varphi(x;s))+u(x,s)).\end{aligned}$$

Here,

$$\begin{aligned}\nu(\varphi(x;s))+u(x,s) &= \frac{1}{T_0}\int_0^{T_0}\{u(\varphi(x;s),t)+u(x,s)\}dt\\&= \frac{1}{T_0}\int_0^{T_0}u(x,s+t)dt.\end{aligned}$$

Since

$$\frac{d}{ds}\left(\frac{1}{T_0}\int_0^{T_0}u(x,s+t)dt\right)=\frac{1}{T_0}u(x,T_0)>0,$$

the map f_0 is a diffeomorphism sending φ to ψ. □

6. Lie G-flows

In this section, we shall state the classification theorem of Lie G-flows proved by E. Ghys.

In Section 4, we proved that the dimension of the syndetic hulls for the holonomy group of a complete affine flow on M with solvable holonomy group is equal to either $\dim M - 1$ or $\dim M$. If it is equal to $\dim M - 1$, then the flow is actually a Lie G-flow.

Definition 6.1 Let $(N, \mathcal{F})$ be a Lie G-foliation with holonomy group Γ. We say that $(N, \mathcal{F})$ is a *classifying space for the pair* (G, Γ), if for any Lie G-foliation $\mathcal{F}'$ with holonomy group Γ on a compact manifold N' there exists a smooth map $f : N' \to N$ such that $f^*\mathcal{F} = \mathcal{F}'$.

The following theorem implies that two Lie G-flows without closed orbits are homotopy equivalent if they have a same holonomy group.

Theorem 6.2 (Haefliger [8]) *Let $\mathcal{F}$ be a Lie G-foliation with holonomy group Γ on a compact manifold N. Then $(N, \mathcal{F})$ is a classifying space for the pair (G, Γ) if and only if all the leaves of $\mathcal{F}$ are contractible.*

Let (M, φ) be a Lie G-flow on a compact manifold M without closed orbits. Suppose there exists an algebraic Lie G-flow $(\Delta \backslash H, \psi)$ as in Example 1.2 with the same holonomy group Γ of φ. Then there exists a homotopy equivalence of M to $\Delta \backslash H$ sending φ to ψ since they are classifying spaces for the pair (G, Γ). Since the minimal sets of the Lie G-flow gives a fiber bundle structure on the manifold (according to Molino's structure theorem [11]), Theorem 5.1 shows that these two flows are conjugate.

However, in general, the existence of an algebraic Lie G-flow for the given pair (G, Γ) is not known. In the case G is nilpotent, we can construct an algebraic Lie G-flow for the given pair (G, Γ) by Malcev's theorem.

Theorem 6.3 (E. Ghys [7]) *Let φ be a Lie G-flow. If G is simply connected and nilpotent, then φ is conjugate to an algebraic flow.*

7. Criterion for cross section

In this section, we are going to explain when a (G, X)-flow has a cross section in terms of the holonomy group and syndetic hull.

Let φ be a non-singular flow on a manifold of dimension q. Recall that a *cross section* of φ is a $(q-1)$-dimensional submanifold Σ such that Σ is transverse to φ and every orbit of φ intersects Σ.

Definition 7.1 An *immersed cross section* Σ of the flow (M^{n+1}, φ) is an immersed copy of a connected closed n-dimensional manifold such that Σ is transverse to φ and every orbit of φ intersects Σ.

It was shown by S. Matsumoto that the existence of immersed cross section is equivalent to that of cross section. The reader refer to S. Matsumoto [9] and [10] for detailed arguments.

Proposition 7.2 (S. Matsumoto [9]) *If the flow has an immersed cross section, then it has a cross section.*

Now let us show the necessary condition for a (G, X)-flow to admit a cross section in terms of the holonomy group.

Proposition 7.3 *Let (M, φ) be a complete (G, X)-flow. Suppose X is contractible. Then φ has a cross section if there exists a subgroup Γ_0 of the holonomy group Γ such that Γ_0 acts freely and properly discontinuously on X and the quotient space $\Gamma_0 \backslash X$ is a compact manifold.*

Proof. Let Π_0 be the inverse image of Γ_0 by the holonomy homomorphism h. The developing map $\mathcal{D} : \tilde{M} \to X$ induces a trivial $\boldsymbol{R}$-bundle projection $\xi : \tilde{M}/\Pi_0 \to \Gamma_0 \backslash X$. Let $s : \Gamma_0 \backslash X \to \tilde{M}/\Pi_0$ be a section of the $\boldsymbol{R}$-bundle and $\pi_0 : \tilde{M}/\Pi_0 \to M$ the covering map. Then the map $\pi_0 \circ s : \Gamma_0 \backslash X \to M$ is an immersion. It gives birth to an immersed cross section of the flow. Thus the flow has a cross section. □

In the case where φ is a $(H, H/H_0)$-flow, we can say Proposition 7.3 as follows.

Lemma 7.4 *Let (M, φ) be a complete $(H, H/H_0)$-flow. Suppose H is simply connected solvable and the holonomy group Γ is a uniform lattice of H. If the commutator subgroup N of H acts freely on H/H_0, then the flow has a cross section.*

Proof. Let $p : H \to H/N$ be the natural projection. Since N is simply connected and nilpotent, $N \cap \Gamma$ is a uniform lattice of N. Thus we have $\operatorname{rank}(N \cap \Gamma) = \dim N$ and $\operatorname{rank} p(\Gamma) = \dim p(H)$.

Since $N \cap H_0 = \{e\}$, the image $p(H_0)$ is a subgroup of $p(H) \cong \boldsymbol{R}^k$

isomorphic to $\boldsymbol{R}$. Let $\bar{\gamma}_1, \ldots, \bar{\gamma}_k$ be generators of $p(\Gamma)$. If $k \neq 1$, then there exists $k-1$ generators $\bar{\gamma}_{i_1}, \ldots, \bar{\gamma}_{i_{k-1}}$ such that $L \cap p(H_0) = \{e\}$, where L is the subgroup spanned by $\bar{\gamma}_{i_1}, \ldots, \bar{\gamma}_{i_{k-1}}$. Let $H_1 = p^{-1}(L)$ and $\Gamma_1 = \Gamma \cap H_1$. Then H_1 acts on H/H_0 freely and transitively and Γ_1 is a uniform lattice of H_1. Hence φ has a cross section.

If $k = 1$, then N acts on H/H_0 freely and transitively and $N \cap \Gamma$ is a uniform lattice of N. Hence φ has a cross section. □

8. Transversely homogeneous flows

In this section, we shall prove our main theorem.

We already proved, in Section 4, that a complete affine flow with solvable holonomy group is actually a Lie G-flow or transversely homogeneous flow with respect to a syndetic hull. In the following, we first examine transversely homogeneous flows.

Let (M^{n+1}, φ) be a $(H, H/H_0)$-flow.

Suppose H is a simply connected solvable group of dimension $n+1$ contained in some $GL(m; \boldsymbol{R})$ and H/H_0 is diffeomorphic to $\boldsymbol{R}^n$. We also assume the holonomy homomorphism is injective and the holonomy group Γ is a uniform lattice of H. Note that Γ is a strongly polycyclic group.

Lemma 8.1 *There exists a bundle structure over S^1 on M such that φ is either tangent or transverse to the fibers.*

Proof. It is enough to check the case $H_0 \subset N$ by Proposition 7.4. Let $p : H \to H/N$ be the natural projection, where N is the commutator subgroup of H. Then $p(H)$ is isomorphic to $\boldsymbol{R}^k$ for some k and $p(\Gamma)$ is a free abelian group of the rank k as in the proof of Lemma 7.4. Take $k-1$ generators of $p(\Gamma)$ and let L be a subgroup of $p(H)$ spanned by these $k-1$ generators. Then $H_1 = p^{-1}(L)$ is a normal subgroup of H containing H_0 and H/H_1 is isomorphic to $\boldsymbol{R}$. Moreover the image of Γ by the natural projection $H \to H/H_1$ is a cyclic group. Since $H_0 \subset H_1$, we can decompose H/H_0 as $\bigcup_{t \in \boldsymbol{R}} H_1(\delta_t(H_0))$, where we denote $\boldsymbol{R} = \{\delta_t \mid t \in \boldsymbol{R}\}$.

Define $\mathcal{D}_1 : \tilde{M} \to \boldsymbol{R}$ by $\mathcal{D}_1(\tilde{x}) = q(\mathcal{D}(\tilde{x}))$, where $q : H/H_0 \to \boldsymbol{R}$ is a projection defined by $q(h_1\delta_t(0)) = t$. The map $\mathcal{D}_1$ is a fiber bundle projection and equivariant with respect to the action of $\pi_1(M)$. Thus $\mathcal{D}_1$ give birth to a bundle structure over S^1 such that the flow is tangent to the fibers. □

Since Γ is a uniform lattice of H, we have an algebraic transversely homogeneous flow ψ on $\Gamma\backslash H$ defined by the fiber bundle projection H to H/H_0. We shall show that the two flows (M,φ) and $(\Gamma\backslash H,\psi)$ are diffeomorphic if H is unipotent.

Proposition 8.2 *Let (M^{n+1},φ) be a $(H,H/H_0)$-flow. Suppose H is a connected unipotent linear group of dimension $n+1$ and $\Gamma = h(\pi_1(M))$ is a uniform lattice of H. Then (M,φ) is diffeomorphic to the homogeneous flow on $\Gamma\backslash H$ defined by H_0.*

To prove this, we need the following.

Lemma 8.3 *If (M,φ) has a cross section, then the flow is diffeomorphic to $(\Gamma\backslash H,\psi)$.*

Proof. Let Σ be a cross section of φ and $\Gamma_1 = h(\pi_1(\Sigma))$. Let H_1 be a syndetic hull of Γ. Since H is unipotent, H_1 is a subgroup of H.

Let δ be an element of Γ such that $\bar{\delta} \in \Gamma/\Gamma_1 \cong \boldsymbol{Z}$ is a generator. We have a decomposition $H_1 \cdot J$ of H such that Γ_1 and $\langle\delta\rangle$ are uniform lattices of H_1 and J respectively. Hence Σ is diffeomorphic to $\Gamma_1\backslash H_1$. Therefore M is a $\Gamma_1\backslash H_1$ bundle over S^1 whose monodromy map is induced by the action of δ on H_1. On the other hand, $\Gamma\backslash H$ is also a $\Gamma_1\backslash H_1$ bundle over S^1 whose monodromy map is induced by the action of δ on H_1. Hence (M,φ) is isomorphic to $(\Gamma\backslash H,\psi)$. □

Proof of Proposition 8.2. We shall prove this by induction of n. Suppose $n=1$. Then M is diffeomorphic to 2-torus and φ has a cross section. Thus the statement is true. We suppose the statement is true for $q < n+1$.

By Lemma 8.3, we may assume φ dose not have a cross section. By Lemma 8.1, M is the total space of a bundle over circle $\xi : M \to S^1$ with respect to short exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma_1 & \longrightarrow & \Gamma & \longrightarrow & \boldsymbol{Z} & \longrightarrow & 0 \\ & & \cap & & \cap & & \cap & & \\ 1 & \longrightarrow & H_1 & \longrightarrow & H & \longrightarrow & \boldsymbol{R} & \longrightarrow & 0 \end{array}$$

and φ is tangent to each fiber F_t. Note that H_1 contains H_0.

Let

$$1 \longrightarrow \pi_1(F) \longrightarrow \pi_1(M) \xrightarrow{\xi_\#} \pi_1(S^1) \longrightarrow 0$$

be a short exact sequence of fundamental groups and Let $\hat{\delta} \in \pi_1(M)$ be an element such that $\xi_{\#}(\hat{\delta}) = 1$ and $\delta = h(\hat{\delta}) \in \Gamma$. We can identify $\tilde{M}$ with $H_1 \times \boldsymbol{R}$ by a trivialization such that $\mathcal{D}((h_1, t)) = h_1\delta_t(0)$. Write the deck transformation $\hat{\delta}(h_1, t) = (\tilde{f}_t(h_1), t+1)$. Then the diffeomorphism $\tilde{f}_t : H_1 \to H_1$ induces a diffeomorphism $f_t : \Gamma_1\backslash H_1 \to \Gamma_1\backslash H_1$ which gives a monodromy map of the bundle $\xi : M \to S^1$.

The manifold $\Gamma\backslash H$ also has a structure of a bundle over circle with fiber $\Gamma_1\backslash H_1$ and ψ is tangent to the fibers as follows. Define $\rho_t : H_1 \to H_1$ by $\rho_t(h_1) = \delta_t h_1 \delta_t^{-1}$. Then H is isomorphic to a semi-direct product $H_1 \times_\rho \boldsymbol{R}$. The monodromy map $\bar{\rho}$ of the bundle $\Gamma\backslash H \to S^1$ is induced by $\rho = \rho_1$. In the following, we shall show that f_t is isotopic to $\bar{\rho}$.

We can deform f_t by an isotopy to f' such that $f'(e) = e$. We claim that $\tilde{f}_t|_{\Gamma_1} = \rho|_{\Gamma_1}$. Indeed, for any $\gamma_1 \in \Gamma_1$, we have $\hat{\delta}\gamma_1(h_1, t) = (\tilde{f}_t(\gamma_1 h_1), t+1)$. Since Γ_1 is normal in $\pi_1(M)$, there exists $\gamma_1' \in \Gamma_1$ such that $\hat{\delta}\gamma_1 = \gamma_1'\hat{\delta}$. Hence, we have $\hat{\delta}\gamma_1(h_1, t) = \gamma_1'\hat{\delta}(h_1, t) = (\gamma_1'\tilde{f}_t(h_1), t+1)$. It does not depend on the choices of h_1 and t. Especially if we choose $h_1 = e$, then $\tilde{f}_t(\gamma_1) = \gamma_1'\tilde{f}_t(e) = \rho(\gamma_1)$.

For any $h_1 \in H_1$, we have

$$\begin{aligned}\mathcal{D}(\hat{\delta}(h_1, 0)) &= \mathcal{D}((\tilde{f}_0(h_1), 1)) \\ &= \tilde{f}_0(h_1)\delta(0)\end{aligned}$$

and

$$\delta(\mathcal{D}(h_1, 0)) = \delta h_1(0).$$

By the equivariance of $\mathcal{D}$ and h, we have

$$\delta h_1(0) = \tilde{f}_0(h_1)\delta(0).$$

Hence $h_1^{-1}\delta^{-1}\tilde{f}_0(h_1)\delta = h_1^{-1}\rho^{-1}(\tilde{f}_0(h_1)) \in H_0$. Set $r(h_1) := h_1^{-1}\rho^{-1}(\tilde{f}_0(h_1))$. Since $\tilde{f}_0|_{\Gamma_1} = \rho|_{\Gamma_1}$, for any $\gamma_1 \in \Gamma_1$, we have

$$\begin{aligned}r(\gamma_1 h_1) &= (\gamma_1 h_1)^{-1}\rho^{-1}(\tilde{f}_0(\gamma_1 h_1)) \\ &= \gamma_1^{-1}h_1^{-1}\rho^{-1}(\rho(\gamma_1)\tilde{f}_0(h_1)) \\ &= h_1^{-1}\rho^{-1}(\tilde{f}_0(h_1)) \\ &= r(h_1).\end{aligned}$$

Since $r(h_1) \in H_0 \cong \boldsymbol{R}$, we can define $r_s : H_1 \to H_0$ by $r_s(h_1) = sr(h_1)$ for any $s \in [0,1]$. Define a map $\tilde{g}_s : H_1 \to H_1$ by $\tilde{g}_s(h_1) = \rho(h_1)\rho(r_s(h_1))$.

For any $s \in [0,1]$, $\tilde{g}_s$ is a diffeomorphism and

$$\begin{aligned}\tilde{g}_s(\gamma_1 h_1) &= \rho(\gamma_1 h_1)\rho(r_s(\gamma_1 h_1)) \\ &= \rho(\gamma_1)\rho(h_1)\rho(r_s(h_1)) \\ &= \rho(\gamma_1)\tilde{g}_s(h_1).\end{aligned}$$

Thus, for any $s \in [0,1]$, $\tilde{g}_s$ induce a diffeomorphism $g_s : \Gamma_1 \backslash H_1 \to \Gamma_1 \backslash H_1$. Since $g_0 = \bar{\rho}$ and $g_1 = f_0$, the two diffeomorphisms $\bar{\rho}$ and f_0 are isotopic. Hence (M, φ) is diffeomorphic to $(\Gamma \backslash H, \psi)$. □

Finally, we shall prove our main theorem.

Theorem 8.4 *A complete affine flow is virtually algebraic if its holonomy homomorphism is injective and its holonomy group is nilpotent.*

Proof. Let (M^{n+1}, φ) be a complete affine flow as in the theorem and Γ the holonomy group of φ. We may assume that a syndetic hull H for Γ is connected. Suppose H is not unipotent. Then there exists an element h of H such that h has the nontrivial semi-simple part. Let $h = u_0 s_0$ be a Jordan decomposition. Since s_0 is non-trivial, the fixed points set E of s_0 is a nonempty proper affine subspace of $\boldsymbol{R}^n$. Since the algebraic hull $A(\Gamma)$ of Γ is nilpotent, s_0 is in the center of $A(\Gamma)$. This implies that E is kept invariant by the action of $A(\Gamma)$, hence by that of Γ. This contradicts Lemma 4.1. Thus Γ and H is unipotent.

By Corollary 4.3, $\dim H$ is equal to either n or $n+1$. If $\dim H = n$, then (M, φ) is actually a Lie H-flow by Lemma 4.2. Hence Theorem 6.3 shows (M, φ) is virtually algebraic. If $\dim H = n+1$, then (M, φ) is a $(H, H/H_0)$-flow by Lemma 4.2, where H_0 is the stabilizer of $0 \in \boldsymbol{R}^n$. Since $\operatorname{rank} \Gamma = n+1 = \dim H$, the subgroup Γ is a uniform lattice of H. Hence Proposition 8.2 implies that (M, φ) is virtually algebraic. □

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