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A homomorphism between an equivariant SK ring and the Burnside ring for $\mathbf{Z}_4$

(Dedicated to Professor Fuichi Uchida on his 60th birthday)

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Abstract. In this paper, we first determine a ring structure of Z_4 equivariant cutting and pasting theory $SK_*^{Z_4}$. Using the result, we obtain a minimal set of generators of $\mathrm{Ker}\,\phi$, where $\phi : SK_*^{Z_4} \to A(Z_4)$ is the natural surjection to the Burnside ring for Z_4.

Key words: cutting and pasting, Burnside ring, slice types.

1. Introduction

Let G be a finite abelian group, $A(G)$ the Burnside ring and SK_*^G the G-equivariant cutting and pasting ring in the sence of [4]. In [6] Kosniowski proposed that we have a natural homomorhism $SK_*^G \to A(G)$ and what we can say about this homomorhism. In [5] Koshikawa has studied it for the case $G = \mathbf{Z}_2$. In this note, we consider the case $G = \mathbf{Z}_4$.

In Section 2, we determine a ring structure of $SK_*^{\mathbf{Z}_4}$ (Theorem 2.13) by calculating the euler characteristic of manifold with some slice types. In Section 3, we obtain a relation between $SK_*^{\mathbf{Z}_4}$ and Burnside ring $A(\mathbf{Z}_4)$ (Theorem 3.9). Finally we mention a transfer map $SK_*^{\mathbf{Z}_4} \to SK_*^{\mathbf{Z}_2}$ (Proposition 3.11).

Throughout this paper, by a G manifold we mean an unoriented compact smooth manifold with smooth G action. Further it usually has no boundary.

2. A ring structure of $SK_*^{\mathbf{Z}_4}$

In this section, we first recall some basic facts about the theory SK_*^G, and we next determine a ring structure of $SK_*^{Z_4}$.

Let M^n be a closed n dimensional G manifold, and let $L \subset M$ satisfy the following properties,

(1) L is a G invariant codimension 1 smooth submanifold of M,

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(2) L has trivial normal bundle in M, and

(3) the normal bundle of L in M is G equivalent to $L \times \mathbf{R}$ with trivial action of G on the real numbers $\mathbf{R}$.

We assume that L separates M, that is $M = N_1 \cup N_2$ (pasting along the common boundaries $L = \partial N_i$) for some G invariant submanifolds N_i of codimension zero. It is no gain in generality to drop this condition, because the union of L with a second copy of L, suitably embedded near L, will separate M.

Let M_1 and M be n-dimensional G manifolds. We say that M and M_1 are obtained from each other by a G equivariant cutting and pasting if M_1 has been obtained from M by the step as mentioned above, that is, $M_1 = N_1 \cup_\varphi N_2$ and $M = N_1 \cup_\psi N_2$ pasting along the common parts $L \subset M_1$ (or M) by some G diffeomorphisms $\varphi, \psi : L \to L$.

Definition 2.1 If M_1^n has been obtained from M^n by a finite sequence of G equivariant cuttings and pastings, then we say that M_1 and M are SK^G equivalent.

This is an equivalence relation on the set of n dimensional G manifolds. The set of equivalence classes forms an abelian semigroup if we use disjoint union as addition, and has a zero given by the empty set $\emptyset$. The Grothendieck group of this semigroup is then denoted by SK_n^G. If $G = \{1\}$, then SK_n^G is denoted by SK_n. We denote by $[M]$ the equivalence class containing a G manifold M. Further we define SK_*^G as $\sum_{n \geq 0} SK_n^G$. Then it is a graded module over $SK_* = \sum_{n \geq 0} SK_n$, where SK_* is the integral polynomial ring over the integers $\mathbf{Z}$ with a generator α represented by the real projective plane $[\mathbf{R}P^2]$ ([6], 2.5.1). The module operation is given by $[\mathbf{R}P^2]^m[M^n] = [(\mathbf{R}P^2)^m \times M^n]$, where we consider $(\mathbf{R}P^2)^m$ has the trivial G action and $(\mathbf{R}P^2)^m \times M^n$ has the diagonal G action. Moreover, SK_*^G is a graded ring with multiplication by $[M^m][N^n] = [M^m \times N^n]$ with unit $[pt]$, where $M^m \times N^n$ has also diagonal G action and pt is the one-point space with trivial action.

If H is a subgroup of G, then H module is a finite dimensional real vector space together with a linear action of H on it. If M is a G manifold and $x \in M$, then there is a G_x module U_x which is equivariantly diffeomorphic to a G_x neighbourhood of x where $G_x = \{g \in G \mid gx = x\}$ is the isotropy subgroup at x. This module U_x decomposes as $U_x = \mathbf{R}^p \oplus V_x$ when G_x acts trivially on $\mathbf{R}^p$ and $V_x^{G_x} = \{v \in V_x \mid gv = v \text{ for any } g \in G_x\} = \{0\}$. We refer to the pair $\sigma_x = [G_x; V_x]$ as the slice type of x. By a G slice type in general, we mean a pair $[H; V]$ of a subgroup H and an H module V such that $V^H = \{0\}$.

There is a partial order on the set of all G slice types given by: $[H; V] \leq [K; W]$ means $[K; W]$ is a slice type of the G manifold $G \times_H V$ where $G \times_H V$ is $G \times V$ factored by the equivalence relation: $(g, x) \sim (gh, h^{-1}x)$ for $h \in H$. If M is a G manifold and $\sigma = [H; V]$ is a slice type, define $M_\sigma = \{x \in M \mid \sigma_x \leq \sigma\}$. Then M_σ is a G invariant submanifold of M with $\dim(M_\sigma) = \dim(M) - \dim(V)$ (cf. [4, p. 37]).

Now let $G = \mathbf{Z}_4$, the cyclic group of order 4 with a generator $i = \sqrt{-1}$. Let $\tilde{\mathbf{R}}$ denote the real numbers with $\mathbf{Z}_4$ (and $\mathbf{Z}_2$) acting by multiplication by -1, while let $\tilde{\mathbf{C}}$ denote the complex numbers with $\mathbf{Z}_4$ acting by multiplication by i. Then, the $\mathbf{Z}_4$ slice types are $\sigma_{-1} = [1; \{0\}]$, $\sigma_j = [\mathbf{Z}_2; \tilde{\mathbf{R}}^j]$, $(j \geq 0)$ and $\sigma_{j,k} = [\mathbf{Z}_4; \tilde{\mathbf{R}}^j \times \tilde{\mathbf{C}}^k]$, $(j, k \geq 0)$. Concerning the partial order, we note that $\sigma_{j,k} \leq \sigma_{2k} \leq \sigma_{-1}$ and $\sigma_{2k+1} \leq \sigma_{-1}$. We can therefore define an invariant submanifold of $\mathbf{Z}_4$ manifold of M as follows: $M_{\sigma_{2k}} = \{x \in M \mid \sigma_x = \sigma_{2k}$ or $\sigma_{j,k} (j \geq 0)\}$, $M_{\sigma_{2k+1}} = \{x \in M \mid \sigma_x = \sigma_{2k+1}\}$ or $M_{\sigma_{j,k}} = \{x \in M \mid \sigma_x = \sigma_{j,k}\}$. We see that $\dim(M_{\sigma_j}) = m - j$ and $\dim(M_{\sigma_{j,k}}) = m - (j + 2k)$ as mentioned above, where $m = \dim(M)$ (cf. [6, p. 121 and p. 211]). Notice that $M_{\sigma_{-1}} = M$.

Let

$$M_i = \mathbf{Z}_4 \times_{\mathbf{Z}_2} \mathbf{R}P(\mathbf{R} \times \tilde{\mathbf{R}}^i), \quad M_{j,k} = \mathbf{R}P(\mathbf{R} \times \tilde{\mathbf{R}}^j) \times \mathbf{R}P(\mathbf{R} \times \tilde{\mathbf{C}}^k),$$

and let $x = [\mathbf{Z}_4]$, $x_i = [M_i]$, $x_{j,k} = [M_{j,k}]$.

Then the SK_* module structure of $SK_*^{\mathbf{Z}_4}$ is as follows.

Proposition 2.2 ([6], 5.4.1) *$SK_*^{\mathbf{Z}_4}$ is a free SK_* module with basis $\mathcal{B} = \{x, x_i, x_{j,k} \ (i, j, k \geq 0)\}$.*

Proposition 2.3 ([6], 5.4.7) *Two n dimensional $\mathbf{Z}_4$ manifolds M, M' are $SK^{\mathbf{Z}_4}$ equivalent if and only if*
(1) $\chi(M) = \chi(M')$ (2) $\chi_i(M) = \chi_i(M')$ $i = 0, 1, \ldots, n$
(3) $\chi_{j,k}(M) = \chi_{j,k}(M')$ $j, k \geq 0, j + 2k \leq n$ *where* $\chi_i(M) = \chi(M_{\sigma_i})$ *and* $\chi_{j,k}(M) = \chi(M_{\sigma_{j,k}})$.

Remark 2.4 Let M be $\mathbf{Z}_4$, M_i or $M_{j,k}$. Then the values $\chi_{i'}(M)$ and $\chi_{j',k'}(M)$ which do not vanish are as follows.

$\chi = 4$ on $\mathbf{Z}_4$, $\chi = \chi_{2i} = 2$ on M_{2i}, $\chi_1 = \chi_{2i+1} = 2$ on M_{2i+1}, $\chi = \chi_{2k} = \chi_{2j,k} = 1$ on $M_{2j,k}$, and $\chi_{1,k} = \chi_{2j+1,k} = 1$ on $M_{2j+1,k}$.

For each M, the manifolds M_{σ_i} and $M_{\sigma_{j,k}}$ are obvious. We therefore obtain the above data.

Proposition 2.5 *Let* $\mathbf{K} = \mathbf{C}$ *or the field* $\mathbf{H}$ *of quaternions and let* $\mathbf{K}P(\mathbf{K} \times \widetilde{\mathbf{K}}^n)$ *be the projective space associated to* $\mathbf{K} \times \widetilde{\mathbf{K}}^n$ *with* $\mathbf{Z}_4$ *action* $id \times i$ $(n \geq 0)$. *Then we have*
(i) $[\mathbf{C}P(\mathbf{C} \times \widetilde{\mathbf{C}}^n)] = x_{0,n} + n\alpha^{n-1}x_{0,1}$, *and*
(ii) $[\mathbf{H}P(\mathbf{H} \times \widetilde{\mathbf{H}}^n)] = x_{0,2n} + n\alpha^{2n-2}x_{0,2}$.

Proof. Note that $\mathbf{C}P(\mathbf{C} \times \widetilde{\mathbf{C}}^n)$ (or $\mathbf{H}P(\mathbf{H} \times \widetilde{\mathbf{H}}^n)$) has the data on slice types as $\chi = n+1$, $\chi_{0,n} = 1$, $\chi_{0,1} = n$ (or $\chi = n+1$, $\chi_{0,2n} = 1$, $\chi_{0,2} = n$) respectively ([3], p. 106). Hence the relation (i) or (ii) follows by comparing the data of both sides (cf. Remark 2.4). □

Example 2.6 We show (i) by an $SK^{\mathbf{Z}_4}$ process as follows.

Put $N_i = A_i + B_i$ $(i = 1, 2)$ where $A_1 = D(\widetilde{\mathbf{C}}^n)$, $A_2 = D(\mathbf{C}) \times_{S^1} S(\widetilde{\mathbf{C}}^n)$, $B_1 = [-1, 1] \times_{Z_2} S(\widetilde{\mathbf{C}}^n)$ and $B_2 = [-1, 1]' \times_{Z_2} S(\widetilde{\mathbf{C}}^n)$. Further, consider $L = L' + L''$ where $L' = L'' = S(\widetilde{\mathbf{C}}^n)$ with natural embeddings $L' = \partial A_i \subset A_i$ and $L'' = \{-1, 1\} \times_{Z_2} S(\widetilde{\mathbf{C}}^n) \subset B_i$. Now let φ, ψ:$L = \partial N_1 \to L = \partial N_2$ be identifications:

$$\varphi : A_1 \supset L' \to L' \subset A_2, \quad B_1 \supset L'' \to L'' \subset B_2,$$
$$\psi : A_1 \supset L' \to L'' \subset B_2, \quad B_1 \supset L'' \to L' \subset A_2.$$

Then

$$N_1 \cup_\varphi N_2 = \mathbf{C}P(\mathbf{C} \times \widetilde{\mathbf{C}}^n) + S^1 \times_{Z_2} S(\widetilde{\mathbf{C}}^n) \quad \text{and}$$
$$N_1 \cup_\psi N_2 = \mathbf{R}P(\mathbf{R} \times \widetilde{\mathbf{C}}^n) + P,$$

where

$$\begin{aligned} P &= D(\mathbf{C}) \times_{S^1} S(\widetilde{\mathbf{C}}^n) \cup [-1, 1] \times_{Z_2} S(\widetilde{\mathbf{C}}^n) \\ &\cong D(\mathbf{C}) \times_{S^1} S(\widetilde{\mathbf{C}}^n) \cup ([-1, 1] \times_{Z_2} S^1) \times_{S^1} S(\widetilde{\mathbf{C}}^n) \\ &\cong \mathbf{R}P(\mathbf{R} \times \mathbf{C}) \times_{S^1} S(\widetilde{\mathbf{C}}^n) \end{aligned}$$

with obvious identifications. Observe P fibers equivariantly over $\mathbf{C}P^{n-1} = S(\widetilde{\mathbf{C}}^n)/S^1$ with fiber $\mathbf{R}P(\mathbf{R} \times \widetilde{\mathbf{C}})$. Hence $[P] = [\mathbf{C}P^{n-1}] \cdot [\mathbf{R}P(\mathbf{R} \times \widetilde{\mathbf{C}})]$ by [6, Theorem 2.4.1] or [4, Lemma (1.5)]. Since $\mathbf{C}P^{n-1}$ is cobordant to

$(\mathbf{R}P^{n-1})^2$ in the unoriented cobordism ring N_* (cf. [7], Lemma 7),

$$\begin{aligned}[\mathbf{C}P^{n-1}] &= [\mathbf{R}P^{n-1}]^2 + \frac{1}{2}(\chi(\mathbf{C}P^{n-1}) - \chi((\mathbf{R}P^{n-1})^2))[S^{2n-2}] \\ &= n[\mathbf{R}P^{2n-2}] \\ &= n\alpha^{n-1}\end{aligned}$$

by $[S^{2n-2}] = 2[\mathbf{R}P^{2n-2}]$ and $[\mathbf{R}P^{2m}] = [\mathbf{R}P^2]^m$ in general (cf. [6 Corollary 2.3.4 and p. 62]). On the other hand, $S^1 \times_{Z_2} S(\widetilde{\mathbf{C}}^n)$ fibers equivariantly over $\mathbf{R}P^1 = S^1/Z_2$ with fiber $S(\widetilde{\mathbf{C}}^n)$, which implies that $[S^1 \times_{Z_2} S(\widetilde{\mathbf{C}}^n)] = [\mathbf{R}P^1]\cdot[S(\widetilde{\mathbf{C}}^n)] = 0$ since $[\mathbf{R}P^1] = 0$ in SK_* ([6], Theorem 2.4.1 (i)). Therefore we have the relation for $\mathbf{C}P(\mathbf{C} \times \widetilde{\mathbf{C}}^n)$. □

Since $[\mathbf{C}P(\mathbf{C} \times \widetilde{\mathbf{C}}^n)] = x_{0,n}$ or $[\mathbf{H}P(\mathbf{H} \times \widetilde{\mathbf{H}}^n)] = x_{0,2n}$ (mod SK_* decomposable), we have the following result.

Corollary 2.7 *The element $x_{0,n}$ (or $x_{0,2n}$) in the basis $\mathcal{B}$ is replaced by $[\mathbf{C}P(\mathbf{C} \times \widetilde{\mathbf{C}}^n)]$ (or $[\mathbf{H}P(\mathbf{H} \times \widetilde{\mathbf{H}}^n)]$) respectively.*

Now we go back to G slice types. Let $\sigma = [H; V]$ be a slice type of $x = [g, w] \in G \times_K W$. Since $G_w = H (\subset K)$, W decomposes as $W = \langle w \rangle \oplus W'$ as an H module, where $\langle w \rangle$ is a submodule generated by w and W' is its complement. We therefore $V = NT(W') = NT(W)$, where $NT(-)$ is the non-trivial part of H module. Let M be a G manifold, and let $\sigma = [H; V]$ and $\sigma' = [H; V']$ be H slice types. If $x \in M_\sigma \cap M_{\sigma'}$, then both σ and σ' be H slice types of $G \times_{G_x} V_x$. Hence $\sigma = \sigma'$ because $V = V' = NT(V_x)$ as H modules. We therefore $M^H = \coprod_\sigma M_\sigma$ (disjoint union), where the sum is taken over all H slice types $\sigma = [H; V]$.

Lemma 2.8 *Let M and N be Z_4 manifolds, then*

$$\chi_i(M \times N) = \sum_{p+q=i} \chi_p(M)\chi_q(N) \quad \textit{and}$$

$$\chi_{j,k}(M \times N) = \sum_{p+q=j, r+s=k} \chi_{p,r}(M)\chi_{q,s}(N).$$

Proof. We first prove that

$$(2.8.1) \qquad (M \times N)_{\sigma_i} = \coprod_{p+q=i} (M_{\sigma_p} \times N_{\sigma_q}) \quad \text{and}$$

$$(M\times N)_{\sigma_{j,k}} = \coprod_{p+q=j,r+s=k} (M_{\sigma_{p,r}}\times N_{\sigma_{q,s}}).$$

Suppose that $H=\mathbf{Z}_2$. Since $(M\times N)_{\sigma_i}=\coprod_{p+q=i}(M_{\sigma_p}\times N_{\sigma_q})$, it sufficies to show that $M_{\sigma_p}\times M_{\sigma_q}\subset (M\times N)_{\sigma_j}$, where $j=p+q$. Let $p=2k$, $q=2l+1$ and put $(x,y)\in M_{\sigma_p}\times N_{\sigma_q}$. There are two cases for the slice type of x, that is, one: $\sigma_x=[\mathbf{Z}_2;\tilde{\mathbf{R}}^{2k}]$ and the other: $\sigma_x=[\mathbf{Z}_4;\tilde{\mathbf{R}}^j\times\tilde{\mathbf{C}}^k]$ for some $j\ge 0$. On the other hand, $\sigma_y=[\mathbf{Z}_2;\tilde{\mathbf{R}}^{2l+1}]$. Then a $\mathbf{Z}_2$ neighbourhood of (x,y) in $M\times N$ is equivariantly diffeomorhic to $\tilde{\mathbf{R}}^{2k}\times\tilde{\mathbf{R}}^{2l+1}$ in the first case and $\mathbf{R}^j\times\tilde{\mathbf{R}}^{2k}\times\tilde{\mathbf{R}}^{2l+1}$ in the second one. Therfore $\sigma_{(x,y)}=[\mathbf{Z}_2;\tilde{\mathbf{R}}^{2k+2l+1}]$ in both cases, and $(x,y)\in(M\times N)_{\sigma_j}$ with $j=p+q$. Similarly we have the same results in another cases, from which the first part of (2.8.1) follows. In a same way, we have the second part. Taking χ for both sides of (2.8.1), we obtain the lemma. □

Proposition 2.9

(1) $x^2=4x$ (2) $xx_{2j+1}=0,\ xx_{2j}=2\alpha^j x$

(3) $xx_{2j+1,l}=0,\ xx_{2j,l}=\alpha^{j+l}x$ (4) $x_{2k}x_{2l}=2x_{2(k+l)}$

(5) $x_{2k}x_{2l+1}=2x_{2k+2l+1}+2\alpha^l x_{2k+1}-2\alpha^{k+l}x_1$

(6) $x_{2k+1}x_{2l+1}=-4\alpha^{k+l+1}x+2x_{2k+2l+2}+2\alpha^l x_{2k+2}+2\alpha^k x_{2l+2}+2\alpha^{k+l}x_2$

(7) $x_{2m}x_{2n,l}=\alpha^n x_{2(m+l)}$ (8) $x_i x_{2n+1,l}=0$

(9) $x_{2m+1}x_{2n,l}=\alpha^n x_{2m+2l+1}+\alpha^{m+n}x_{2l+1}-\alpha^{m+n+l}x_1$

(10) $x_{2m,j}x_{2n,l}=x_{2(m+n),j+l}$

(11) $x_{2m,j}x_{2n+1,l}=x_{2m+2n+1,j+l}+\alpha^n x_{2m+1,j+l}-\alpha^{m+n}x_{1,j+l}$

(12) $x_{2m+1,j}x_{2n+1,l}=-2\alpha^{m+n+1}x_{2j+2l}+\alpha^n x_{2m+2,j+l}+\alpha^m x_{2n+2,j+l}$
$+\alpha^{m+n}x_{2,j+l}+x_{2m+2n+2,j+l}$

Proof. We prove (12) by Proposition 2.3. Let

$$[M_{2m+1,j}][M_{2n+1,l}] = a[\mathbf{R}P^2]^{2t}[\mathrm{Z}_4]+\sum_i b_i[\mathbf{R}P^2]^i[M_{2(t-i)}] + \sum_{q,r} c_{q,r}[\mathbf{R}P^2]^q[M_{2(t-q-r),r}]$$

where $a, b_i, c_{q,r}\in\mathbf{Z}$, $t=m+n+1+j+l$ and $0\le i\le t$, $0\le q+r\le t$.

The euler characteristics of the left side are $\chi=0$, $\chi_{2m+2n+2,j+l}=\chi_{2m+2,j+l}=\chi_{2n+2,j+l}=\chi_{2,j+l}=1$ and the others $\chi_{h,k}=0$. On the other hand, those of the right side are $\chi=4a+2\sum_i b_i+\sum_{q,r}c_{q,r}$, $\chi_{2m+2n+2,j+l}=c_{0,j+l}$, $\chi_{2m+2,j+l}=c_{n,j+l}$, $\chi_{2n+2,j+l}=c_{m,j+l}$, $\chi_{2,j+l}=c_{m+n,j+l}$, $\chi_{2j+2l}=$

$2b_{m+n+1}+4$, $\chi_{2(m+n+1+j+l-i)} = 2b_i$ $(0 \le i \le t)$ and the others $\chi_{h,k} = 0$. (cf. Remark 2.4 and Lemma 2.8).

Therefore $c_{0,j+l} = c_{n,j+l} = c_{m,j+l} = c_{m+n,j+l} = 1$, $b_{m+n+1} = -2$, $a = 0$ and the other coefficients are 0. Hence we can obtain (12). In the similar way we have the rest equalities. □

Lemma 2.10 *Let* $c_n = [\mathbf{C}P(\mathbf{C} \times \widetilde{\mathbf{C}}^n)]$ *and* $h_n = [\mathbf{H}P(\mathbf{H} \times \widetilde{\mathbf{H}}^n)]$ *in* $SK_*^{Z_4}$, *then the following relations hold.*

(1) $c_m \cdot c_n = c_{m+n} + m\alpha^{m-1}c_{n+1} + n\alpha^{n-1}c_{m+1} + mn\alpha^{m+n-2}c_2 - (2mn + m + n)\alpha^{m+n-1}c_1 \quad (m+n \ge 2)$,

(2) $h_m \cdot h_n = h_{m+n} + m\alpha^{2(m-1)}h_{n+1} + n\alpha^{2(n-1)}h_{m+1} + mn\alpha^{2(m+n-2)}h_2 - (2mn + m + n)\alpha^{2(m+n-1)}h_1 \quad (m+n \ge 2)$,

(3) $c_{2m+1}^2 = h_{2m+1} + 2(2m+1)\alpha^{2m}h_{m+1} - (2m+1)\alpha^{4m}h_1 \quad (m \ge 0)$,

(4) $h_m = c_{2m} + m\alpha^{2m-2}c_2 - 2m\alpha^{2m-1}c_1 \quad (m \ge 1)$,

and $c_0 = h_0 = 1$.

The proofs are obtained from Proposition 2.5 and 2.9 (10) straightforwardly, so we omit them here. From this, we have the following proposition.

Proposition 2.11 *Let* $\mathcal{C}$ *(or* $\mathcal{H}$*) be an* SK_* *submodule generated by the class* $\{c_n \mid n \ge 0\}$ *(or* $\{h_n \mid n \ge 0\}$*) respectively, then it is an* SK_* *subalgebra of* $SK_*^{Z_4}$ *and* $\mathcal{H} \subset \mathcal{C}$.

Next we consider an SK_* algebra structure of $SK_*^{Z_4}$. We first reduce the following equalities.

Lemma 2.12

(i) $x_{2m} = x_0(x_{0,1})^m, \quad m \ge 1$

(ii) $x_{2m+3} = x_3(x_{0,1})^m - (x_3 - \alpha x_1)\sum_{i=1}^{m}\alpha^i(x_{0,1})^{m-i}, \quad m \ge 1$

(iii) $x_{2m,j} = (x_{2,0})^m(x_{0,1})^j, \quad m \ge 0,\ j \ge 0$

(iv) $x_{2m+3,j} = (x_{0,1})^j\left\{(x_{2,0})^m x_{3,0} - (x_{3,0} - \alpha x_{1,0})\sum_{i=1}^{m}\alpha^i(x_{2,0})^{m-i}\right\}, \quad m \ge 1,\ j \ge 0$

Proof. We use the equalities in Proposition 2.9. From (7) we obtain (i) by induction on m, while from (10) we obtain (iii) by induction on j and m. Next let us put $(n, l) = (0, 1)$ on (9), then we have $x_{2m+3} = x_{2m+1}x_{0,1} - (x_3 - \alpha x_1)\alpha^m$. From this, (ii) follows by induction on m. Finally,

$x_{2m+3,j} = x_{0,j}x_{2m+3,0}$ from (11). Moreover, from (12) we have $x_{2m+3,0} = (x_{2,0})^m x_{3,0} - (x_{3,0} - \alpha x_{1,0}) \sum_{i=1}^{m} \alpha^i (x_{2,0})^{m-i}$ as (ii). These imply (iv).

$\square$

Since $SK_*^{Z_4}$ is freely generated over SK_* by these $x, x_i, x_{j,k}$ $(i, j, k \geq 0)$, we have the following.

Theorem 2.13 *As an SK_*-algebra $SK_*^{Z_4} \cong \mathcal{P}/\mathcal{I}$, where $\mathcal{P}$ is an SK_* polynomial ring with indeterminates x, x_0, x_1, x_3, $x_{0,1}$, $x_{1,0}$, $x_{2,0}$ and $x_{3,0}$, and $\mathcal{I}$ is an ideal generated by the relations induced from Proposition* 2.9 (*or Lemma* 2.12).

Let $p = x_{2,0}^3$ and $q = x_{3,0}^2 - 2\alpha x_{2,0}^2 - \alpha^2 x_{2,0} + 2\alpha^3 x_0$ for example, then $p = q = x_{6,0}$ in $SK_*^{Z_4}$ from Proposition 2.9 (10) and (12). Hence $p - q \in \mathcal{I}$. Further it is easy to see that the above eight indeterminates supply a minimal set of generators.

3. The relation between $SK_*^{Z_4}$ and $A(Z_4)$.

We define another equivalence relation as follows. Let M and N be closed smooth G manifolds, then $M \sim N$ if and only if the H fixed point sets M^H and N^H for all subgroups H of G have the same euler characteristics $\chi(M^H)$ and $\chi(N^H)$. Denote by $A(G)$ the set of equivalence classes under this equivalence relation, and denote by $[M] \in A(G)$ the class of M (we use conveniently same notation as the element of SK_*^G). The disjoint union and the cartesian product of G manifolds induce an addition and multiplication on $A(G)$. Then $A(G)$ becomes a commutative ring with identity $[pt]$.

Definition 3.1 We call $A(G)$ the Burnside ring of G.

Let M be a G manifold and H be a subgroup of G. Then we define $M_H = \{x \in M \mid G_x = H\}$. Now we note that we consider only G a finite abelian group. So the next formula is the special case of tom Dieck's one ([2], 5.5.1).

Proposition 3.2 *$A(G)$ is the free abelian group with basis $\{[G/H] \mid H \subset G\}$ and any element $[M] \in A(G)$ have the relation $[M] = \sum_{H \subset G} \chi(M_H/G)$ $[G/H]$.*

By this fomula, $A(\mathbf{Z}_p) \cong \mathbf{Z}[x]/(x^2 - px)$ for any prime integer p ([5], Lemma 6). On the other hand, we have the following.

Lemma 3.3

$$A(\mathbf{Z}_{2^n}) \cong \mathbf{Z}[z_1, z_2, \dots, z_n]/(z_i z_{i+j} - 2^{n-(i+j-1)} z_i)$$

where $z_i = [\mathbf{Z}_{2^n}/\mathbf{Z}_{2^{i-1}}]$ $(i = 1, \dots, n+1)$.

Proof. From Proposition 3.2, $A(\mathbf{Z}_{2^n})$ is a free abelian group generated by z_i, where $z_{n+1} = 1$. Put $M = (\mathbf{Z}_{2^n}/\mathbf{Z}_{2^{i-1}})$, then $\chi(M^{\mathbf{Z}_{2^{k-1}}}) = 2^{n-k+1}$ for $1 \le k \le i$ or 0 for $i+1 \le k \le n+1$. Hence we obtain $z_i z_{i+j} = 2^{n-(i+j-1)} z_i$ by comparing euler characteristics of both sides. □

Definition 3.4 Let $[M] \in SK_*^{\mathbf{Z}_4}$, then $[M]$ can be naturally regarded as an element of $A(\mathbf{Z}_4)$. We denote this correspondence by $\phi : SK_*^{\mathbf{Z}_4} \to A(\mathbf{Z}_4)$. Then ϕ is well-defined because $\chi(M^{\mathbf{Z}_2}) = \sum_i \chi_i(M)$ and $\chi(M^{\mathbf{Z}_4}) = \sum_{j,k} \chi_{j,k}(M)$, and is a ring homomorphism.

The generators of $SK_*^{\mathbf{Z}_4}$ are mapped by ϕ as follws.

Lemma 3.5 $\phi(x) = u$, $\phi(x_{2i}) = v$, $\phi(x_{2i+1}) = 2v - u$, $\phi(x_{2i,j}) = 1$ *and,* $\phi(x_{2i+1,j}) = 2-v$ *for* $i, j \ge 0$, *where* $u = [\mathbf{Z}_4]$, $v = [\mathbf{Z}_4/\mathbf{Z}_2]$ *and* $1 = [\mathbf{Z}_4/\mathbf{Z}_4]$.

Proof. $\phi(x) = u$ is a trivial. Let $\phi(x_{2i+1}) = au + bv + c$ for $a, b, c \in \mathbf{Z}$. Since $\chi(M_{2i+1}) = 0$, $\chi({M_{2i+1}}^{\mathbf{Z}_2}) = 4$ and $\chi({M_{2i+1}}^{\mathbf{Z}_4}) = 0$ from Remark 2.4, we have $4a+2b+c = 0$, $2b+c = 4$ and $c = 0$. So we have $\phi(x_{2i+1}) = 2v - u$. Similarly we ontain another equalities. □

Next let us calculate $\operatorname{Ker}\phi$.

Lemma 3.6 $\operatorname{Ker}\phi$ *is freely generated by* $P(l,i,j) = \alpha^l x_{2i+1,j} - x_{1,0}$, $Q(p,h,k) = \alpha^p x_{2h,k} - x_{0,0}$, $R(q,t) = \alpha^q x_{2t+1} - x_1$, $S(r,w) = \alpha^r x_{2w} + x_{1,0} - 2x_{0,0}$ *and,* $T(s) = \alpha^s x + x_1 + 2x_{1,0} - 4x_{0,0}$ *where* $i, j, k, l, p, q, r, s, t, w \ge 0$.

Proof. For any fixed $n \ge 0$, let $[M]$ be in $\operatorname{Ker}\phi$ and let it be an SK_* linear combination as follows.

$$\begin{aligned}[M] = &\sum_{i,j,l} a_l^{i,j} \alpha^l x_{2i+1,j} + \sum_{h,k,p} b_p^{h,k} \alpha^p x_{2h,k} + \sum_{q,t} c_q^t \alpha^q x_{2t+1} \\ &+ \sum_{r,w} d_r^w \alpha^r x_{2w} + \sum_s e_s \alpha^s x, \quad \text{for } a_l^{i,j}, b_p^{h,k}, c_q^t, d_r^w, e_s \in \mathbf{Z},\end{aligned}$$

where the suffix are taken over $0 \le i+j+l, h+k+p, q+t, r+w, s \le n$.

Now $\phi(\alpha)=1$, so by Lemma 3.5,

$$\begin{aligned}\phi([M]) &= \Big(-\sum_{i,j,l} a_l^{i,j} + 2\sum_{q,t} c_q^t + \sum_{r,w} d_r^w\Big)v \\ &\quad + \Big(2\sum_{i,j,l} a_l^{i,j} + \sum_{h,k,p} b_p^{h,k}\Big) + \Big(-\sum_{q,t} c_q^t + \sum_s e_s\Big)u.\end{aligned}$$

Then we have the following simultaneous equations with rank 3:

$$\begin{cases} \sum_{i,j,l} a_l^{i,j} - \sum_{r,w} d_r^w - 2\sum_s e_s = 0 \\ \sum_{h,k,p} b_p^{h,k} + 2\sum_{r,w} d_r^w + 4\sum_s e_s = 0 \\ \sum_{q,t} c_q^t - \sum_s e_s = 0. \end{cases}$$

Now let $\bar{a}_l^{i,j}$ be the vector whose (i,j,l)-the coordinate $a_l^{i,j}=1$ and the others are zero. Similary we define the vectors for another letters. Then the vectors $\bar{a}_l^{i,j}-\bar{a}_0^{0,0}$, $\bar{b}_p^{h,k}-\bar{b}_0^{0,0}$, $\bar{c}_q^t-\bar{c}_0^0$, $\bar{d}_r^w+\bar{a}_0^{0,0}-2\bar{b}_0^{0,0}$ and $\bar{e}^s+2\bar{a}_0^{0,0}-4\bar{b}_0^{0,0}+\bar{c}_0^0$ are linearly independent solutions. This gives the result. □

Since $SK_* \subset SK_*^{Z_4}$, we may consider $A(Z_4)$ as SK_* algebra via ϕ ([1], Chapter 2). In this case, for $[M]\in SK_*$ and $[N]\in A(Z_4)$, $[M][N]=\phi([M])[N]=[M\times N]$ and ϕ is algebra homomorphism.

Now we will reduce the above generators in order to get the minimal set of generators of $\operatorname{Ker}\phi$ as SK_* subalgebra.

Let for $i\geq 0$ $A_i=P(i,0,0)$, $B_i=P(0,i,0)+S(0,0)$, $C_i=Q(i,0,0)$, $D_i=Q(0,i,0)$, $E_i=Q(0,0,i)$, $F_i=R(i,0)$, $G_i=R(0,i)-2S(0,0)+T(0)$, $H_i=S(i,0)-S(0,0)$, $I_i=S(0,i)$ and $J_i=T(i)$. Then we can reduce these relations as follows.

Lemma 3.7

$$A_i = A_1\sum_{s=1}^{i}\alpha^{i-s} \tag{3.1}$$

$$B_i = B_1(D_1+1)^{i-1} + (2D_1-H_1)\sum_{s=0}^{i-2}(D_1+1)^s, \quad i\geq 2 \tag{3.2}$$

$$C_i = C_1\sum_{s=1}^{i}\alpha^{i-s} \tag{3.3}$$

$$D_i = \sum_{s=0}^{i-1} \binom{i}{s} D_1^{i-s}, \quad i \geq 1 \tag{3.4}$$

$$E_i = \sum_{s=0}^{i-1} \binom{i}{s} E_1^{i-s}, \quad i \geq 1 \tag{3.5}$$

$$F_i = F_1 \sum_{s=1}^{i} \alpha^{i-s} \tag{3.6}$$

$$\begin{aligned} G_i =& (E_1+1)^i G_0 + (2I_1 - J_1 + G_0) \sum_{s=0}^{i-1} (E_1+1)^{i-1-s} \\ & - (G_1 - 2H_1 + J_1 - 2I_0 - G_0 - \alpha G_0) \sum_{s=1}^{i-1} (E_1+1)^{i-1-s} \alpha^s, \\ & i \geq 1 \end{aligned} \tag{3.7}$$

$$H_i = H_1 \sum_{s=1}^{i} \alpha^{i-s} \tag{3.8}$$

$$I_i = I_1 + (I_1 - I_0) \sum_{s=1}^{i-1} (E_1+1)^s, \quad i \geq 1 \tag{3.9}$$

$$J_i = \alpha^{i-1} J_1 + \sum_{k=1}^{i-1} \alpha^{i-1-k} (4C_1 - 2A_1 - F_1), \quad i \geq 1 \tag{3.10}$$

Proof. We can easily obtain (3.1), (3.2), (3.3), (3.6), (3.8) by induction on i. We have (3.4) by the relation

$$D_{i+1} = D_i D_1 + D_i + D_1$$

Similarly we obtain (3.5). Next

$$\begin{aligned} G_{i+1} &= x_{2i+3} - 2x_0 + x \\ &= x_3 (x_{0,1})^i - (x_3 - \alpha x_1) \sum_{s=1}^{i} \alpha^s (x_{0,1})^{i-s} - 2x_0 + x \\ &= x_{0,1} x_{2i+1} - (x_3 - \alpha x_1)\alpha^i - 2x_0 + x \\ &= (x_{0,1} - x_{0,0}) G_i + G_i + 2(x_2 - x_0) - (\alpha x - x) - (x_3 - \alpha x_1)\alpha^i \\ &= (E_1 + 1) G_i + (2I_1 - J_1 + G_0) \\ &\quad - (G_1 - 2H_1 + J_1 - 2I_0 - G_0 - \alpha G_0)\alpha^i \end{aligned}$$

Then we can obtain (3.7) by induction on i. By Lemma 2.9 and induction, we obtain

$$\begin{aligned} I_{i+1} &= x_{1,0} - 2x_{0,0} + x_{2(i+1)} \\ &= x_{1,0} - 2x_{0,0} + x_{0,1}x_{2i} \\ &= x_{1,0} - 2x_{0,0} + x_{0,1}(I_i - x_{1,0} + 2x_{0,0}) \\ &= I_0 + (E_1+1)I_i - (E_1+1)I_0 + I_1 - I_0 \\ &= (I_1 - I_0 - I_0E_1) + (E_1+1)I_i \end{aligned}$$

So we obtain (3.9). Finally we deform J_{i+1} as follows by induction.

$$\begin{aligned} J_{i+1} &= 2x_{1,0} - 4x_{0,0} + x_1 + \alpha^{i+1}x \\ &= 2x_{1,0} - 4x_{0,0} + x_1 + \alpha(J_i - 2x_{1,0} + 4x_{0,0} - x_1) \\ &= 2(x_{1,0} - \alpha x_{1,0}) - 4(x_{0,0} - \alpha x_{0,0}) + (x_1 - \alpha x_1) + \alpha J_i \\ &= \alpha J_i + (-2A_1 + 4C_1 - F_1) \end{aligned}$$

Then we obtain (3.10). □

Next we have the following lemma.

Lemma 3.8

(1) $\alpha^l x_{2i+1,j} - x_{1,0} = \alpha^l B_i(E_j+1) + 2\alpha^l E_j + 2C_l - B_0 - H_l - \alpha^l(I_1 - B_0)\sum_{s=0}^{j-1}(E_1+1)^s, \quad j \geq 1$

(2) $\alpha^p x_{2h,k} - x_{0,0} = \alpha^p(D_hE_k + D_h + E_k) + C_p$

(3) $\alpha^q x_{2t+1} - x_1 = \alpha^q(G_t - G_0) + F_q$

(4) $x_{1,0} - 2x_{0,0} + \alpha^r x_{2w} = \alpha^r I_w + 2C_r - A_r$

(5) $2x_{1,0} - 4x_{0,0} + x_1 + \alpha^s x = \alpha^{s-1}J_1 - (2A_1 - 4C_1 + F_1)\sum_{i=1}^{s-1}\alpha^{s-1-i}, \quad s \geq 1$

Proof. By lemma 2.12 $x_{2i+1,j} = B_iE_j + B_i + 2x_{0,j} - x_{2j}$. Let $D'_j = x_{2j} - x_0$, then $\alpha^l x_{2i+1,j} - x_{1,0} = \alpha^l B_i(E_j+1) + 2\alpha^l E_j + 2C_l - B_0 - H_l - \alpha^l D'_j$. If $j \geq 1$, then $D'_j = (I_1 - B_0)\sum_{s=0}^{j-1}(E_1+1)^s$ by induction on j. So we have (1). Similarly we obtain (2) from $x_{2h,k} = D_hE_k + D_h + E_k + x_{0,0}$. We can easily obtain (3) and (4). (5) is (3.10) of Lemma 3.7. □

Therefore, by Lemma 3.7 and 3.8, we have the following.

Theorem 3.9 *If $\mathcal{S}$ is an SK_*-subalgebra of $SK_*^{\mathbf{Z}_4}$ generated by $P(1,0,0)$, $P(0,1,0)$, $Q(1,0,0)$, $Q(0,1,0)$, $Q(0,0,1)$, $R(1,0)$, $R(0,1)$, $S(1,0)$, $S(0,1)$, $S(0,0)$, $T(0)$, $T(1)$, then the sequence*

$$0 \to \mathcal{S} \xrightarrow{\iota} SK_*^{\mathbf{Z}_4} \xrightarrow{\phi} A(\mathbf{Z}_4) \to 0$$

is a short exact sequence and splits as ring, where ι is an inclusion homomorphism. Further, the above class supply a minimal set of generators.

Proof. By the above argument $\mathcal{S} = \operatorname{Ker}\phi$, so the exactness is trivial. The split map $\psi : A(\mathbf{Z}_4) \to SK_*^{\mathbf{Z}_4}$ is give by $\psi(1) = x_{0,0}$, $\psi(u) = x$, and $\psi(v) = x_0$. By Proposition 2.9 (1), (2), (4) and Lemma 3.5, we see that ψ is a ring homomorphism. □

Remark. The transfer homomorphism

Let $y = [\mathbf{Z}_2] \in SK_0^{\mathbf{Z}_2}$, $y_i = [\mathbf{R}P(\mathbf{R} \times \tilde{\mathbf{R}}^i)] \in SK_i^{\mathbf{Z}_2}$. Then $SK_*^{\mathbf{Z}_2}$ is a free SK_* module with basis $\{y\} \cup \{y_i \mid i \geq 0\}$ (cf. [6, 5.3.1]). As a ring structure of $SK_*^{\mathbf{Z}_2}$, we have the following.

Proposition 3.10 ([5], Theorem 3) *For any integers $m, n \geq 0$,*

(1) $y^2 = 2y$ (2) $yy_{2m+1} = 0$ (3) $yy_{2m} = \alpha^m y$

(4) $y_{2m} = y_2^m$ (5) $y_{2m+1}y_{2n} = y_{2m+2n+1} + \alpha^m y_{2n+1} - \alpha^{m+n} y_1$

(6) $y_{2m+1}y_{2n+1} = \alpha^{m+n}y_2 + \alpha^m y_2^{n+1} + \alpha^n y_2^{m+1} + y_2^{m+n+1} - 2\alpha^{m+n+1}y.$

Let $t : SK_*^{\mathbf{Z}_4} \longrightarrow SK_*^{\mathbf{Z}_2}$ be a transfer map (restriction map) and let $e : SK_*^{\mathbf{Z}_2} \longrightarrow SK_*^{\mathbf{Z}_4}$ be an extension map, that is $e([M]) = [\mathbf{Z}_4 \times_{\mathbf{Z}_2} M]$, then we have the following result.

Proposition 3.11

(1) $et(x) = 2x$, $et(x_i) = 2x_i$, $et(x_{2i,j}) = \alpha^i x_{2j}$ *and* $et(x_{2i+1,j}) = 0$.

(2) $te(y) = 2y$, $te(y_i) = 2y_i$.

Proof. By 5.3.7 in [6], $t(x) = 2y$, $t(x_i) = 2y_i$ and $t(x_{2i,j}) = \alpha^i y_{2j}$. On the other hand $\chi(M_{2i+1,j}) = 0$ and $\chi(M_{2i+1,j}^{\mathbf{Z}_2}) = 0$ so $x_{2i+1,j} = [M_{2i+1,j}] = 0$ in $SK_*^{\mathbf{Z}_2}$. This implies that $t(x_{2i+1,j}) = 0$. On the extension map, $e(y) = x$ and $e(y_i) = x_i$ are trivial. Therefore we have (1) and (2). □

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