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Singular integrals with rough kernels on product spaces

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Abstract. Suppose that $\Omega(x', y') \in L^1(S^{n-1} \times S^{m-1})$ is a homogeneous function of degree zero satisfying the mean zero property (1.1), and that $h(s, t)$ is a bounded function on $\mathbb{R}\times\mathbb{R}$. The singular integral operator Tf on the product space $\mathbb{R}^n \times \mathbb{R}^m$ $(n \geq 2, m \geq 2)$ is defined by

$$Tf(\xi, \eta) = \text{p.v.} \int_{\mathbb{R}^n\times\mathbb{R}^m} h(|x|, |y|)|x|^{-n}|y|^{-m}\Omega(x', y')f(\xi - x, \eta - y)dx\, dy.$$

We prove that the operator Tf is bounded in $L^p(\mathbb{R}^n \times \mathbb{R}^m)$, $p \in (1, \infty)$, provided that Ω is a function in certain block space $B_q^{0,1}(S^{n-1} \times S^{m-1})$ for some $q > 1$. The result answers a question posed in [JL].

We also study singular integral operators along certain surfaces.

Key words: singular integrals, rough kernel, block spaces, product spaces.

1. Introduction

Let $\mathbb{R}^N$ $(N = n$ or $m)$, $N \geq 2$, be the N-dimensional Euclidean space and S^{N-1} be the unit sphere in $\mathbb{R}^N$ equipped with normalized Lebesgue measure $d\sigma = d\sigma(\cdot)$. For nonzero points $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, we define $x' = x/|x|$ and $y' = y/|y|$. For $n \geq 2$, $m \geq 2$, let $\Omega(x', y') \in L^1(S^{n-1}\times S^{m-1})$ be a homogeneous function of degree zero, and satisfy

$$\int_{S^{n-1}} \Omega(x', y')d\sigma(x') = \int_{S^{m-1}} \Omega(x', y')d\sigma(y') = 0. \tag{1.1}$$

Let $h(s, t)$ be a locally integrable function on $\mathbb{R} \times \mathbb{R}$. The singular integral operator Tf on the product space $\mathbb{R}^n \times \mathbb{R}^m$ is defined by

$$(Tf)(x, y) = \text{p.v.} \int_{\mathbb{R}^n\times\mathbb{R}^m} K(\xi, \eta)f(x - \xi, y - \eta)d\xi\, d\eta \tag{1.2}$$

where $K(x, y) = h(|x|, |y|)\Omega(x', y')|x|^{-n}|y|^{-m}$ and f is a test function in $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$. If $h = 1$ and Ω satisfies some regularity conditions, then it is known that the operator T is bounded in $L^p(\mathbb{R}^n \times \mathbb{R}^m)$, $1 < p < \infty$ (see [Fe]). That the L^p-boundedness of T continues to hold under the weaker

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condition $\Omega \in L^q(S^{n-1} \times S^{m-1})$ was obtained by Duoandikoetxea in the following theorem.

Theorem A (see [Du]) *Suppose $n \geq 2$, $m \geq 2$, that Ω is a homogeneous function of degree zero satisfying* (1.1), *and that h satisfies*

$$\sup_{S>0,\, R>0} S^{-1}R^{-1} \int_0^R \int_0^S |h(s,t)|^2 ds\, dt < \infty. \tag{1.3}$$

Then the operator T is bounded in $L^p(\mathbb{R}^n \times \mathbb{R}^m)$, $1 < p < \infty$, provided $\Omega \in L^q(S^{n-1} \times S^{m-1})$ for some $q > 1$.

In order to weaken the condition $\Omega \in L^q$, Jiang and Lu introduced the block function spaces $B_q^{0,1}$ on $S^{n-1} \times S^{m-1}$ and proved the following L^2-boundedness theorem.

Theorem B (see [JL]) *Suppose $n \geq 2$, $m \geq 2$, and that Ω is a homogeneous function of degree zero and satisfies* (1.1). *If h is a bounded function, then the operator T is bounded in $L^2(\mathbb{R}^n \times \mathbb{R}^m)$ provided $\Omega \in B_q^{0,1}(S^{n-1} \times S^{m-1})$ for some $q > 1$, where $B_q^{0,1}$ are certain block spaces strictly containing the L^r spaces for all $r > 1$.*

It seems that the method in [JL] works only on the case $p = 2$, since it is mainly based on Plancherel's theorem. So Jiang and Lu asked the following question.

Question Under the hypothesis on Ω in Theorem B, is the operator T bounded on L^p for all $p \in (1, \infty)$?

The main purpose of this paper is to solve this problem. We have

Theorem 1 *Suppose that Ω is a homogeneous function of degree zero satisfying* (1.1), *and that h is a bounded function. Then T is bounded in $L^p(\mathbb{R}^n \times \mathbb{R}^m)$, provided that $\Omega \in B_q^{0,1}(S^{n-1} \times S^{m-1})$ for some $q > 1$.*

Remark. The conclusion in Theorem 1 remains valid when the condition $h \in L^\infty$ is replaced by the weaker condition (1.3) on h. The proof of this fact can be obtained by using a slight modification of our argument.

This paper is organized as follows. In the second section we will review the definition of the block spaces. After proving the L^p boundedness property for certain maximal functions in Section 3 and obtaining an L^2 estimate in Section 4, we will prove Theorem 1 in Section 5. Finally, in Section 6, we will discuss the L^p boundedness for the singular integral operators along surfaces. Throughout this paper, we always use letter C to denote positive constants that may vary at each occurrence but is independent of the essential variables.

2. Block spaces

First we review the definition of the block spaces.

A *q-block* on $S^{n-1} \times S^{m-1}$ is an L^q $(1 < q \leq \infty)$ function $b(\cdot,\cdot)$ that satisfies the following conditions (a) and (b).

(a) $\operatorname{supp}(b) \subseteq Q$ where Q is an interval on $S^{n-1} \times S^{m-1}$. Precisely,

$$\begin{aligned} &Q = Q_1(\xi',\alpha) \times Q_2(\eta',\beta), \quad \text{where} \\ &Q_1(\xi',\alpha) = \{x' \in S^{n-1} : |x'-\xi'| < \alpha \text{ for some } \xi' \in S^{n-1} \text{ and } \alpha \in (0,1]\}, \\ &Q_2(\eta',\beta) = \{y' \in S^{m-1} : |y'-\eta'| < \beta \text{ for some } \eta' \in S^{m-1} \text{ and } \beta \in (0,1]\}. \end{aligned}$$

(b) $\|b\|_q \leq |Q|^{(1/q-1)}$, where $|Q|$ is the volume of Q.

The block spaces $B_q^{0,1}$ on $S^{n-1} \times S^{m-1}$ are defined by

$$\begin{aligned} B_q^{0,1} = \Big\{\Omega \in L^1(S^{n-1} \times S^{m-1}) : \Omega(x',y') = \sum_\mu C_\mu b_\mu(x',y'), \\ \text{where each } b_\mu \text{ is a } q\text{-block supported in an interval } Q^\mu, \\ \text{and } M_q^{0,1}(\{C_\mu\}) < \infty\Big\} \end{aligned}$$

where

$$M_q^{0,1}(\{C_\mu\}) = \sum_\mu |C_\mu|\{1 + (\log^+ 1/|Q^\mu|)^2\}. \tag{2.1}$$

The "norm" $M_q^{0,1}(\Omega)$ of $\Omega \in B_q^{0,1}$ is defined by $M_q^{0,1}(\Omega) = \inf\{M_q^{0,1}(\{C_\mu\})\}$ where the infimum is taken over all q-block decompositions of Ω.

The block spaces were invented by M.H. Taibleson and G. Weiss in the study of the convergence of the Fourier series (see [TW]). Later on, these spaces and their applications were studied by many authors [Lo] [So] [MTW], et al. For further information, readers may see the book [LTW]. In particular, it was noted by Keitoku and Sato that $\bigcup_{r>1} L^r(S^{n-1}) \subseteq B_q^{0,1}(S^{n-1})$ for any fixed $q > 1$, and the inclusion is proper (see [KS]).

Suppose $n \geq 2$, $m \geq 2$ and that $b(\cdot,\cdot)$ is a q-block on $S^{n-1} \times S^{m-1}$ with

$\mathrm{supp}(b) \subseteq Q_1(\xi', \alpha) \times Q_2(\eta', B)$. We let

$$F_b(s,t) = (1-s^2)^{(n-3)/2}(1-t^2)^{(m-3)/2}\chi_{\{|s|<1,\,|t|<1\}}(s,t)\Theta(s,t) \tag{2.2}$$

where

(i) if $n > 2$ and $m > 2$,

$$\Theta(s,t) = \iint_{S^{n-2}\times S^{m-2}} |b(s,(1-s^2)^{1/2}\tilde{x}, t, (1-t^2)^{1/2}\tilde{y})| d\sigma(\tilde{x}) d\sigma(\tilde{y});$$

(ii) if $n = 2$ and $m > 2$ then $\Theta(s,t)$ is defined by

$$\int_{S^{m-2}} \Big(|b(s,(1-s^2)^{1/2}, t, (1-t^2)^{1/2}\tilde{y})| + |b(s,-(1-s^2)^{1/2}, t, (1-t^2)^{1/2}\tilde{y})| \Big) d\sigma(\tilde{y});$$

(iii) if $n > 2$ and $m = 2$ then $\Theta(s,t)$ is defined by

$$\int_{S^{m-1}} \Big(|b(s,(1-s^2)^{1/2}\tilde{x}, t, (1-t^2)^{1/2})| + |b(s,(1-s^2)^{1/2}\tilde{x}, t, -(1-t^2)^{1/2})| \Big) d\sigma(\tilde{x});$$

(iv) if $m = n = 2$, then $\Theta(s,t)$ is defined by

$$\begin{aligned} &|b(s,(1-s^2)^{1/2}, t, (1-t^2)^{1/2})| + |b(s,-(1-s^2)^{1/2}, t, (1-t^2)^{1/2})| \\ &\quad + |b(s,(1-s^2)^{1/2}, t, -(1-t^2)^{1/2})| \\ &\quad + |b(s,-(1-s^2)^{1/2}, t, -(1-t^2)^{1/2})|. \end{aligned}$$

Lemma 2.1 *For any q-block $b(\cdot,\cdot)$ supported in $Q_1(\xi',\alpha)\times Q_2(\eta',\beta)$, and for the function F_b defined in* (2.2)*, there exists a number $d \in (1,q]$ such that, up to a constant factor independent of $b(\cdot,\cdot)$, F_b is a d-block on $\mathbb{R}\times\mathbb{R}$. More precisely, F_b is a function on $\mathbb{R}\times\mathbb{R}$ which satisfies the following conditions* (2.3) *and* (2.4).

$$\mathrm{supp}(F_b) \subseteq I = I_1 \times I_2 \tag{2.3}$$

where $I_1 = (\xi_1' - 2r(\xi'), \xi_1' + 2r(\xi'))$, $I_2 = (\eta_1' - 2\rho(\eta'), \eta_1' + 2\rho(\eta'))$ *with* $r(\xi') = |\xi|^{-1}|B_\alpha \xi|$, $B_\alpha\xi = (\alpha^2\xi_1, \alpha\xi_2, \alpha\xi_3, \ldots, \alpha\xi_n)$ *and* $\rho(\eta') = |\eta|^{-1}|A_\beta\eta|$, $A_\beta\eta = (\beta^2\eta_1, \beta\eta_2, \ldots, \beta\eta_m)$.

$$\|F_b\|_d \leq C|I|^{1/d-1} \tag{2.4}$$

where C is a constant independent of $b(\cdot,\cdot)$ and ξ and η are any non-zero vectors such that $\xi' = \xi/|\xi|$, $\eta' = \eta/|\eta|$.

Proof. The proof of this lemma is essentially the same as the proof of the one parameter case in [FP]. For the sake of completeness and rigor, we present its proof for the main case $n > 2$ and $m > 2$. Also without loss of generality, we assume that $0 < \alpha \le 1/4$ and $0 < \beta \le 1/4$. Let $\xi' = (\xi_1', (1-\xi_1'^2)^{1/2}\widetilde{\zeta})$ for some $\widetilde{\zeta} \in S^{n-2}$ and let $\eta' = (\eta_1', (1-\eta_1'^2)\widetilde{\eta})$ for some $\widetilde{\eta} \in S^{m-2}$. If $F_b \ne 0$ then

$$(s, (1-s^2)^{1/2}\widetilde{x}) \in Q_1(\xi', \alpha), \quad (t, (1-t^2)^{1/2}\widetilde{y}) \in Q_2(\eta', \beta)$$

for some $\widetilde{x} \in S^{n-2}$ and $\widetilde{y} \in S^{m-2}$. Therefore we have

$$2\xi_1' s + 2(1-s^2)^{1/2}(1-\xi_1'^2)^{1/2}\langle\widetilde{\zeta}, \widetilde{x}\rangle \ge 2 - \alpha^2$$

for some $\widetilde{x} \in S^{n-2}$ and

$$2\eta_1' t + 2(1-t^2)^{1/2}(1-\eta_1'^2)^{1/2}\langle\widetilde{\eta}, \widetilde{y}\rangle \ge 2 - \beta^2$$

for some $\widetilde{y} \in S^{m-2}$.

Since $\langle\widetilde{\zeta}, \widetilde{x}\rangle \le 1$ and $\langle\widetilde{\eta}, \widetilde{y}\rangle \le 1$, we obtain

$$\begin{aligned}(s-\xi_1')^2 + |(1-s^2)^{1/2} - (1-\xi_1'^2)^{1/2}|^2 &\le \alpha^2,\\ (t-\eta_1')^2 + |(1-t^2)^{1/2} - (1-\eta_1'^2)^{1/2}|^2 &\le \beta^2.\end{aligned} \tag{2.5}$$

(2.5) implies that

$$|s-\xi_1'| \le \alpha, \qquad |t-\eta_1'| \le \beta; \tag{2.6}$$

$$\begin{aligned}&|(1-s^2)^{1/2} - (1-\xi_1'^2)^{1/2}| \le \alpha,\\ &|(1-t^2)^{1/2} - (1-\eta_1'^2)^{1/2}| \le \beta;\end{aligned} \tag{2.7}$$

and

$$|s-\xi_1'| \le 2|\xi|^{-1}|B_\alpha\xi|; \tag{2.8}$$

$$|t-\eta_1'| \le 2|\eta|^{-1}|A_\beta\eta|, \tag{2.9}$$

where $\xi' = \xi/|\xi|$ and $\eta' = \eta/|\eta|$. Inequalities (2.6) and (2.7) follow from (2.5) trivially. The proof of (2.9) is similar to those of (2.8). To see (2.8) we shall consider the following two cases.

Case a: $|\xi_1'| > 3/4$. Then by (2.6) and (2.7) we have

$$|s+\xi_1'| \geq 2|\xi_1'| - |s-\xi_1'| > 1$$

and

$$\begin{aligned}|s-\xi_1'| &\leq |s^2-\xi_1'^2| \\ &= |(1-s^2)^{1/2}-(1-\xi_1'^2)^{1/2}|\,|2(1-\xi_1'^2)^{1/2}+(1-s^2)^{1/2}-(1-\xi_1'^2)^{1/2}| \\ &\leq \alpha^2+2\alpha(1-\xi_1'^2)^{1/2} \leq 2|\xi|^{-1}|B_\alpha\xi|.\end{aligned}$$

Case b: $|\xi_1'| \leq 3/4$. Then $1/2 \leq (1-\xi_1'^2)^{1/2}$. By (2.6) we find

$$|s-\xi_1'| \leq \alpha < \alpha^2+2\alpha(1-\xi_1'^2)^{1/2} \leq 2|\xi|^{-1}|B_\alpha\xi|,$$

which proves (2.8).

By letting $r(\xi') = |\xi|^{-1}|B_\alpha\xi|$ and $\rho(\eta') = |\eta|^{-1}|A_\beta\eta|$, we see that (2.3) is satisfied.

It remains to verify (2.4). To this end, we consider the following three cases.

Case 1: $(1-\xi_1'^2)^{1/2} \leq 99\alpha$ and $(1-\eta_1'^2)^{1/2} \leq 99\beta$. By (2.2), (2.7) and Hölder's inequality, we find that $\|F_b\|_q$ is dominated by

$$\begin{aligned}&C\alpha^{(n-3)/q'}\beta^{(m-3)/q'}\Big\{\int_{-1}^{1}\int_{-1}^{1}(1-s^2)^{(n-3)/2}(1-t^2)^{(m-3)/2}|\Theta(s,t)|^q ds\,dt\Big\}^{1/q} \\ &\quad\leq C\alpha^{(n-3)/q'}\beta^{(m-3)/q'}\|b\|_{L^q(S^{n-1}\times S^{m-1})} \leq C\alpha^{-2/q'}\beta^{-2/q'} \\ &\quad= C|I|^{-1/q'}.\end{aligned}$$

Case 2: $(1-\xi_1'^2)^{1/2} > 99\alpha$ and $(1-\eta_1'^2) > 99\beta$. By (2.7) we find

$$\begin{aligned}(1-\xi_1'^2)^{1/2}/2 &\leq (1-s^2)^{1/2} \leq 2(1-\xi_1'^2)^{1/2}, \\ (1-\eta_1'^2)^{1/2}/2 &\leq (1-t^2)^{1/2} \leq 2(1-\eta_1'^2)^{1/2}.\end{aligned} \tag{2.10}$$

For $\varepsilon > 0$, $\delta > 0$, let

$$\begin{aligned}\Gamma(\varepsilon) &= \{x \in \mathbb{R}^{n-1} : 1-\varepsilon \leq \langle x,\tilde{\zeta}\rangle \leq 1\}, \\ \Gamma(\delta) &= \{y \in \mathbb{R}^{m-1} : 1-\delta \leq \langle y,\tilde{\eta}\rangle \leq 1\},\end{aligned}$$

and $\Gamma(\varepsilon,\delta) = \Gamma(\varepsilon)\times\Gamma(\delta)$. When ε and δ are small we have

$$\int_{S^{n-2}\cap\Gamma(\varepsilon)} d\sigma(\tilde{x}) \cong \varepsilon^{(n-2)/2}, \quad \int_{S^{m-2}\cap\Gamma(\delta)} d\sigma(\tilde{y}) \cong \delta^{(m-2)/2}.$$

By the support condition of $b(\cdot,\cdot)$, we find

$$\begin{aligned}
&\{(\widetilde{x},\widetilde{y}) \in S^{n-2}\times S^{m-2} : b(s,(1-s^2)^{1/2}\widetilde{x},t,(1-t^2)^{1/2}\widetilde{y}) \neq 0\}\\
&\quad \subseteq \{\widetilde{x}\in S^{n-2} : 2\xi_1's + 2(1-s^2)^{1/2}(1-\xi_1'^2)^{1/2}\langle\widetilde{\zeta},\widetilde{x}\rangle \geq 2-\alpha^2\}\\
&\qquad \times \{\widetilde{y}\in S^{m-2} : 2\eta_1't + 2(1-t^2)^{1/2}(1-\eta_1'^2)^{1/2}\langle\widetilde{\eta},\widetilde{y}\rangle \geq 2-\beta^2\}\\
&\quad \subseteq \{\widetilde{x}\in S^{n-2} : 1-(1-\xi_1'^2)^{-1/2}(1-s^2)^{-1/2}\alpha^2/2 \leq \langle\widetilde{\zeta},\widetilde{x}\rangle \leq 1\}\\
&\qquad \times \{\widetilde{y}\in S^{m-2} : 1-(1-\eta_1'^2)^{-1/2}(1-t^2)^{-1/2}\beta^2/2 \leq \langle\widetilde{\eta},\widetilde{y}\rangle \leq 1\}\\
&\quad = \Gamma(\varepsilon,\delta)
\end{aligned}$$

where $\varepsilon = \alpha^2/2(1-\xi_1'^2)^{-1/2}(1-s^2)^{-1/2}$ and $\delta = \beta^2/2(1-\eta_1'^2)^{1/2}(1-t^2)^{-1/2}$. Thus by Hölder's inequality and (2.10), we find that $\|F_b\|_q$ is dominated by

$$\begin{aligned}
&\leq C(1-\xi_1'^2)^{(n-3)/2q'}(1-\eta_1'^2)^{(m-3)/2q'}\\
&\qquad\qquad \left\{\int_{\{S^{n-2}\times S^{m-2}\}\cap\Gamma(\varepsilon,\delta)} d\sigma(\widetilde{x})d\sigma(\widetilde{y})\right\}^{1/q'}\|b\|_{L^q}\\
&\leq C\{(1-\xi_1'^2)^{-1/2}\alpha^{-1}(1-\eta_1'^2)^{-1/2}\beta^{-1}\}^{q'} = C|I|^{-1+1/q}.
\end{aligned}$$

Case 3:
(i) $(1-\xi_1'^2)^{1/2} \leq 99\alpha$ and $(1-\eta_1'^2)^{1/2} > 99\beta$, or
(ii) $(1-\xi_1'^2)^{1/2} > 99\alpha$ and $(1-\eta_1'^2)^{1/2} \leq 99\beta$.

The proofs of (i) and (ii) are exactly the same, we prove (i) only. By inspecting the proofs for Cases 1 and 2, we find

$$\|F_b\|_q \leq C\alpha^{(n-3)/q'}(1-\eta_1'^2)^{(m-3)/2q'}\|b\|_q\left\{\int_{S^{m-2}\cap\Gamma(\delta)} d\sigma(\widetilde{y})\right\}^{1/q'}$$

where $\delta = (1-\eta_1'^2)^{-1/2}(1-t^2)^{-1/2}\beta/2 \cong (1-\eta_1'^2)^{-1}\beta/2$. So we easily see

$$\|F_b\|_q \leq C\{\alpha^{-2}\beta^{-1}(1-\eta_1'^2)^{-1/2}\}^{1/q'} = C|I|^{-1+1/q}.$$

□

3. Certain maximal functions

Let the functions h and $\Omega = \sum C_\mu b_\mu$ be as in Theorem 1. Let $E_{k,j} = \{(x,y)\in\mathbb{R}^n\times\mathbb{R}^m : 2^k \leq |x| < 2^{k+1}, 2^j \leq |y| < 2^{j+1}\}$ and $\mathbb{Z}^2 = \mathbb{Z}\times\mathbb{Z}$. We define the following functions and operators.

$$B_\mu(f)(x,y) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^m} h(|\xi|,|\eta|)|\xi|^{-n}|\eta|^{-m}b_\mu(\xi',\eta')f(x-\xi,y-\eta)d\xi\,d\eta;$$

$$\widehat{\sigma}_{\Omega,k,j}(\xi,\eta)=\int_{E_{k,j}}h(|x|,|y|)|x|^{-n}|y|^{-m}\Omega(x',y')e^{-i\{\langle\xi,x\rangle+\langle\eta,y\rangle\}}dx\,dy;$$
$$|\sigma_{b_\mu,k,j}|\widehat{\ }(\xi,\eta)=\int_{E_{k,j}}|x|^{-n}|y|^{-m}|b_\mu(x',y')|e^{-i\{\langle x,\xi\rangle+\langle y,\eta\rangle\}}dx\,dy;$$
$$|\sigma_{\Omega,k,j}|\widehat{\ }(\xi,\eta)=\int_{E_{k,j}}|x|^{-n}|y|^{-m}|\Omega(x',y')|e^{-i\{\langle x,\xi\rangle+\langle y,\eta\rangle\}}dx\,dy;$$
$$\sigma^*_{b_\mu}f(x,y)=\sup_{(k,j)\in\mathbb{Z}^2}||\sigma_{b_\mu,k,j}|*f(x,y)|;$$
$$\sigma^*_{\Omega}f(x,y)=\sup_{(k,j)\in\mathbb{Z}^2}||\sigma_{\Omega,k,j}|*f(x,y)|.$$

Clearly, we have

$$T(f)=\sum_\mu C_\mu B_\mu(f)\quad\text{and}\quad T(f)=\sum_k\sum_j\sigma_{k,j}*f.$$

Also, we can write

$$B_\mu(f)=\sum_k\sum_j\sigma_{b_\mu,k,j}*f,$$

where

$$\widehat{\sigma}_{b_\mu,k,j}(\xi,\eta)=\int_{E_{k,j}}h(|x|,|y|)|x|^{-n}|y|^{-m}b_\mu(x',y')e^{-i\{\langle\xi,x\rangle+\langle\eta,y\rangle\}}dx\,dy.$$

We define $\|\sigma_{\Omega,k,j}\|=\int_{E_{k,j}}|x|^{-n}|y|^{-m}|\Omega(x',y')|dx\,dy$. It is easy to see that $\|\sigma_{\Omega,k,j}\|=\||\sigma_{\Omega,k,j}|\|\le C$ and $\|\sigma_{b_\mu,k,j}\|=\||\sigma_{b_\mu,k,j}|\|\le C$ uniformly for k, j and b_μ, and both $|\sigma_{\Omega,k,j}|$ and $|\sigma_{b_\mu,k,j}|$ are positive.

Next we will prove that the operators σ^*_b are bounded in $L^p(\mathbb{R}^n\times\mathbb{R}^m)$, $1<p<\infty$, and the bounds are independent of the blocks b. Suppose that $b(\cdot,\cdot)$ is a q-block supported in $Q_1(1,\alpha)\times Q_2(\tilde{1},\beta)$ where $1=(1,0,0,\dots,0)\in S^{n-1}$ and $\tilde{1}=(1,0,\dots,0)\in S^{m-1}$. Let $I_{k,j}=(2^k,2^{k+1})\times(2^j,2^{j+1})$. For any $\xi\ne0$ and $\eta\ne0$, we choose rotations O and $\widetilde{O}$ such that $O(\xi)=1|\xi|$ and $\widetilde{O}(\eta)=\tilde{1}|\eta|$. Let $x'=(u,x'_2,\dots,x'_n)$ and $y'=(v,y'_2,\dots,y'_m)$. Then by the method of rotation due to Calderón and Zygmund, we have

$$|\sigma_{b,k,j}|\widehat{\ }(\xi,\eta)=\int_{I_{k,j}}s^{-1}t^{-1}\int_{S^{n-1}\times S^{m-1}}|b(O^{-1}(x'),\widetilde{O}^{-1}(y'))|\times e^{-i\{s|\xi|\langle1,x'\rangle+t|\eta|\langle\tilde{1},\eta\rangle\}}d\sigma(x')d\sigma(y')ds\,dt$$

where O^{-1} and $\widetilde{O}^{-1}$ are the inverses of O and $\widetilde{O}$, respectively. We denote $b_R(x',y')=b(O^{-1}(x'),\widetilde{O}^{-1}(y'))$. Then it easy to see that b_R is a q-block supported in the interval $Q_1(\xi',\alpha)\times Q_2(\eta',\beta)$. Thus we have

$$|\sigma_{b,k,j}|\widehat{\ }(\xi,\eta) = \int_{I_{k,j}} s^{-1}t^{-1}\int_{\mathbb{R}\times\mathbb{R}} F_b(u,v) \times e^{-iu|\xi|s}e^{-iv|\eta|t}du\,dv\,ds\,dt \tag{3.1}$$

where F_b is the function defined in (2.2).

We will use Theorem 1 in [Du] to prove the L^p boundedness of σ_b^*. For this purpose, we also need to study some other maximal functions.

By Lemma 2.1 we know that the function F_b in (3.1) is a q-block on $\mathbb{R}\times\mathbb{R}$ with support in the interval $I_1\times I_2$, where $I_1=(\xi_1'-2r(\xi'),\xi_1'+2r(\xi'))$, $I_2=(\eta_1'-2\rho(\eta'),\eta_1'+2\rho(\eta'))$, $\xi'=(\xi_1',\ldots,\xi_n')$ and $\eta'=(\eta_1',\ldots,\eta_m')$. We define the functions $\lambda_{b,k,j}$, $\Lambda_{b,k,j}$ and $\Pi_{b,k,j}$ on $\mathbb{R}\times\mathbb{R}^m$, $\mathbb{R}^n\times\mathbb{R}$ and $\mathbb{R}\times\mathbb{R}$, respectively, by

$$\lambda_{b,k,j}*f(\theta,V) = \int_{E_{k,j}} |x|^{-n}|y|^{-m}b(O^{-1}x',y')f(\theta-|x|,V-y)dx\,dy,$$

$$\Lambda_{b,k,j}*f(U,\zeta) = \int_{E_{k,j}} |x|^{-n}|y|^{-m}b(x',\widetilde{O}^{-1}y')f(U-x,\zeta-|y|)dx\,dy,$$

$$\Pi_{b,k,j}*f(\theta,\zeta) = \int_{E_{k,j}} |x|^{-n}|y|^{-m}b(O^{-1}x',\widetilde{O}^{-1}y')f(\theta-|x|,\zeta-|y|)dx\,dy.$$

Then it is easy to see that

$$\widehat{\lambda}_{b,k,j}(\xi_1,\eta) = C\int_{I_{k,j}} s^{-1}t^{-1}\int_{\mathbb{R}}\int_{\mathbb{R}} F_b(u,v)e^{-it|\eta|v}e^{-is\xi_1}du\,dv\,ds\,dt,$$

$$\widehat{\Lambda}_{b,k,j}(\xi,\eta_1) = C\int_{I_{k,j}} s^{-1}t^{-1}\int_{\mathbb{R}}\int_{\mathbb{R}} F_b(u,v)e^{-is|\xi|u}e^{-it\eta_1}du\,dv\,ds\,dt,$$

$$\widehat{\Pi}_{b,k,j}(\xi_1,\eta_1) = \int_{I_{k,j}} s^{-1}t^{-1}e^{-it\eta_1}e^{-is\xi_1}ds\,dt\int_{\mathbb{R}}\int_{\mathbb{R}} F_b(u,v)du\,dv.$$

Now we define the maximal functions

$$\lambda_b^*f_1 = \sup_{(k,j)\in\mathbb{Z}^2}|\lambda_{b,k,j}*f_1|,\quad \Lambda_b^*f_2 = \sup_{(k,j)\in\mathbb{Z}^2}|\Lambda_{b,k,j}*f_2|,$$

and

$$\Pi_b^*f_3 = \sup_{k,j}|\Pi_{b,k,j}*f_3|.$$

Proposition 3.1 *For $1 < p < \infty$, λ_b^*, Λ_b^* and Π_b^* are bounded operators in $L^p(\mathbb{R} \times \mathbb{R}^m)$, $L^p(\mathbb{R}^n \times \mathbb{R})$ and $L^p(\mathbb{R} \times \mathbb{R})$, respectively. These bounds are independent of the blocks $b(\cdot, \cdot)$.*

Proof. By the definition of Π_b^*, for any positive function f_3 on $\mathbb{R} \times \mathbb{R}$,

$$\begin{aligned}\sup_{(k,j)\in\mathbb{Z}^2} &|\Pi_{b,k,j} * f_3(u,v)| \\ &\leq C \sup_{r>0,\, \rho>0} r^{-1}\rho^{-1} \int_0^r \int_0^\rho f_3(u-s, v-t)ds\, dt \\ &\leq CM(f_3)(u,v)\end{aligned}$$

where Mf is the Hardy-Littlewood maximal function on $\mathbb{R}\times\mathbb{R}$. This proves the L^p boundedness of Π_b^*.

Now we turn to prove the L^p boundedness of Λ_b^* and λ_b^*. Since the proofs for these two operators are exactly the same, we will prove it for Λ_b^* only. By the definition of $\Lambda_{b,k,j}$, it is easy to see that for non-negative functions f_2 on $\mathbb{R}^n \times \mathbb{R}$,

$$\begin{aligned}|\Lambda_{b,k,j} &* f_2(\xi, v)| \\ &\leq C \int_{2^k \leq |x| < 2^{k+1}} |x|^{-n} \int_{S^{m-1}} |b(x', y')| d\sigma(y') \\ &\qquad\qquad \times \left\{ \int_{2^j}^{2^{j+1}} f_2(\xi - x, v - t)t^{-1}dt \right\} dx.\end{aligned}$$

Since

$$\sup_{j\in\mathbb{Z}} \left| \int_{2^j}^{2^{j+1}} f_2(\xi - x, v - t)t^{-1}dt \right| \leq M_1 f_2(\xi - x, v)$$

with M_1 being the one-dimensional Hardy-Littlewood maximal function acting on the v-variable, we only need to prove that $\sup_k |\nu_{b,k} * f|$ is bounded in $L^p(\mathbb{R}^n)$ and the bound is independent of b, where

$$\widehat{\nu}_{b,k}(\xi) = C \int_{2^k \leq |x| < 2^{k+1}} |x|^{-n} \int_{S^{m-1}} |b(x', y')| d\sigma(y') e^{-i\langle \xi, x\rangle} dx.$$

But one easily verifies that

$$\widetilde{b}(x') = \int_{S^{m-1}} |b(x', y')| d\sigma(y')$$

is a q-block on S^{n-1}. This from the proof of (ii) in Lemma 3.1 of [FP] we obtain the L^p boundedness of $f \to \sup_{k\in\mathbb{Z}} |\nu_{b,k} * f|$. Proposition 3.1 is proved. $\square$

Proposition 3.2 *σ_b^* and σ_Ω^* are bounded in $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ and the bound of σ_b^* is independent of the block $b(\cdot,\cdot)$.*

Proof. Clearly we only need to prove the L^p boundedness of σ_b^*. Also without loss of generality, we assume that the support of b is contained in $Q_1(1,\alpha) \times Q_2(\widetilde{1},\beta)$. Let $\lambda_{b,k,j}$, $\Lambda_{b,k,j}$ and $\Pi_{b,k,j}$ be the corresponding functions as in Proposition 3.1. We first prove the following estimates.

$$\begin{aligned} &| \, |\sigma_{b,k,j}|\widehat{\ }(\xi,\eta) - \widehat{\lambda}_{b,k,j}(\xi_1,\eta) - \widehat{\Lambda}_{b,k,j}(\xi,\eta_1) + \widehat{\Pi}_{b,k,j}(\xi_1,\eta_1)| \\ &\quad \le C|2^k B_\alpha \xi| \, |2^j A_\beta \eta|, \end{aligned} \tag{3.2}$$

$$| \, |\sigma_{b,k,j}|\widehat{\ }(\xi,\eta) - \widehat{\lambda}_{b,k,j}(\xi_1,\eta)| \le C|2^k B_\alpha \xi| \, |2^j A_\beta \eta|^{-1/d'}, \tag{3.3}$$

$$| \, |\sigma_{b,k,j}|\widehat{\ }(\xi,\eta) - \widehat{\Lambda}_{b,k,j}(\xi,\eta_1)| \le C|2^k B_\alpha \xi|^{-1/d'} |2^j A_\beta \eta|, \tag{3.4}$$

$$| \, |\sigma_{b,k,j}|\widehat{\ }(\xi,\eta)| \le C|2^k B_\alpha \xi|^{-1/d'} |2^j A_\beta \eta|^{-1/d'} \tag{3.5}$$

for some $d > 1$, where d' is the conjugate index of d, and C is a constant independent of k, j and the blocks b.

To prove (3.2), by definitions we have that

$$\begin{aligned} &| \, |\sigma_{b,k,j}|\widehat{\ }(\xi,\eta) - \widehat{\lambda}_{b,k,j}(\xi_1,\eta) - \widehat{\Lambda}_{b,k,j}(\xi,\eta_1) + \widehat{\Pi}_{b,k,j}(\xi_1,\eta_1)| \\ &\le \Bigg| \int_{I_{k,j}} s^{-1}t^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}} F_b(u,v)\{e^{-iu|\xi|s} - e^{-i|\xi|\xi_1' s}\} \\ &\qquad\qquad \times \{e^{-iv|\eta|t} - e^{-i|\eta|\eta_1' t}\} du\, dv\, ds\, dt \Bigg|. \end{aligned} \tag{3.6}$$

By Lemma 2.1 we know that F_b is a q-block on $\mathbb{R}\times\mathbb{R}$ supported in the interval $I_1 \times I_2 = (\xi_1' - 2r(\xi'),\, \xi_1' + 2r(\xi')) \times (\eta_1' - 2\rho(\eta'), \eta_1' + 2\rho(\eta'))$. So (3.2) follows easily from (3.6).

The proofs of (3.3) and (3.4) are similar, we will prove (3.3) only. Let $\widehat{F}_b^{(i)}$ be the Fourier transform of $F_b(\cdot,\cdot)$ about the i-th variable, $i = 1,2$. Then

$$\begin{aligned} &| \, |\sigma_{b,k,j}|\widehat{\ }(\xi,\eta) - \widehat{\lambda}_{b,k,j}(\xi_1,\eta)| \\ &\le \int_{I_{k,j}} s^{-1}t^{-1} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} F_b(u,v) e^{-it|\eta|v} dv \right| |e^{-is|\xi|u} - e^{-i\xi_1 s}| du\, ds\, dt \end{aligned}$$

$$\leq C|2^k B_\alpha \xi| \int_{I_1} \int_{2^j|\eta|}^{2^{j+1}|\eta|} t^{-1} |\widehat{F}_b^{(2)}(u,t)| dt\, du.$$

Thus by Hölder's inequality and the Hausdorff-Young inequality, we have

$$\begin{aligned} & |\, |\sigma_{b,k,j}|\widehat{\ }(\xi,\eta) - \widehat{\lambda}_{b,k,j}(\xi_1,\eta)| \\ & \quad \leq C|2^k B_\alpha \xi| (2^j|\eta|)^{-1/d'} \int_{I_1} \|F_b(u,\cdot)\|_{L^d(\mathbb{R})} du \\ & \quad \leq C|2^k B_\alpha \xi| (2^j|\eta|)^{-1/d'} r(\xi')^{1/d'} \|F_b\|_{L^d(\mathbb{R}\times\mathbb{R})}. \end{aligned}$$

Now by (2.4), the term in the previous line is dominated by

$$C|2^k B_\alpha \xi| (2^j|\eta|\rho(\eta'))^{-1/d'} = C|2^k B_\alpha \xi|\, |2^j A_\beta \eta|^{-1/d'}.$$

(3.3) is proved.

To prove (3.5), by (3.1) we have

$$|\, |\sigma_{b,k,j}|\widehat{\ }(\xi,\eta)| \leq C \int_{2^k|\xi|}^{2^{k+1}} \int_{2^j|\eta|}^{2^{j+1}|\eta|} s^{-1} t^{-1} |\widehat{F}_b(s,t)| ds\, dt.$$

Thus (3.5) follows easily by Hölder's inequality, the Hausdorff-Young inequality and Lemma 2.1.

Let $\Phi \in \mathcal{S}(\mathbb{R}^n)$, $\Psi \in \mathcal{S}(\mathbb{R}^m)$ be positive radial functions such that $\widehat{\Phi}(0) = \widehat{\Psi}(0) = 1$ and define $\widehat{\Phi}_k(\xi) = \widehat{\Phi}(2^k B_\alpha \xi)$, $\widehat{\Psi}_j(\eta) = \widehat{\Psi}(2^j A_\beta \eta)$. Then, we define the measures $\Gamma_{b,k,j}$ by

$$\begin{aligned} \widehat{\Gamma}_{b,k,j}(\xi,\eta) \;=\; & |\sigma_{b,k,j}|\widehat{\ }(\xi,\eta) - \widehat{\Phi}_k(\xi)\widehat{\lambda}_{b,k,j}(\xi_1,\eta) - \widehat{\Psi}_j(\eta)\widehat{\Lambda}_{b,k,j}(\xi,\eta_1) \\ & + \widehat{\Phi}_k(\xi)\widehat{\Psi}_j(\eta)\widehat{\Pi}_{b,k,j}(\xi_1,\eta_1). \end{aligned}$$

Let $t^{\mp\alpha} = \inf(t^\alpha, t^{-\alpha})$. We can prove the following estimate for $\Gamma_{b,k,j}$.

$$|\widehat{\Gamma}_{b,k,j}(\xi,\eta)| \leq C|2^k B_\alpha \xi|^{\mp v} |2^j A_\beta \eta|^{\mp\varphi} \tag{3.7}$$

for some $v, \varphi > 0$, where the constant C is independent of k, j and the block b. In fact, by the definition of $\widehat{\Gamma}_{b,k,j}$

$$\begin{aligned} & |\widehat{\Gamma}_{b,k,j}(\xi,\eta)| \\ & \quad \leq |\{\widehat{\lambda}_{b,k,j}(\xi_1,\eta) - \widehat{\Pi}_{b,,k,j}(\xi_1,\eta_1)\}\{(1 - \widehat{\Phi}_k(\xi)\}| \\ & \qquad + |\{\widehat{\Lambda}_{b,k,j}(\xi,\eta_1) - \widehat{\Pi}_{b,k,j}(\xi_1,\eta_1)\}\{1 - \widehat{\Psi}_j(\eta)\}| \\ & \qquad + |\, |\sigma_{b,k,j}|\widehat{\ }(\xi,\eta) - \widehat{\lambda}_{b,k,j}(\xi_1,\eta) - \widehat{\Lambda}_{b,k,j}(\xi,\eta_1) + \widehat{\Pi}_{b,k,j}(\xi_1,\eta_1)| \\ & \qquad + |(1 - \widehat{\Phi}_k(\xi))(1 - \widehat{\Psi}_j(\eta))\widehat{\Pi}_{b,k,j}(\xi_1,\eta_1)|. \end{aligned}$$

By the definitions of $\widehat{\lambda}_{b,k,j}$, $\widehat{\Lambda}_{b,k,j}$ and $\widehat{\Pi}_{b,k,j}$, it is easy to see that

$$|\widehat{\lambda}_{b,k,j}(\xi_1,\eta)-\widehat{\Pi}_{b,k,j}(\xi_1,\eta_1)| \leq C|2^j A_\beta \eta|, \tag{3.8}$$

$$|\widehat{\Lambda}_{b,k,j}(\xi,\eta_1)-\widehat{\Pi}_{b,k,j}(\xi_1,\eta_1)| \leq C|2^k B_\alpha \xi| \tag{3.9}$$

where the constant C is independent of k, j and $b(\cdot,\cdot)$. Thus by (3.2) and the choice of Φ and Ψ we have

$$|\widehat{\Gamma}_{b,k,j}(\xi,\eta)| \leq C|2^k B_\alpha \xi|\,|2^j A_\beta \eta|. \tag{3.10}$$

Next, we have

$$\begin{aligned} |\widehat{\Gamma}_{b,k,j}(\xi,\eta)| \leq\ & |\,|\sigma_{b,k,j}|\widehat{\ }(\xi,\eta)-\widehat{\lambda}_{b,k,j}(\xi_1,\eta)| \\ & +|\widehat{\lambda}_{b,k,j}(\xi_1,\eta)\{1-\widehat{\Phi}_k(\xi)\}| \\ & +|\widehat{\Psi}_j(\eta)\{\widehat{\Lambda}_{b,k,j}(\xi,\eta_1)-\widehat{\Pi}_{b,k,j}(\xi_1,\eta_1)\}| \\ & +|\{1-\widehat{\Phi}_k(\xi)\}\widehat{\Psi}_j(\eta)\widehat{\Pi}_{b,k,j}(\xi_1,\eta_1)| \\ =\ & J_1+J_2+J_3+J_4. \end{aligned}$$

By (3.3), we know $J_1 \leq C|2^k B_\alpha \xi|\,|2^j A_\beta \eta|^{1/d'}$. By (3.9), $J_3 \leq C|2^j A_\beta \eta|^{-1}$ $|2^k B_\alpha \xi|$. Also it is easy to see, by the choice of Φ and Ψ, that $J_4 \leq C|2^k B_\alpha \xi|\,|2^j A_\beta \eta|^{-1}$. Following the proof of (3.3) we find

$$\begin{aligned} J_2 &\leq C|2^k B_\alpha \xi| \int_{\mathbb{R}} \int_{2^j}^{2^{j+1}} \left| \int_{\mathbb{R}} F_b(u,v) e^{-it|\eta| v} dv \right| t^{-1} dt\, du \\ &\leq C|2^k B_\alpha \xi| \int_{\mathbb{R}} \int_{2^j|\eta|}^{2^{j+1}|\eta|} t^{-1} |\widehat{F}_b^{(2)}(u,t)| dt\, du \\ &\leq |2^k B_\alpha \xi|\,|2^j A_\beta \eta|^{-1/d'}, \end{aligned}$$

which shows that

$$|\widehat{\Gamma}_{b,k,j}(\xi,\eta)| \leq C|2^k B_\alpha \xi|\,|2^j A_\beta \eta|^{-1/d'}. \tag{3.11}$$

Similarly, we can prove

$$|\widehat{\Gamma}_{b,k,j}(\xi,\eta)| \leq C|2^k B_\alpha \xi|^{-1/d'}|2^j A_\beta \eta|. \tag{3.12}$$

By the definition of $\widehat{\lambda}_{b,k,j}$ and $\widehat{\Lambda}_{b,k,j}$, it is easy to see

$$\begin{aligned} |\widehat{\lambda}_{b,k,j}(\xi_1,\eta)| &\leq C|2^j A_\beta \eta|^{-1/d'}, \\ |\widehat{\Lambda}_{b,k,j}(\xi,\eta_1)| &\leq C|2^k B_\alpha \xi|^{-1/d'} \end{aligned} \tag{3.13}$$

where C is independent of $b(\cdot,\cdot)$, k, j and (ξ,η).

Thus by the definition of Φ and Ψ, we have

$$|\widehat{\Gamma}_{b,k,j}(\xi,\eta)| \le C|2^k B_\alpha \xi|^{-1/d'}|2^j A_\beta \eta|^{-1/d'}. \tag{3.14}$$

Therefore, (3.7) follows from (3.10)–(3.12) and (3.14).

Now by a minor modification of the proof of Theorem 1 in [Du] and Proposition 3.1, we obtain the L^p boundedness of σ_b^* and that the bound is independent of the block b. Proposition 3.2 is proved. □

We also need to study two more maximal functions. We define $A_{b,k,j}$ and $B_{b,k,j}$ by

$$A_{b,k,j} * f(x,y) = \int_{E_{k,j}} h(|\xi|,|\eta|)b(\xi',\eta')|\xi|^{-n}|\eta|^{-m} f(x-\xi,y)d\xi\, d\eta$$

$$B_{b,k,j} f(x,y) = \int_{E_{k,j}} h(|\xi|,|\eta|)b(\xi',\eta')|\xi|^{-n}|\eta|^{-m} f(x,y-\eta)d\xi\, d\eta.$$

It is easy to see that

$$\widehat{A}_{b,k,j}(\xi,\eta) = \int_{E_{k,j}} h(|x|,|y|)|x|^{-n}|y|^{-m} b(x',y')e^{-i\langle x,\xi\rangle}dx\,dy,$$

$$\widehat{B}_{b,k,j}(\xi,\eta) = \int_{E_{k,j}} h(|x|,|y|)|x|^{-n}|y|^{-m} b(x',y')e^{-i\langle y,\eta\rangle}dx\,dy.$$

Now we define the functions $\tau_{b,k,j}$ and $\Sigma_{b,k,j}$ by

$$\widehat{\tau}_{b,k,j}(\xi,\eta) = \widehat{\sigma}_{b,k,j}(\xi,\eta) - \widehat{A}_{b,k,j}(\xi,\eta)$$

and

$$\widehat{\Sigma}_{b,k,j}(\xi,\eta) = \widehat{\sigma}_{b,k,j}(\xi,\eta) - \widehat{B}_{b,k,j}(\xi,\eta).$$

Then for any non-negative function f

$$|A_{b,k,j} * f(U,V)| \le C|D_{b,k,j} * f(U,V)|$$

$$|B_{b,k,j} * f(U,V)| \le |G_{b,k,j} * f(U,V)|$$

where both $D_{b,k,j}$ and $G_{b,k,j}$ are positive and

$$\widehat{D}_{b,k,j}(\xi,\eta) = \int_{E_{k,j}} |x|^{-n}|y|^{-m}|b(x',y')|e^{-i\langle x,\xi\rangle}dx\,dy$$

$$\widehat{G}_{b,k,j}(\xi,\eta) = \int_{E_{k,j}} |x|^{-n}|y|^{-m}|b(x',y')|e^{-i\langle y,\eta\rangle}dx\,dy.$$

Proposition 3.3 *Let*

$$G_b^* f = \sup_{(k,j)\in\mathbb{Z}^2} |G_{b,k,j} * f|, \quad D_b^* f = \sup_{(k,j)\in\mathbb{Z}^2} |D_{b,k,j} * f|.$$

Then both G_b^* and D_b^* are L^p bounded.

Proof. Since the proofs for these two operators are the same, we will prove G_b^* only. In fact, for a non-negative function f

$$\begin{aligned} G_{b,k,j} * f(U,V) &= \int_{E_{k,j}} |x|^{-n}|y|^{-m}|b(x',y')|f(U,V-y)dx\,dy \\ &\le C\int_{2^j\le|y|\le 2^{j+1}} |y|^{-m}\widetilde{b}(y')f(U,V-y)dy \end{aligned}$$

where

$$\widetilde{b}(y') = \int_{S^{n-1}} |b(x',y')|d\sigma(x').$$

Since $\widetilde{b}$ is a q-block on S^{m-1}, the L^p boundedness of G_b^* can be found in [FP]. It is easy to see that $\tau_{b,k,j}$ and $\Sigma_{b,k,j}$ are bounded by positive measures. More precisely, for any non-negative function f

$$\begin{aligned} |\tau_{b,k,j} * f| &\le \{|\sigma_{b,k,j}| + D_{b,k,j}\} * f, \\ |\Sigma_{b,k,j} * f| &\le \{|\sigma_{b,k,j}| + G_{b,k,j}\} * f. \end{aligned} \tag{3.15}$$

Thus by Propositions 3.2 and 3.3, we have

$$\begin{aligned} &\Big\| \sup_{(k,j)\in\mathbb{Z}^2} |\tau_{b,k,j} * f| \Big\|_p \\ &\quad\le C\Big\| \sup_{(k,j)\in\mathbb{Z}^2} \{|\sigma_{b,k,j}| + D_{b,k,j}\} * f \Big\|_p \le C\|f\|_p, \\ &\Big\| \sup_{(k,j)\in\mathbb{Z}^2} |\Sigma_{b,k,j} * f| \Big\|_p \\ &\quad\le C\Big\| \sup_{(k,j)\in\mathbb{Z}^2} \{|\sigma_{b,k,j}| + G_{b,k,j}\} * f \Big\|_p \le C\|f\|_p \end{aligned} \tag{3.16}$$

where C is independent of the block $b(\cdot,\cdot)$. □

Now, we can obtain the following lemma.

Lemma 3.4 (see page 189 in [Du]) *Let $\mathcal{T}_{k,j}$ be one of the operators $\sigma_{\Omega,k,j}$,*

$\sigma_{b,k,j}$, $\Sigma_{b,k,j}$ and $\tau_{b,k,j}$. For arbitrary functions $g_{k,j}$,

$$\left\|\left(\sum_{k,j}|\mathcal{T}_{k,j}*g_{k,j}|^2\right)^{1/2}\right\|_p \le C\left\|\left(\sum_{k,j}|g_{k,j}|^2\right)^{1/2}\right\|_p \tag{3.17}$$

for any $p\in(1,\infty)$, where the constant C is independent of the block b.

Proof. Using Proposition 3.2, (3.15) and (3.16), the lemma is an easy corollary of Lemma 1 in [Du]. □

4. An L^2 estimate

The main purpose of this section is to obtain the following lemma.

Lemma 4.1 *Let $\Omega=\sum C_\mu b_\mu$ be a block function in Theorem 1, where each $b=b_\mu$ is a q-block with* supp$(b)\subseteq Q$*. Then,*

(i) $|\widehat{\sigma_{\Omega,k,j}}(\xi,\eta)|\le C|2^k\xi|\,|2^j\eta|$;

(ii) $|\widehat{\tau}_{b,k,j}(\xi,\eta)|\le C|2^k\xi|^{1/\log|Q|}|2^j\eta|$ *if* $|Q|<e^{q/1-q}$;

(iii) $|\widehat{\tau}_{b,k,j}(\xi,\eta)|\le C|2^k\xi|^{-1/q'}|2^j\eta|$ *if* $|Q|\ge e^{q/1-q}$;

(iv) $|\widehat{\Sigma}_{b,k,j}(\xi,\eta)|\le C|2^k\xi|\,|2^j\eta|^{1/\log|Q|}$ *if* $|Q|<e^{q/1-q}$;

(v) $|\widehat{\Sigma}_{b,k,j}(\xi,\eta)|\le C|2^k\xi|\,|2^j\eta|^{1/q'}$ *if* $|Q|\ge e^{q/1-q}$;

(vi) $|\widehat{\sigma}_{b,k,j}(\xi,\eta)|\le C\{|2^k\xi|\,|2^j\eta|\}^{1/\log|Q|}$ *if* $|Q|<e^{q/1-q}$;

(vii) $|\widehat{\sigma}_{b,k,j}(\xi,\eta)|\le C\{|2^k\xi|\,|2^j\eta|\}^{-1/q'}$ *if* $|Q|\ge e^{q/1-q}$

where C is a constant independent of $k,j\in\mathbb{Z}$, $(\xi,\eta)\in\mathbb{R}^{n+m}$ and the block $b(\cdot,\cdot)$.

For the sake of simplicity, we prove the case $n>2$ and $m>2$ only. The proof for other cases are similar, with only minor modifications.

By the mean zero property (1.1) of Ω, we have

$$\begin{aligned}|\widehat{\sigma_{\Omega,k,j}}(\xi,\eta)| &= \bigg|\int_{I_{k,j}}h(s,t)t^{-1}s^{-1}\int_{S^{n-1}\times S^{m-1}}\Omega(x',y')\\ &\qquad\times\{e^{-it\langle\eta,y'\rangle}-i\}\{e^{-is\langle\xi,x'\rangle}-1\}d\sigma(x')d\sigma(y')ds\,dt\bigg|\\ &\le C\|\Omega\|_{L^1(S^{n-1}\times S^{m-1})}|\xi|\,|\eta|\int_{I_{k,j}}|h(s,t)|ds\,dt.\end{aligned}$$

So we obtain (i).

We turn to prove (ii). Fixing any $\xi \neq 0$ and $\eta \neq 0$, by the rotation method, without loss of generality, we may write

$$\begin{aligned}|\widehat{\tau_{b,k,j}}(\xi,\eta)| &= \Big|\int_{I_{k,j}} h(s,t)s^{-1}t^{-1}\int_{S^{n-1}\times S^{m-1}} b(x',y')e^{-is|\xi|\langle 1,x'\rangle} \\ &\qquad\qquad \times \{e^{-it|\eta|\langle \tilde{1},y'\rangle}-1\}d\sigma(x')d\sigma(y')ds\,dt\Big| \\ &\leq C\int_{2^j}^{2^{j+1}} |\eta| \int_{S^{m-1}}\int_{2^k}^{2^{k+1}} s^{-1} \\ &\qquad\qquad \times \Big|\int_{S^{n-1}} b(x',y')e^{-is|\xi|u}d\sigma(x')\Big| d\sigma(\eta')ds\,dt.\end{aligned}$$

Thus $|\widehat{\tau}_{b,k,j}(\xi,\eta)|$ is dominated by

$$C|2^j\eta|\int_{S^{m-1}}\int_{2^k|\xi|}^{2^{k+1}|\xi|} s^{-1}\Big|\int_{\mathbb{R}}\Delta_{y'}(u)e^{-isu}du\Big|\,ds\,d\sigma(y')$$

where

$$\Delta_{y'}(u)=(1-u^2)^{(n-3)/2}\chi_{\{|u|<1\}}(u)\int_{S^{n-2}} b(u,(1-u^2)^{1/2}\tilde{x},y')d\sigma(\tilde{x}).$$

Therefore,

$$|\widehat{\tau}_{b,k,j}(\xi,\eta)| \leq C|2^j\eta|\int_{S^{m-1}}\int_{2^k|\xi|}^{2^{k+1}|\xi|} s^{-1}|\widehat{\Delta}_{y'}(s)|ds\,d\sigma(y').$$

Pick a number ω in the interval $(1,2)$ such that $\omega < q$. By Hölder's inequality we have

$$|\widehat{\tau}_{b,k,j}(\xi,\eta)| \leq C|2^j\eta|\int_{S^{m-1}}\left\{\int_{2^k|\xi|}^{2^{k+1}|\xi|} s^{-\omega}dt\right\}^{1/\omega}\|\widehat{\Delta}_{y'}\|_{L^{\omega'}}d\sigma(y')$$

Thus by the Hausdorff-Young inequality, we find that $|\widehat{\tau}_{b,k,j}(\xi,\eta)|$ is dominated by

$$\begin{aligned}&C|2^j\eta|(\omega-1)^{-1/\omega}(|2^k\xi|)^{1-\omega}-|2^{k+1}\xi|^{1-\omega})^{1/\omega}\int_{S^{m-1}}\|\Delta_{y'}\|_{\omega}d\sigma(y') \\ &\quad \leq C|2^j\eta|(\omega-1)^{-1/\omega}|2^k\xi|^{-1/\omega'}(1-2^{1-\omega})^{1/\omega}\int_{S^{m-1}}\|\Delta_{y'}\|_{\omega}d\sigma(y'). \qquad (4.1)\end{aligned}$$

By Hölder's inequality again, we have

$$\begin{aligned}\int_{S^{m-1}} \|\Delta_{y'}\|_{L^\omega(\mathbb{R})} d\sigma(y') &\le C\|b\|_{L^\omega(S^{n-1}\times S^{m-1})} \\ &\le C\|b\|_{L^q(S^{n-1}\times S^{m-1})}|Q|^{1/\omega-1/q} \\ &\le C|Q|^{-1/\omega'}. \end{aligned} \tag{4.2}$$

Now combining (4.1) and (4.2) and taking $\omega = \log|Q|/(1+\log|Q|)$, we easily obtain (ii). Switching the variables ξ and η in the proof of (ii), we obtain the estimate (iv). If $|Q| \ge e^{q/(1-q)}$, taking $\omega = q$ in the proofs of (4.1) and (4.2), then we obtain that

$$|\widehat{\tau}_{b,k,j}(\xi,\eta)| \le C|2^j\eta|\,|2^k\xi|^{-1/q'}|Q|^{-1/q'} \le C_q|2^j\eta|\,|2^k\xi|^{-1/q'}$$

where the constant C depends only on $q > 1$. Thus (iii) is proved. Similarly we can prove (v). Since the proofs of (vi) and (vii) are similar, we will prove (vi) only. By the method of rotation

$$|\widehat{\sigma}_{b,k,j}(\xi,\eta)| \le C\int_{2^k|\xi|}^{2^{k+1}|\xi|}\int_{2^j|\eta|}^{2^{j+1}|\eta|} s^{-1}t^{-1}|\widehat{F}_b(s,t)|ds\,dt.$$

Again we use Hölder's inequality and the Hausdorff-Young inequality to obtain

$$|\widehat{\sigma}_{b,k,j}(\xi,\eta)| \le C\left\{\int_{2^k|\xi|}^{2^{k+1}|\xi|}\int_{2^j|\eta|}^{2^{j+1}|\eta|} s^{-\omega}t^{-\omega}ds\,dt\right\}^{1/\omega}\|F_b\|_\omega.$$

Using the proof in (4.2), we obtain $\|F_b\|_\omega \le C|Q|^{-1/\omega'}$. Therefore

$$|\widehat{\sigma}_{b,k,j}(\xi,\eta)| \le C(\omega-1)^{-2/\omega}\omega'^{-2}|Q|^{-1/\omega'}|2^k\xi|^{-1/\omega'}|2^j\eta|^{-1/\omega'}.$$

Letting $\omega = \log|Q|/\{\log|Q|+1\}$, we obtain (vi).

5. Proof of Theorem 1.

Our proof is based on the method used in [Du]. For a given block function $\Omega = \sum c_\mu b_\mu$, by Lemma 4.1, without loss of generality, we assume that the supports Q_μ of b_μ are uniformly small such that

$$|Q_\mu| < e^{q/(1-q)} \quad \text{and} \quad \log(\log(1/|Q_\mu|)) \ge 1.$$

Take two radial Schwartz functions, $\Phi^1 \in \mathcal{S}(\mathbb{R}^n)$, $\Phi^2 \in \mathcal{S}(\mathbb{R}^m)$ such that

$$0 \le (\Phi^i)\hat{} \le 1, \quad i = 1,2; \quad \sum_k (\Phi^1)\hat{}(2^k s)^2 = \sum_j (\Phi^2)\hat{}(2^j t)^2 = 1,$$

$$\operatorname{supp}(\Phi^i)\hat{} \subseteq \{2^{-1} < |\xi_i| \le 2\}, \quad i = 1,2.$$

If Φ_k^1 and Φ_j^2 are defined by $(\Phi_k^1)\hat{}(\xi) = (\Phi^1)\hat{}(2^k\xi)$ and $(\Phi_j^2)\hat{}(\eta) = (\Phi^2)\hat{}(2^j\eta)$, then

$$Tf = \sum_{k,j}\sum_{\ell,\nu} \sigma_{k,j} * (\Phi_{k+\ell}^1 \otimes \Phi_{j+\nu}^2) * (\Phi_{k+\ell}^1 \otimes \Phi_{j+\nu}^2) * f = \sum_{\ell,\nu} T_{\ell,\nu} f.$$

Thus

$$\|Tf\|_p \le \sum_{\ell\ge 0}\sum_{\nu\ge 0} \|T_{\ell\nu} f\|_p + \sum_{\ell<0}\sum_{\nu\ge 0} \|T_{\ell,\nu} f\|_p + \sum_{\ell\ge 0}\sum_{\nu<0} \|T_{\ell,\nu} f\|_p + \sum_{\ell<0}\sum_{\nu<0} \|T_{\ell,\nu} f\|_p.$$

By Lemma 3.4, (i) in Lemma 4.1 and the proof of Theorem 2 in [Du], it is easy to see

$$\sum_{\ell\ge 0}\sum_{\nu\ge 0} \|T_{\ell,\nu} f\|_p \le C\|f\|_p. \tag{5.1}$$

For $\ell < 0$ and $\nu \ge 0$, by the cancellation condition of Ω and the definition of $\tau_{\Omega,k,j}$, we have

$$T_{\ell,\nu} f = \sum_{k,j} \tau_{\Omega,k,j} * (\Phi_{k+\ell}^1 \otimes \Phi_{j+\nu}^2) * (\Phi_{k+\ell}^1 \otimes \Phi_{j+\nu}^2) * f.$$

Thus

$$\sum_{\ell<0}\sum_{\nu\ge 0} \|T_{\ell,\nu} f\|_p \le \sum_{\ell<0}\sum_{\nu\ge 0}\sum_{\mu} |C_\mu| \|I_{b_\mu,\ell,\nu} f\|_p$$

where

$$I_{b_\mu,\ell,\nu} f = \sum_{k,,j} \tau_{b_\mu,k,j} * (\Phi_{k+\ell}^1 \otimes \Phi_{j+\nu}^2) * (\Phi_{k+\ell}^1 \otimes \Phi_{j+\nu}^2) * f$$

By Lemma 3.4 and the Littlewood-Paley theorem, one has

$$\|I_{b_\mu,\ell,\nu} f\|_{p_0} \le C\|f\|_{p_0} \quad \text{for any} \quad 1 < p_0 < \infty, \tag{5.2}$$

where C is independent of b_μ, ℓ, and ν. On the other hand, by Plancherel's theorem.

$$\|I_{b_\mu,\ell,\nu}f\|_2 \leq \sum_{k,j} \int_{\Delta_{k,j,\ell,\nu}} |\widehat{\tau}_{b_\mu,k,j}(\xi,\eta)|^2 |\widehat{f}(\xi,\eta)|^2 d\xi\, d\eta$$

where

$$\begin{aligned}\Delta_{k,j,\ell,\nu} = \{(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^m : 2^{-k-\ell-1} \leq |\xi| < 2^{-k-\ell+1}, \\ 2^{-j-\nu-1} \leq |\eta| < 2^{-j-\nu+1}\}.\end{aligned}$$

Thus by (iv) of Lemma 4.1, we know that if $(\xi,\eta) \in \Delta_{k,j,\ell,\nu}$ then

$$|\widehat{\tau}_{b_\mu,k,j}(\xi,\eta)| \leq C|2^k\xi|\,|2^j\eta|^{1/\log|Q_\mu|} \leq C2^{-\ell}2^{-\nu/\log|Q_\mu|}.$$

Therefore, it is easy to see

$$\|I_{b_\mu,\ell,\nu}\|_{L^2\to L^2} \leq C2^{-\ell}2^{-\nu/\log|Q_\mu|}. \tag{5.3}$$

We now use interpolation to obtain

$$\|I_{b_\mu,\ell,\nu}f\|_p \leq C2^{-\nu\theta/\log|Q_\mu|}2^{-\ell\theta}\|f\|_p \tag{5.4}$$

for some $\theta > 0$. This shows that

$$\begin{aligned}\sum_{\nu<0}\sum_{\ell\geq 0}\|T_{\ell,\nu}f\|_p &\leq C\sum_{\nu<0}\sum_{\ell\geq 0}\sum_{\mu}|C_\mu|2^{-\nu\theta/\log|Q_\mu|}2^{-\theta\ell}\|f\|_p \\ &\leq C\|f\|_p\sum_{\mu}|C_\mu|\log(1/|Q_\mu|). \qquad (5.5)\end{aligned}$$

Clearly, the constant C above is independent of the essential variables. Similarly, by (ii) in Lemma 4.1, we can prove

$$\sum_{\nu\geq 0}\sum_{\ell<0}\|T_{\ell,\nu}f\|_p \leq C\|f\|_p\sum_{\mu}|C_\mu|\log(1/|Q_\mu|). \tag{5.6}$$

Finally, using (vi) in Lemma 4.1 and the same argument in (5.5), we find

$$\sum_{\ell<0}\sum_{\nu<0}\|T_{\ell,\nu}f\|_p \leq C\|f\|_p\sum_{\mu}|C_\mu|(\log(1/|Q|))^2. \tag{5.7}$$

Now the theorem follows by (5.1), (5.5), (5.6) and (5.7).

6. Singular integrals along surfaces

Let $K(x,y)$ be the kernel as in (1.2) and let $\gamma(s,t)$ be a real valued function on $\mathbb{R}\times\mathbb{R}$. For $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$, $(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^m$ and $z \in \mathbb{R}$, we define the singular integral operator $\tau_\gamma f$ along the surface $\mathcal{L} = (\xi, \eta, \gamma(|\xi|, |\eta|))$ by

$$T_\gamma f(x,y,z) = \text{p.v.} \int_{\mathbb{R}^n\times\mathbb{R}^m} K(\xi,\eta) f(x-\xi, y-\eta, z-\gamma(|\xi|,|\eta|)) d\xi\, d\eta. \quad (6.1)$$

In the one parameter case, the L^p boundedness of such kind of operators $T_\gamma f(x,z)$ was studied by a number of authors ([Ch], [FLP], [KWWZ], et. al.). Our main purpose of this section is to study the L^p boundedness of $T_\gamma f(x,y,z)$.

For the functions $\widehat{\sigma}_{k,j}(\xi,\eta)$, $|\sigma_{b,k,j}|\widehat{\ }(\xi,\eta), \ldots$, in the Section 3, we define their associated functions along the surface $\mathcal{L}$ by adding a multiplier factor $e^{-i\zeta\gamma(|x|,|y|)}$ (or $e^{-i\zeta\gamma(s,t)}$ in the case of spherical coordinate) in their integrands and denote these new functions by $\widehat{\sigma}_{\gamma,k,j}(\xi,\eta,\zeta)$, $|\sigma_{\gamma,b,k,j}|\widehat{\ }(\xi,\eta,\zeta)$ and so on, where $\zeta \in \mathbb{R}$. More precisely, we define

$$\widehat{\sigma}_{\gamma,k,j}(\xi,\eta,\zeta) = \int_{E_{k,j}} h(|x|,|y|)|x|^{-n}|y|^{-m}\Omega(x',y')e^{-i\{\langle\xi,x\rangle+\langle\eta,y\rangle\}}e^{-i\zeta\gamma(|x|,|y|)}dx\, dy,$$

$$|\sigma_{\gamma,b,k,j}|\widehat{\ }(\xi,\eta,\zeta) = \int_{E_{k,j}} |x|^{-n}|y|^{-m}|b(x',y')|e^{-i\{\langle\xi,x\rangle+\langle\eta,y\rangle\}}e^{-i\zeta\gamma(|x|,|y|)}dx\, dy,$$

$$\widehat{\lambda}_{\gamma,b,k,j}(\xi,\eta,\zeta) = \int_{I_{k,j}} s^{-1}t^{-1}\int_{\mathbb{R}}\int_{\mathbb{R}} F_b(u,v)e^{-it|\eta|v}e^{-is\xi_1}e^{-i\zeta\gamma(s,t)}du\, dv\, ds\, dt$$

and so on, where $(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^m$ and $\zeta \in \mathbb{R}$. We have the following L^2 boundedness theorem.

Theorem 2 *For any real valued function $\gamma(s,t)$, there is a constant C independent of f and γ such that $\|T_\gamma f\|_{L^2(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R})} \le C\|f\|_{L^2(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R})}$.*

Proof. By inspecting the proof of Lemma 4.1, it is easy to see that the estimates in Lemma 4.1 also hold for the corresponding functions $\widehat{\sigma}_{\gamma,\Omega,k,j}(\xi,\eta,\zeta)$, $\widehat{\tau}_{\gamma,b,k,j}(\xi,\eta,\zeta)$, $\widehat{\Sigma}_{\gamma,b,k,j}(\xi,\eta,\zeta)$ and $\widehat{\sigma}_{\gamma,b,k,j}(\xi,\eta,\zeta)$ and the constant C in Lemma 4.1 is independent of γ and ζ. Thus the theorem follows easily by Plancherel's theorem and Lemma 4.1.

By inspecting the proof of Theorem 1, it is also easy to obtain the following L^p boundedness theorem. □

Theorem 3 *Suppose that Ω is a homogeneous function of degree zero satisfying* (1.1), *and h is a bounded function. Suppose also that for any $p \in (1, \infty)$*

$$\Big\| \sup_{(k,j)\in\mathbb{Z}^2} |\lambda_{\gamma,b,k,j} * f_1| \Big\|_{L^p(\mathbb{R}\times\mathbb{R}^m\times\mathbb{R})} \le C\|f_1\|_{L^p(\mathbb{R}\times\mathbb{R}^m\times\mathbb{R})}, \tag{6.2}$$

$$\Big\| \sup_{(k,j)\in\mathbb{Z}^2} |\Lambda_{\gamma,b,k,j} * f_2| \Big\|_{L^p(\mathbb{R}^n\times\mathbb{R}\times\mathbb{R})} \le C\|f_2\|_{L^p(\mathbb{R}^n\times\mathbb{R}\times\mathbb{R})}, \tag{6.3}$$

$$\Big\| \sup_{(k,j)\in\mathbb{Z}^2} |\Pi_{\gamma,b,k,j} * f_3| \Big\|_{L^p(\mathbb{R}\times\mathbb{R}\times\mathbb{R})} \le C\|f_3\|_{L^p(\mathbb{R}\times\mathbb{R}\times\mathbb{R})}, \tag{6.4}$$

$$\Big\| \sup_{(k,j)\in\mathbb{Z}^2} |G_{\gamma,b,k,j} * f| \Big\|_{L^p(\mathbb{R}^{n+m}\times\mathbb{R})} \le C\|f\|_{L^p(\mathbb{R}^{n+m}\times\mathbb{R})}, \tag{6.5}$$

$$\Big\| \sup_{(k,j)\in\mathbb{Z}^2} |D_{\gamma,b,k,j} * f| \Big\|_{L^p(\mathbb{R}^{n+m}\times\mathbb{R})} \le C\|f\|_{L^p(\mathbb{R}^{n+m}\times\mathbb{R})} \tag{6.6}$$

where C is a constant independent of the block function b.

Then for any $p \in (1, \infty)$, we have $\|T_\gamma f\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R})} \le C\|f\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R})}$ provided $\Omega \in B_q^{0,1}(S^{n-1}\times S^{m-1})$.

To prove the L^p boundedness property of the maximal operators in Theorem 3, we only need to study the following three lower dimensional maximal functions. Let u, v, $z \in \mathbb{R}$, we define

$$\begin{aligned} &M_\gamma h(u, v, z) \\ &\quad = \sup_{R>0,\, S>0} R^{-1}S^{-1} \int_{R/2}^{R}\int_{S/2}^{S} |h(u-s, v-t, z-\gamma(s,t)|\,ds\,dt, \\ &\mathcal{M}_\gamma g(u, z) \\ &\quad = \sup_{R>0,\, S>0} R^{-1}S^{-1} \int_{R/2}^{R}\int_{S/2}^{S} |g(u-s, z-\gamma(s,t)|\,ds\,dt, \\ &\mu_\gamma g(v, z) \\ &\quad = \sup_{R>0,\, S>0} R^{-1}S^{-1} \int_{R/2}^{R}\int_{S/2}^{S} |g(v-t, z-\gamma(s,t)|\,ds\,dt. \end{aligned}$$

Theorem 4 *Let Ω and h be the functions as in Theorem* 3. *Suppose that for any $p \in (1, \infty)$*

$$\|M_\gamma h\|_{L^p(\mathbb{R}^3)} \le C\|h\|_{L^p(\mathbb{R}^3)}, \tag{6.7}$$

$$\|\mathcal{M}_\gamma g\|_{L^p(\mathbb{R}^2)} \leq C\|g\|_{L^p(\mathbb{R}^2)}, \tag{6.8}$$

$$\|\mu_\gamma g\|_{L^p(\mathbb{R}^2)} \leq C\|g\|_{L^p(\mathbb{R}^2)}. \tag{6.9}$$

Then the operator $T_\gamma f$ is bounded in $L^p(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R})$ for any $p \in (1, \infty)$.

Proof. For $x, \xi \in \mathbb{R}^n$, $y, \eta \in \mathbb{R}^m$ and $u, v, z \in \mathbb{R}$, we define

$$M^{(1)}_{\xi,\gamma} f(x, v, z) = \sup_{R>0,\, S>0} R^{-1}S^{-1} \int_{R/2}^{R} \int_{S/2}^{S} |f(x - s\xi', v - t, z - \gamma(s,t))|ds\, dt,$$

$$M^{(2)}_{\eta,\gamma} f(u, y, z) = \sup_{R>0,\, S>0} R^{-1}S^{-1} \int_{R/2}^{R} \int_{S/2}^{S} |f(u - s, y - t\eta', z - \gamma(s,t))|ds\, dt,$$

$$\mathcal{M}_{\xi,\gamma} f(x, z) = \sup_{R>0,\, S>0} R^{-1}S^{-1} \int_{R/2}^{R} \int_{S/2}^{S} |f(x - s\xi', z - \gamma(s,t))|ds\, dt,$$

$$\mu_{\eta,\gamma} f(y, z) = \sup_{R>0,\, S>0} R^{-1}S^{-1} \int_{R/2}^{R} \int_{S/2}^{S} |f(y - t\eta', z - \gamma(s,t))|ds\, dt.$$

By the method of rotation and (6.7), it is easy to see that for any $p \in (1, \infty)$

$$\|M^{(1)}_{\xi,\gamma} f\|_{L^p(\mathbb{R}^n\times\mathbb{R}\times\mathbb{R})} \leq C\|f\|_{L^p(\mathbb{R}^n\times\mathbb{R}\times\mathbb{R})}, \tag{6.10}$$

$$\|M^{(2)}_{\eta,\gamma} f\|_{L^p(\mathbb{R}\times\mathbb{R}^m\times\mathbb{R})} \leq C\|f\|_{L^p(\mathbb{R}\times\mathbb{R}^m\times\mathbb{R})} \tag{6.11}$$

where the constant C is independent of $\xi' \in S^{n-1}$ and $\eta' \in S^{m-1}$. Similarly, using the rotation method we have, for any $p \in (1, \infty)$,

$$\|\mathcal{M}_{\xi,\gamma} f\|_{L^p(\mathbb{R}^n\times\mathbb{R})} \leq C\|f\|_{L^p(\mathbb{R}^n\times\mathbb{R})} \tag{6.12}$$

by (6.8) and

$$\|\mu_{\eta,\gamma} f\|_{L^p(\mathbb{R}^m\times\mathbb{R})} \leq C\|f\|_{L^p(\mathbb{R}^m\times\mathbb{R})} \tag{6.13}$$

by (6.9), where the constant C is independent of the unit vectors ξ' and η'. Thus to prove the theorem, it suffices to show that the inequalities in (6.10) to (6.13) imply all the inequalities in (6.2) to (6.6). By the definition of $\widehat{\Pi}_{\gamma,b,k,j}$, it is easy to see

$$\sup_{(k,j)\in\mathbb{Z}^2} |\Pi_{\gamma,b,k,j} * f_3(u,v,z)| \leq CM_\gamma f_3(u,v,z).$$

This proves (6.4). Next by the definition of $\widehat{\Lambda}_{\gamma,b,k,j}$,

$$\begin{aligned}
&|\Lambda_{\gamma,b,k,j} * f_2(x,v,z)| \\
&\leq C\int_{S^{n-1}} \widetilde{b}(\xi')\Big|\int_{2^k}^{2^{k+1}}\int_{2^j}^{2^{j+1}} s^{-1}t^{-1}f_2(x-t\xi', v-t, z-\gamma(s,t))ds\,dt\Big|\sigma(\xi') \\
&\leq C\int_{S^{n-1}} \widetilde{b}(\xi')M^{(1)}_{\xi,\gamma}f_2(x,v,z)d\sigma(\xi')
\end{aligned}$$

where

$$\widetilde{b}(\xi') = \int_{S^{m-1}} |b(\xi',\eta')|d\sigma(\eta')$$

is a q-block on S^{n-1}. Thus for any $p \in (1,\infty)$, by Hölder's inequality we have

$$\begin{aligned}
&\sup_{(k,j)\in\mathbb{Z}^2} |\Lambda_{\gamma,b,k,j} * f_2(x,v,z)| \\
&\qquad\qquad \leq C\left\{\int_{S^{n-1}} \widetilde{b}(\xi')(M^{(1)}_{\xi,\gamma}f_2(x,v,z))^p d\sigma(\xi')\right\}^{1/p}.
\end{aligned}$$

Thus (6.3) follows easily from (6.10).

Using the exactly same argument, we can prove (6.2) by the inequality in (6.11).

By the definition of $\widehat{D}_{\gamma,b,k,j}$, it is easy to see that

$$\begin{aligned}
&|D_{\gamma,b,k,j} * f(x,y,z)| \\
&\leq C\int_{S^{n-1}} \widetilde{b}(\xi')\int_{2^k}^{2^{k+1}}\int_{2^j}^{2^{j+1}} |f(x-t\xi', y, z-\gamma(s,t))|s^{-1}t^{-1}ds\,dt\,d\sigma(\xi')
\end{aligned}$$

where $\widetilde{b}$ is a q-block on S^{n-1}. Thus

$$\begin{aligned}
&\left\{\sup_{(k,j)\in\mathbb{Z}^2} |D_{\gamma,b,k,j} * f(x,y,z)|\right\}^p \\
&\leq C\int_{S^{n-1}} \widetilde{b}(\xi')\Big(\sup_{(k,j)\in\mathbb{Z}^2}\int_{2^k}^{2^{k+1}}\int_{2^j}^{2^{j+1}} |f(x-t\xi', y, z-\gamma(s,t))|ds\,dt\Big)^p d\sigma(\xi').
\end{aligned}$$

Therefore, (6.6) follows easily from (6.12). Similarly, we can use (6.13) to prove (6.5). The theorem is proved. □

Example. Let $\gamma(s,t) = s^\alpha t^\beta$, the singular integral along the surface $(\xi, \eta, |\xi|^\alpha |\eta|^\beta)$ is defined by

$$T_\gamma f(x,y,z) = \text{p.v.} \int_{\mathbb{R}^n \times \mathbb{R}^m} K(\xi,\eta) f(x-\xi, y-\eta, z-|\xi|^\alpha |\eta|^\beta) d\xi\, d\eta,$$

where $K(\xi,\eta)$ is the kernel as in (1.2) and $\alpha > 0$, $\beta > 0$. Then by inspecting the proof of Corollary 3 in [Du], it is easy to see that the maximal functions M_γ, μ_γ and $\mathcal{M}_\gamma$ satisfy the inequalities (6.7)–(6.9). So by Theorem 4, T_γ is bounded in $L^p(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R})$ for all $p \in (1,\infty)$. We noticed that the one parameter case of this T_γ was studied in [Ch] under a stronger condition $\Omega \in L^q(S^{n-1})$.

It would be interesting to know more functions $\gamma(s,t)$ such that the maximal functions M_γ, $\mathcal{M}_\gamma$ and μ_γ are bounded in L^p. This question will be studied in a forthcoming paper.

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