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The Connes spectrum for actions of compact Kac algebras and factoriality of their crossed products

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Abstract. In this paper, we introduce the Connes spectrum for actions of compact Kac algebras on von Neumann algebras. Among other things, it is shown that the crossed product by an action of a compact Kac algebra is a factor if and only if the action is centrally ergodic and has full Connes spectrum.

Key words: von Neumann algebras, Kac algebras, actions, Connes spectrum, crossed products.

Introduction

It is widely acknowledged that the Arveson-Connes theory of the spectrum for actions of locally compact abelian groups on von Neumann algebras was highly successful and was a principal tool for the structure analysis of factors of type III (see [C], [CT]). It is no doubt that to look at the (Connes) spectrum is always very essential to have a deep understanding of such actions. Later, this theory was effectively extended by Olesen-Pedersen (see [Ped]) (also by Kishimoto [Ki]) to the case of abelian actions on C^*-algebras in order to investigate the (ideal) structure (i.e., primeness or simplicity) of the crossed product algebras. Definitions in the case of a non-abelian (compact) group action were presented both in the C^* and W^*-situations [EvS], [K], [GLP]. It seems however that the definition employed in [GLP] is a "best" one in the C^*-case in the sense explained in the introduction of [GLP]. At the same time, this spectrum theory was generalized also to the case of group coactions on operator algebras in [K], [N] (see also [Q] for the discrete case). So one would naturally expect that there should be a unified approach to both situations. Our purpose of this article is to extend this generalization program as far as the case of an action of a compact Kac algebra on a von Neumann algebra. As noted in the introduction of [GLP], in a "good" definition of the spectrum, the kind of result one would expect to generalize in the W^*-case is the theorem of Connes and Takesaki in [CT,

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Corollary 3.4], which states that the crossed product by an abelian action is a factor if and only if the action is centrally ergodic and has full Connes spectrum. We shall undertake our program in this spirit.

The organization of the paper is as follows. In Section 1, we introduce notation used in the sections that follow. We also briefly recall fundamental facts on (compact) Kac algebras. In Section 2, we pursue an analogue of Landstad's argument in [L], algebras of spherical functions. The goal of this section is to show that the map Ψ_π defined there, which is seemingly just a linear map, is actually a $*$-isomorphism. This is a crucial step towards our goal. In Section 3, we introduce the Connes spectrum of an action of a compact Kac algebra. This definition is suggested by the one given in [GLP] in the C^*-case. The point is that what is important in dealing with non-abelian actions is to look at the eigenspaces associated with an action rather than to look at the spectral subspaces. With this definition, we prove our main theorem that the crossed product by a compact Kac algebra action is a factor if and only if the action is centrally ergodic and has full Connes spectrum, which does generalize the Connes-Takesaki's theorem mentioned above. In the course of a proof, we obtain also a generalization of the theorem of Paschke [P, Corollary 3.2]. In Section 4, we consider several examples of compact Kac algebra actions and discuss (compute) their (Connes) spectrum. In the discussion, we compare the existing definitions with ours.

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1. Notation and Fundamentals on compact Kac algebras

In this section, we introduce notation which will be necessary for our discussion that follows. We also briefly review basic results on (compact) Kac algebras and their representations. For the general theory of Kac algebras, we refer to [ES], the notation of which we mainly adopt as well.

From now on, all von Neumann algebras are assumed to have separable preduals.

A Kac algebra is a quadruple $\mathbb{K} = (\mathcal{M}, \Gamma, \kappa, \varphi)$ [ES, Definition 2.2.5] in which:

(Ki) $(\mathcal{M}, \Gamma, \kappa)$ is a co-involutive Hopf von Neumann algebra [ES, Definition 1.2.5];

(Kii) φ is a faithful normal semifinite weight on $\mathcal{M}$, called a Haar measure (weight) of $\mathbb{K}$;

(Kiii) $(id_{\mathcal{M}} \otimes \varphi)\Gamma(x) = \varphi(x) \cdot 1 \quad (x \in \mathcal{M}_+)$;

(Kiv) $(id_{\mathcal{M}} \otimes \varphi)((1 \otimes y^*)\Gamma(x)) = \kappa((id_{\mathcal{M}} \otimes \varphi)(\Gamma(y^*)(1 \otimes x))) \quad (x, y \in \mathfrak{N}_\varphi)$;

(Kv) $\kappa \circ \sigma_t^\varphi = \sigma_{-t}^\varphi \circ \kappa \ (t \in \mathbf{R})$.

We say that $\mathbb{K}$ is compact if $\varphi(1) < \infty$. In this case, it turns out that φ is a trace with $\varphi \circ \kappa = \varphi$. Whenever we deal with a compact Kac algebra, we always normalize a Haar measure: $\varphi(1) = 1$. Let us fix a Kac algebra $\mathbb{K} = (\mathcal{M}, \Gamma, \kappa, \varphi)$. We always think of $\mathcal{M}$ as represented on the Hilbert space $L^2(\varphi)$ obtained from φ. Let Λ_φ denote the canonical injection of $\mathcal{M}$ into $L^2(\varphi)$. Then the equation

$$W(\Lambda_\varphi(x) \otimes \Lambda_\varphi(y)) = \Lambda_{\varphi\otimes\varphi}(\Gamma(y)(x \otimes 1)) \qquad (x, y \in \mathfrak{N}_\varphi)$$

defines a unitary on $L^2(\varphi) \otimes L^2(\varphi)$, called the fundamental unitary of $\mathbb{K}$ [ES, Proposition 2.4.2], and denoted by $W(\mathbb{K})$ if an unnecessary confusion may occur. It implements $\Gamma : \Gamma(x) = W(1 \otimes x)W^* \ (x \in \mathcal{M})$.

The main feature of the theory is the construction of the dual Kac algebra $\hat{\mathbb{K}} = (\hat{\mathcal{M}}, \hat{\Gamma}, \hat{\kappa}, \hat{\varphi})$ [ES, Chap. 3]. The fundamental unitary $W(\hat{\mathbb{K}})$ of $\hat{\mathbb{K}}$ is $\Sigma\, W(\mathbb{K})^* \Sigma$ [ES, Theorem 3.7.3], where Σ in general stands for the flip (twist) operator: $\Sigma(\xi \otimes \eta) = \eta \otimes \xi$. There are other Kac algebras canonically attached to $\mathbb{K}$, such as $\mathbb{K}'$ = the commutant of $\mathbb{K}$, $\mathbb{K}^\sigma$ = the opposite of $\mathbb{K}$, etc. (see [ES]).

The predual $\mathcal{M}_*$ becomes an involutive Banach algebra. We shall be mainly concerned with (nondegenerate) representations of $\mathcal{M}_*$. By [ES, Theorem 3.1.4], any representation μ of $\mathcal{M}_*$ on a Hilbert space $\mathcal{H}_\mu$ admits a generator, i.e., there is a unitary V on $\mathcal{H}_\mu \otimes L^2(\varphi)$ such that $\mu(\omega) = (id \otimes \omega)(V)$. The representation λ that has $W(\hat{\mathbb{K}})$ as a generator is called the regular (Fourier) representation of $\mathbb{K}$. It generates the dual Kac algebra $\hat{\mathcal{M}}$. We denote by $\mathfrak{D}(\mathbb{K})$ the set of all unitary equivalence classes of irreducible representaions of $\mathcal{M}_*$, and call it the unitary dual of $\mathbb{K}$. In the following sections, we often fix a complete set $\mathrm{Irr}(\mathbb{K})$ of representatives of the unitary dual $\mathfrak{D}(\mathbb{K})$.

An action of $\mathbb{K}$ on a von Neumann algebra $\mathcal{A}$ is a unital injective $*$-homomorphism α from $\mathcal{A}$ into $\mathcal{A} \bar{\otimes} \mathcal{M}$ satisfying $(\alpha \otimes id_{\mathcal{M}}) \circ \alpha = (id_{\mathcal{A}} \otimes \Gamma) \circ \alpha$. The crossed product $\mathcal{A} \rtimes_\alpha \mathbb{K}$ of $\mathcal{A}$ by the action α is by definition the von Neumann algebra generated by $\alpha(\mathcal{A})$ and $\mathbf{C} \otimes \hat{\mathcal{M}}'$. Once α is given, we may associate a new action $\bar{\alpha}$ of $\mathbb{K}$ on $\mathcal{A}\bar{\otimes}\mathcal{L}(L^2(\varphi))$ (or, more generally, on $\mathcal{A}\bar{\otimes}\mathcal{B}$ with $\mathcal{B}$ another von Neumann algebra), defined by $\bar{\alpha} = (id_{\mathcal{A}} \otimes \sigma) \circ (\alpha \otimes id)$, where $\sigma = \operatorname{Ad}\Sigma$. We call $\bar{\alpha}$ the amplified action of α. The fixed-point algebra $\mathcal{A}^\alpha$ of α is defined to be the set $\{x \in \mathcal{A} : \alpha(x) = x \otimes 1\}$. For a subspace B of $\mathcal{A}$, we say that B is α-invariant if $\alpha(B) \subseteq B \otimes \mathcal{M}$.

In the remainder of this section, let $\mathbb{K} = (\mathcal{M}, \Gamma, \kappa, \varphi)$ be a compact Kac algebra with the normalized Haar measure φ. For each $\pi \in \operatorname{Irr}(\mathbb{K})$, take its generator $V(\pi) \in \mathcal{L}(\mathcal{H}_\pi) \otimes \mathcal{M}$. For representation theory of compact Kac algebras, refer to [ES, §6]. We write $V(\pi)$ in the form:

$$V(\pi) = \sum_{i,j=1}^{d(\pi)} e_{i,j} \otimes V(\pi)_{i,j},$$

where $d(\pi) = \dim \mathcal{H}_\pi$ and $\{e_{i,j}\}$ is a system of matrix units of $\mathcal{L}(\mathcal{H}_\pi)$. Then it is known that (1) the linear subspace generated by $\{V(\pi)_{i,j} : 1 \leq i, j \leq d(\pi), \pi \in \operatorname{Irr}(\mathbb{K})\}$ is σ-weakly dense in $\mathcal{M}$, and that (2) the system $\{\sqrt{d(\pi)}\,\Lambda_\varphi(V(\pi)_{i,j}) : 1 \leq i, j \leq d(\pi), \pi \in \operatorname{Irr}(\mathbb{K})\}$ is a complete orthonormal basis of $L^2(\varphi)$ – the Peter-Weyl theorem and the Schur's orthogonality relations. For $V(\pi)_{i,j}$, we have the following useful identity: $\Gamma(V(\pi)_{i,j}) = \sum_{k=1}^{d(\pi)} V(\pi)_{i,k} \otimes V(\pi)_{k,j}$. The element χ_π of $\mathcal{M}$ given by $d(\pi)\sum_{i=1}^{d(\pi)} V(\pi)_{i,i}$ is called the (normalized) character of π, which is independent of the choice of the matrix units $\{e_{i,j}\}$.

Let α be an action of $\mathbb{K}$ on a von Neumann algebra $\mathcal{A}$. It is well-known that $E_\alpha = (id \otimes \varphi) \circ \alpha : \mathcal{A} \to \mathcal{A}$ is a faithful normal conditional expectation from $\mathcal{A}$ onto the fixed-point algebra $\mathcal{A}^\alpha$. Take a faithful normal state ω_0 on $\mathcal{A}^\alpha$. Then set $\psi = \omega_0 \circ E_\alpha$. From now on, we shall always think of $\mathcal{A}$ as represented on $L^2(\psi) = L^2(\mathcal{A})$, the Hilbert space obtained from ψ by the GNS construction.

For each $\pi \in \operatorname{Irr}(\mathbb{K})$, we define the π-eigenspace $\mathcal{A}^\alpha(\pi)$ of α to be the set given by

$$\mathcal{A}^\alpha(\pi) = \{X \in \mathcal{A} \otimes \mathcal{L}(\mathcal{H}_\pi) : \bar{\alpha}(X) = X_{12}V(\pi)_{23}\},$$

where, in this case, $\bar{\alpha}$ is the amplified action $\bar{\alpha} = (id_{\mathcal{A}} \otimes \sigma) \circ (\alpha \otimes id_{\mathcal{L}(\mathcal{H}_\pi)})$. On the other hand, with $E_\pi = (id \otimes \chi_\pi^* \varphi) \circ \alpha$, we set $\mathcal{A}_\pi = E_\pi(\mathcal{A})$. This subspace is called the π-spectral subspace of α. Set $\mathcal{Q} = \mathcal{A} \otimes \mathcal{L}(\mathcal{H}_\pi)$. Other than the amplified action $\bar{\alpha}$ on $\mathcal{Q}$ as above, there is another action β_π of $\mathbb{K}$ on $\mathcal{Q}$ defined by $\beta_\pi = \operatorname{Ad} V(\pi)_{23} \circ \bar{\alpha}$. The fixed-point subalgebra $\mathcal{Q}^{\beta_\pi}$ of

$\mathcal{Q}$ by this action β_π shall play a vital role in the discussion of the following sections.

2. An analogue of an theorem of Landstad – Algebras of spherical functions

In the case of a compact group action on a C^*-algebra, it turned out in [L] that the idea (analogue) of algebras of spherical functions in representation theory was very useful in determining (post)liminality of crossed product algebras. We shall see in this section that this idea is equally effective also in the case of a compact Kac algebra action. In fact, Corollary 2.9 will be a key step towards our final goal.

Throughout the rest of this paper, we fix a Kac algebra $\mathbb{K} = (\mathcal{M}, \Gamma, \kappa, \varphi)$ and an action α of $\mathbb{K}$ on a von Neumann algebra $\mathcal{A}$.

We believe that the next result is widely known to specialists. The author, however, cannot find a literature that actually contains its proof in the case of a Kac algebra action. So we provide a proof below for readers' convenience.

Lemma 2.1 (c.f. [LPRS]) *Let $\mathbb{K}$ be a general (not necessarily compact) Kac algebra. Then the crossed product $\mathcal{A} \rtimes_\alpha \mathbb{K}$ is the σ-strong* closure of the linear span of the set $\{\alpha(a)(1 \otimes \lambda^\sigma(\omega)) : a \in \mathcal{A}, \omega \in \mathcal{M}_*\}$. Here λ^σ stands for the regular reprepsentation of the opposite Kac algebra $\mathbb{K}^\sigma$ that has $W(\hat{\mathbb{K}}')$ as its generator.*

Proof. Let $\widetilde{\mathcal{A}}_0$ be the linear span of the set $\{\alpha(a)(1 \otimes \lambda^\sigma(\omega)) : a \in \mathcal{A}, \omega \in \mathcal{M}_*\}$. By the definition of a crossed product, it suffices to show that the algebra $\operatorname{alg}\widetilde{\mathcal{A}}_0$ generated by $\widetilde{\mathcal{A}}_0$ is contained in the σ-strong* closure of $\widetilde{\mathcal{A}}_0$. And for this, it suffices to prove that every element of the form $(1 \otimes \lambda^\sigma(\omega))\alpha(a)$ $(a \in \mathcal{A}, \omega \in \mathcal{M}_*)$ lies in the σ-strong* closure of $\widetilde{\mathcal{A}}_0$. So, let $a \in \mathcal{A}$ and $\omega \in \mathcal{M}_*$. Then we have

$$\begin{aligned}
&(1 \otimes \lambda^\sigma(\omega))\alpha(a) \\
&\quad= (id \otimes \omega \circ \kappa \otimes id)(W(\mathbb{K}^\sigma)_{23})\alpha(a) \\
&\quad= (id \otimes \omega \circ \kappa \otimes id)(W(\mathbb{K}^\sigma)_{23}\alpha(a)_{13}) \\
&\quad= (id \otimes \omega \circ \kappa \otimes id)((id \otimes \sigma)((id \otimes \Gamma) \circ \alpha(a))W(\mathbb{K}^\sigma)_{23}) \\
&\quad= (id \otimes \omega \circ \kappa \otimes id)((id \otimes \sigma)((\alpha \otimes id) \circ \alpha(a))W(\mathbb{K}^\sigma)_{23}),
\end{aligned}$$

which is approximately in the sense of σ-strong* topology

$$\begin{aligned}\sum_i (id\otimes\omega\circ\kappa\otimes id)((id\otimes\sigma)(\alpha(a_i)\otimes x_i)W(\mathbb{K}^\sigma)_{23})\\ &= \sum_i (id\otimes id\otimes\omega\circ\kappa)((\alpha(a_i)\otimes x_i)(\Sigma W(\mathbb{K}^\sigma)\Sigma)_{23})\\ &= \sum_i \alpha(a_i)(id\otimes(\kappa(x_i)\omega)\circ\kappa\otimes id)(W(\mathbb{K}^\sigma)_{23})\\ &= \sum_i \alpha(a_i)(1\otimes\lambda^\sigma((\kappa(x_i)\omega))).\end{aligned}$$

This completes the proof. □

Throughout the remainder of this paper, we always assume that $\mathbb{K}$ is a compact Kac algebra. We shall freely use the notation introduced in Section 1. In particular, for each $\pi\in\mathrm{Irr}(\mathbb{K})$, we fix an orthonormal basis $\{\varepsilon_i^\pi\}_{i=1}^{d(\pi)}$ of the representation space $\mathcal{H}_\pi$ and the corresponding matrix units $\{e_{i,j}^\pi\}$ once and for all. Thus, for example, the generator $V(\pi)$ of π is expressed in the form

$$V(\pi) = \sum_{i,j}^{d(\pi)} e_{i,j}^\pi \otimes V(\pi)_{i,j}. \tag{2.2}$$

We will often drop the index π and simply write ε_i, $e_{i,j}$ for ε_i^π, $e_{i,j}^\pi$ if no confusion ocurrs.

Let π be in $\mathrm{Irr}(\mathbb{K})$. It is easily verified that the unitary $V(\pi)^*$ satisfies $(id\otimes\Gamma^\sigma)(V(\pi)^*) = V(\pi)_{12}^*V(\pi)_{13}^*$, where Γ^σ of course denotes the coproduct of the opposite Kac algebra $\mathbb{K}^\sigma$ of $\mathbb{K}$. So it defines a (nondegenerate) representation π^σ of $\mathbb{K}^\sigma$ on $\mathcal{H}_\pi$ with $V(\pi)^*$ as its generator. With this notation, following the idea of [L], we define a linear map Ψ_π from $\mathcal{Q}^{\beta_\pi}$ into $\mathcal{A}\bar{\otimes}\mathcal{L}(L^2(\varphi))$ by

$$\Psi_\pi(X) = d(\pi)(id\otimes id\otimes \mathrm{Tr})((\alpha\otimes id_{\mathcal{L}(\mathcal{H}_\pi)})(X)(1\otimes(\lambda^\sigma\times\pi^\sigma)(\varphi))) \quad (X\in\mathcal{Q}^{\beta_\pi}).$$

By Lemma 3.2.1 of [ES], we have $\Sigma(\lambda^\sigma\times\pi^\sigma)(\varphi)\Sigma = V(\pi)^*(1\otimes\lambda^\sigma(\varphi))V(\pi)$. From this and a simple calculation, it follows that the map Ψ_π can be transformed as follows:

$$\Psi_\pi(X) = d(\pi)(id\otimes\mathrm{Tr}\otimes id)(V(\pi)_{23}^*(X\otimes\lambda^\sigma(\varphi))V(\pi)_{23}), \tag{2.3}$$

where $\lambda^\sigma \times \pi^\sigma$ indicates the Kronecker product of λ^σ and π^σ (see [ES, Theorem 1.4.3]). We find from this that Ψ_π is $*$-preserving. Note that, since $(\lambda^\sigma \times \pi^\sigma)(\mathcal{M}_*) \subseteq \hat{\mathcal{M}}' \otimes \mathcal{L}(\mathcal{H}_\pi)$, the range of the map Ψ_π is contained in the crossed product $\mathcal{A} \rtimes_\alpha \mathbb{K}$. We want to identify the range of Ψ_π more precisely. To do so, we need some preparatory results.

For the next lemma, we fix the following notation. For a pair (ξ, η) of vectors in a Hilbert space, we let $t_{\xi,\eta}$ denote the operator of rank one defined by $t_{\xi,\eta}\zeta = (\zeta \mid \eta)\xi$.

Lemma 2.4 *Let π be in* $\mathrm{Irr}(\mathbb{K})$. *We have*

$$\sum_{j=1}^{d(\pi)} V(\pi)^*_{i,j}\lambda(\varphi)V(\pi)_{k,j} = \sum_{j=1}^{d(\pi)} t_{\Lambda_\varphi(V(\pi)^*_{i,j}),\,\Lambda_\varphi(V(\pi)^*_{k,j})} = \lambda(V(\pi)^*_{i,k}\varphi)$$

for any $i, k = 1, 2, \ldots, d(\pi)$.

Proof. The first identity is straightforward, since $\lambda(\varphi)\Lambda_\varphi(x) = \varphi(x)\Lambda_\varphi(1)$.

For the second identity, we first note that $V(\pi)^*_{i,k}\varphi = \omega_{\Lambda_\varphi(V(\pi)^*_{i,k}),\,\Lambda_\varphi(1)}$. Hence, by Lemma 6.1.1 and 6.2.3 of [ES], one has

$$(\lambda(V(\pi)^*_{i,k}\varphi)\Lambda_\varphi(x) \mid \Lambda_\varphi(y)) = (I\Lambda_\varphi(V(\pi)^*_{i,k}) \mid \Lambda_\varphi(y) \otimes \hat{J}\Lambda_\varphi(x)),$$

where I stands for the isometry defined in Lemma 6.2.3 of [ES]. By the definition of I, for any $x, y \in \mathcal{M}$, we have

$$\begin{aligned}
&(\lambda(V(\pi)^*_{i,k}\varphi)\Lambda_\varphi(x) \mid \Lambda_\varphi(y)) \\
&= (\Lambda_{\varphi\otimes\varphi}(\Gamma(V(\pi)^*_{i,k})) \mid \Lambda_\varphi(y) \otimes \Lambda_\varphi(\kappa(x)^*)) \\
&= \sum_{j=1}^{d(\pi)} (\Lambda_\varphi(V(\pi)^*_{i,j}) \otimes \Lambda_\varphi(V(\pi)^*_{j,k})) \mid \Lambda_\varphi(y) \otimes \Lambda_\varphi(\kappa(x)^*)) \\
&= \sum_{j=1}^{d(\pi)} (\Lambda_\varphi(x) \mid \Lambda_\varphi(V(\pi)^*_{k,j}))\,(\Lambda_\varphi(V(\pi)^*_{i,j}) \mid \Lambda_\varphi(y)) \\
&= \sum_{j=1}^{d(\pi)} (t_{\Lambda_\varphi(V(\pi)^*_{i,j}),\,\Lambda_\varphi(V(\pi)^*_{k,j})}\Lambda_\varphi(x) \mid \Lambda_\varphi(y)).
\end{aligned}$$

This completes the proof. $\square$

Corollary 2.5 *We have*

$$\lambda^\sigma(V(\pi)^*_{i,j}\varphi) = \sum_{k=1}^{d(\pi)} t_{\Lambda_\varphi(V(\pi)^*_{k,j}),\,\Lambda_\varphi(V(\pi)^*_{k,i})}$$

for any $i, j = 1, 2, \ldots, d(\pi)$. *Hence, if we set* $E^\pi_{i,j} = d(\pi)\lambda^\sigma(V(\pi)^*_{j,i}\varphi)$, *then* $\{E^\pi_{i,j}\}$ *forms a set of* $d(\pi) \times d(\pi)$ *matrix units with* $\sum_{i=1}^{d(\pi)} E^\pi_{i,i} = \lambda^\sigma(\chi^*_\pi\varphi)$.

Proof. The first identity follows from the fact that $\lambda^\sigma(\omega) = \hat{J}\lambda(\omega)^*\hat{J}$ for any $\omega \in \mathcal{M}_*$, where $\hat{J}$ is the modular conjugation of the Haar measure $\hat{\varphi}$ given by $\hat{J}\Lambda_\varphi(x) = \Lambda_\varphi(\kappa(x)^*)$. The second statement results immediately from the first one. □

Let $X \in \mathcal{Q}^{\beta_\pi}$. With (2.3) and the expression

$$X = \sum_{i,j=1}^{d(\pi)} X_{i,j} \otimes e^\pi_{i,j},$$

we find from a direct computation that

$$\Psi_\pi(X) = d(\pi) \sum_{i,j,k=1}^{d(\pi)} X_{i,j} \otimes V(\pi)^*_{i,k}\lambda^\sigma(\varphi)V(\pi)_{j,k}.$$

From this and Lemma 2.4, it follows that

$$\Psi_\pi(X) = d(\pi) \sum_{i.j}^{d(\pi)} X_{i,j} \otimes \lambda(V(\pi)^*_{i,j}\varphi). \tag{2.6}$$

Since $\{d(\pi)\lambda(V(\pi)^*_{i,j}\varphi)\}$ forms a set of $d(\pi) \times d(\pi)$ matrix units by Lemma 2.4, one can easily verify by (2.6) that Ψ_π is a homomorphism. Next, note that $\Psi_\pi(1) = d(\pi)(1 \otimes (id \otimes \mathrm{Tr}) \circ (\lambda^\sigma \times \pi^\sigma)(\varphi))$. Since $\lambda^\sigma \times \pi^\sigma$ has $W(\hat{\mathbb{K}}')_{13}V(\pi)^*_{23}$ as its generator, we have

$$\begin{aligned} &d(\pi)(id \otimes \mathrm{Tr}) \circ (\lambda^\sigma \times \pi^\sigma)(\varphi) \\ &\quad = d(\pi)(id \otimes \mathrm{Tr} \otimes \varphi)(W(\hat{\mathbb{K}}')_{13}V(\pi)^*_{23}) \\ &\quad = d(\pi) \sum_{i,j=1}^{d(\pi)} \mathrm{Tr}(e_{j,i})(id \otimes \varphi)(W(\hat{\mathbb{K}}')(1 \otimes V(\pi)^*_{i,j})) = \lambda^\sigma(\chi^*_\pi\varphi). \end{aligned}$$

Hence we obtain $\Psi_\pi(1) = 1 \otimes \lambda^\sigma(\chi^*_\pi\varphi)$. Finally, suppose that $\Psi_\pi(X) = 0$. Then, by (2.3), we have $(id \otimes \mathrm{Tr} \otimes \varphi)(V(\pi)^*_{23}(X^*X \otimes \lambda^\sigma(\varphi))V(\pi)_{23}) = 0$.

Since $V(\pi)(\mathrm{Tr}\otimes\varphi)V(\pi)^*$ is faithful, it follows that $X = 0$. As Ψ_π is certainly normal, we conclude that Ψ_π is a normal injective $*$-homomorphism from $\mathcal{Q}^{\beta_\pi}$ into the reduced von Neumann algebra $q_\pi(\mathcal{A} \rtimes_\alpha \mathbb{K})q_\pi$, where $q_\pi = 1 \otimes \lambda^\sigma(\chi_\pi^*\varphi)$. In particular, the range of Ψ_π is a von Neumann subalgebra of $q_\pi(\mathcal{A} \rtimes_\alpha \mathbb{K})q_\pi$.

We set $p_\pi = \lambda^\sigma(\chi_\pi^*\varphi)$. Inside the von Neumann algebra $q_\pi(\mathcal{A} \rtimes_\alpha \mathbb{K})q_\pi$, we have a factor $\mathbf{C} \otimes \hat{\mathcal{M}}'\mathbf{p}_\pi$ of type $I_{d(\pi)}$, which we denote by $\mathcal{F}_\pi$. By Corollary 2.5, we may take $\{1 \otimes E^\pi_{i,j}\}$ for a system of matrix units for $\mathcal{F}_\pi$. Let $\mathcal{F}^c_\pi$ be the relative commutant of $\mathcal{F}_\pi$ in $q_\pi(\mathcal{A} \rtimes_\alpha \mathbb{K})q_\pi$. Note that the range of Ψ_π is then contained in $\mathcal{F}^c_\pi$ from (2.6). Now the equation

$$F_\pi(T) = \frac{1}{d(\pi)} \sum_{i,j=1}^{d(\pi)} (1 \otimes E^\pi_{j,i})\, T\, (1 \otimes E^\pi_{i,j}) \quad (T \in q_\pi(\mathcal{A} \rtimes_\alpha \mathbb{K})q_\pi)$$

defines a faithful normal conditional expectation F_π from $q_\pi(\mathcal{A} \rtimes_\alpha \mathbb{K})q_\pi$ onto $\mathcal{F}^c_\pi$ (see [St, pp. 126–127] for example). Let $\widetilde{\mathcal{A}}_0$ be the strongly dense subspace of the crossed product $\mathcal{A} \rtimes_\alpha \mathbb{K}$ introduced in the proof of Lemma 1. Then, from normality of F_π, the set $F_\pi(q_\pi\widetilde{\mathcal{A}}_0 q_\pi)$ forms a σ-weakly dense subspace of $\mathcal{F}^c_\pi$.

Let T be in $q_\pi\widetilde{\mathcal{A}}_0 q_\pi$. By definition, T has the form $T = \sum_{i,j=1}^{d(\pi)} q_\pi\alpha(a_{i,j})$ $q_\pi(1 \otimes E^\pi_{i,j})$, where $a_{i,j} \in \mathcal{A}$. With this expression, we have

$$F_\pi(T) = d(\pi) \sum_{i,j,k=1}^{d(\pi)} (1 \otimes \lambda^\sigma(V(\pi)^*_{j,k}\varphi))\alpha(a_{i,j})(1 \otimes \lambda^\sigma(V(\pi)^*_{k,i}\varphi)).$$

To continue this calculation, we need the following lemma.

Lemma 2.7 *We have*

$$\sum_{k=1}^{d(\pi)} \lambda^\sigma(V(\pi)^*_{j,k}\varphi)\, z\, \lambda^\sigma(V(\pi)^*_{k,i}\varphi) = \sum_{m,n=1}^{d(\pi)} \varphi(V(\pi)_{n,j} z V(\pi)^*_{m,i})\lambda(V(\pi)^*_{n,m}\varphi)$$

for any $z \in \mathcal{M}$.

Proof. This follows from Corollary 2.5 by a direct computation. So it is left to readers. □

By Lemma 2.7, we have

$$F_\pi(T) = d(\pi) \sum_{m,n=1}^{d(\pi)} \Big(\sum_{i,j=1}^{d(\pi)} (id \otimes V(\pi)^*_{m,i}\varphi V(\pi)_{n,j}) \circ \alpha(a_{i,j}) \Big) \otimes \lambda(V(\pi)^*_{n,m}\varphi). \tag{2.8}$$

With $X_{n,m} = \sum_{i,j=1}^{d(\pi)} (id \otimes V(\pi)^*_{m,i}\varphi V(\pi)_{n,j}) \circ \alpha(a_{i,j})$, we define

$$X = \sum_{m,n=1}^{d(\pi)} X_{n,m} \otimes e^\pi_{n,m} \in \mathcal{A} \otimes \mathcal{L}(\mathcal{H}_\pi) = \mathcal{Q}.$$

We assert that X belongs to $\mathcal{Q}^{\beta_\pi}$. To prove this, let us note first that the identity $(\alpha \otimes id) \circ \alpha = (id \otimes \Gamma) \circ \alpha$ implies $\alpha(X_{n,m}) = \sum_{i,j=1}^{d(\pi)} (id \otimes (id \otimes V(\pi)^*_{m,i}\varphi V(\pi)_{n,j}) \circ \Gamma) \circ \alpha(a_{i,j})$. From this and (2.2), it follows that

$$\begin{aligned}\beta_\pi(X) &= V(\pi)_{23}\bar{\alpha}(X)V(\pi)^*_{23} \\ &= \sum_{i,j,k,\ell,m,n=1}^{d(\pi)} (id \otimes \sigma) \\ &\circ \Big((1 \otimes V(\pi)_{k,\ell}(id \otimes V(\pi)^*_{m,i}\varphi V(\pi)_{\ell,j}) \circ \Gamma)(\alpha(a_{i,j}))V(\pi)^*_{n,m} \otimes e^\pi_{k,n} \Big).\end{aligned}$$

Since

$$\begin{aligned}&\sum_{\ell,m=1}^{d(\pi)} V(\pi)_{k,\ell}(id \otimes V(\pi)^*_{m,i}\varphi V(\pi)_{\ell,j}) \circ \Gamma)(a \otimes x)V(\pi)^*_{n,m} \\ &= \sum_{\ell,m=1}^{d(\pi)} a \otimes V(\pi)_{k,\ell}(id \otimes V(\pi)^*_{m,i}\varphi V(\pi)_{\ell,j}) \circ \Gamma(x)V(\pi)^*_{n,m} \\ &= \sum_{\ell,m=1}^{d(\pi)} a \otimes (id \otimes \varphi)((V(\pi)_{k,\ell} \otimes V(\pi)_{\ell,j})\Gamma(x)(V(\pi)^*_{n,m} \otimes V(\pi)^*_{m,i})) \\ &= a \otimes (id \otimes \varphi) \circ \Gamma(V(\pi)_{k,j}xV(\pi)^*_{n,i}) \\ &= a \otimes \varphi(V(\pi)_{k,j}xV(\pi)^*_{n,i}) \cdot 1 \\ &= (id \otimes V(\pi)^*_{n,i}\varphi V(\pi)_{k,j} \otimes id)(a \otimes x \otimes 1)\end{aligned}$$

for any $a \in \mathcal{A}$ and $x \in \mathcal{M}$, we have

$$\begin{aligned}\beta_\pi(X) &= \sum_{i,j,k,n=1}^{d(\pi)} (id \otimes \sigma) \\ &\quad \circ \Big((id \otimes V(\pi)^*_{n,i}\varphi V(\pi)_{k,j} \otimes id)(\alpha(a_{i,j}) \otimes 1) \otimes e^\pi_{k,n}\Big) \\ &= \sum_{i,j,k,n=1}^{d(\pi)} (id \otimes V(\pi)^*_{n,i}\varphi V(\pi)_{k,j})(\alpha(a_{i,j})) \otimes e^\pi_{k,n} \otimes 1 = X_{12}\end{aligned}$$

Hence X belongs to $\mathcal{Q}^{\beta_\pi}$ as asserted. By comparing (2.6) with (2.8), we find that $F_\pi(T) = \Psi_\pi(X)$. This shows that the range of Ψ_π contains a σ-weakly dense subspace $F_\pi(q_\pi \widetilde{\mathcal{A}}_0 q_\pi)$ of $\mathcal{F}^c_\pi$. Since the ragne is contained in $\mathcal{F}^c_\pi$ as noted before, we conclude that $\Psi_\pi(\mathcal{Q}^{\beta_\pi})$ is precisely $\mathcal{F}^c_\pi$.

We summarize the results obtained in the above discussion in the next theorem.

Theorem 2.9 *For each π in* Irr($\mathbb{K}$), *the map Ψ_π defined in* (2.3) *is a $*$-isomorphism from the fixed-point algebra $\mathcal{Q}^{\beta_\pi} = (\mathcal{A} \otimes \mathcal{L}(\mathcal{H}_\pi))^{\beta_\pi}$ onto the relative commutant $\mathcal{F}^c_\pi$ of $\mathcal{F}_\pi$ in $q_\pi(\mathcal{A} \rtimes_\alpha \mathbb{K})q_\pi$.*

Corollary 2.10 *For each $\pi \in$* Irr($\mathbb{K}$), *the following are equivalent*:

(1) *$q_\pi(\mathcal{A} \rtimes_\alpha \mathbb{K})q_\pi$ is a factor*;

(2) *$\mathcal{F}^c_\pi$ is a factor*;

(3) *$\mathcal{Q}^{\beta_\pi} = (\mathcal{A} \otimes \mathcal{L}(\mathcal{H}_\pi))^{\beta_\pi}$ is a factor.*

Proof. Since $\mathcal{F}_\pi$ is a type $I_{d(\pi)}$ subfactor of $q_\pi(\mathcal{A} \rtimes_\alpha \mathbb{K})q_\pi$, we have a canonical tensor product deomposition $q_\pi(\mathcal{A} \rtimes_\alpha \mathbb{K})q_\pi \cong \mathcal{F}^c_\pi \otimes \mathcal{F}_\pi$ (see [St, pp. 126–127]). The assertion now follows immediately from Theorem 2.9. □

3. Connes spectrum and factoriality

In this section, we first introduce the (Arveson) spectrum and the Connes spectrum of an action of a compact Kac algebra. After we do that, we shall devote ourselves to proving the mian theorem of this paper that characterizes factoriality of the crossed product by a compact Kac algebra action, in terms of the Connes spectrum.

We will retain the notation established in the previous sections.

Before we give the next definition, note that X^*Y always belongs to

$\mathcal{Q}^{\beta_\pi}$ for any $X, Y \in \mathcal{A}^\alpha(\pi)$.

Definition 3.1 (see [GLP]) We denote by $\mathrm{Sp}(\alpha)$ the set of all $[\pi]$ ($\pi \in \mathrm{Irr}(\mathbb{K})$) in the unitary dual $\mathfrak{D}(\mathbb{K})$ of $\mathbb{K}$ such that $\mathcal{A}^\alpha(\pi)^*\mathcal{A}^\alpha(\pi)$ is σ-weakly dense in $\mathcal{Q}^{\beta_\pi} = (\mathcal{A} \otimes \mathcal{L}(\mathcal{H}_\pi))^{\beta_\pi}$. We call this set the (Arveson) spectrum of α.

In the next definition, note that, for each projection e in $\mathcal{A}^\alpha$, the map $\alpha^e : \mathcal{A}_e \to \mathcal{A}_e \bar{\otimes} \mathcal{M}$ defined by $\alpha^e(exe) = (e \otimes 1)\alpha(x)(e \otimes 1)$ is an action of $\mathbb{K}$ on the reduced von Neumann algebra $\mathcal{A}_e$.

Definition 3.2 (see [GLP]) We set $\Gamma(\alpha) = \bigcap\{\mathrm{Sp}(\alpha^e) : e$ is a non-zero projection in $\mathcal{A}^\alpha\}$. We call this set the Connes spectrum of α.

Lemma 3.3 *We have*

$$\Gamma(\alpha) = \bigcap\{\mathrm{Sp}(\alpha^e) : e \textit{ is a non-zero central projection in } \mathcal{A}^\alpha\}.$$

In particular, if $\mathcal{A}^\alpha$ is a factor, then $\Gamma(\alpha) = \mathrm{Sp}(\alpha)$.

Proof. It suffices to prove that $\mathrm{Sp}(\alpha^e) \supseteq \mathrm{Sp}(\alpha^{z(e)})$ for any non-zero projection e in $\mathcal{A}^\alpha$, where $z(e)$ is the central support of e in $\mathcal{A}^\alpha$. So let us take a non-zero projection e in $\mathcal{A}^\alpha$, and set $f = z(e)$. For an operator a in $\mathcal{A}$, we denote by $\tilde{a}$ the operator given by $\tilde{a} = a \otimes 1 \in \mathcal{Q}$. Before we proceed to a proof, note first that, since $\mathcal{A}^{\alpha^e}(\pi) = \tilde{e}\mathcal{A}^\alpha(\pi)\tilde{e}$, one always has $\mathcal{A}^{\alpha^e}(\pi)^*\mathcal{A}^{\alpha^e}(\pi) = \tilde{e}\mathcal{A}^\alpha(\pi)^*\tilde{e}\mathcal{A}^\alpha(\pi)\tilde{e}$. Notice also that $\mathcal{Q}^{\beta_\pi^e} = \tilde{e}\mathcal{Q}^{\beta_\pi}\tilde{e}$.

Now we suppose that $[\pi]$ belongs to $\mathrm{Sp}(\alpha^f)$. So we have

$$\overline{\tilde{f}\mathcal{A}^\alpha(\pi)^*\tilde{f}\mathcal{A}^\alpha(\pi)\tilde{f}}^{\sigma\text{-w}} = \tilde{f}\mathcal{Q}^{\beta_\pi}\tilde{f}.$$

Since $\tilde{e} \leq \tilde{f}$, one immediately sees that

$$\overline{\tilde{e}\mathcal{A}^\alpha(\pi)^*\tilde{f}\mathcal{A}^\alpha(\pi)\tilde{e}}^{\sigma\text{-w}} = \tilde{e}\mathcal{Q}^{\beta_\pi}\tilde{e}.$$

Hence it remains to show that

$$\overline{\mathcal{A}^\alpha(\pi)^*\tilde{f}\mathcal{A}^\alpha(\pi)}^{\sigma\text{-w}} = \overline{\mathcal{A}^\alpha(\pi)^*\tilde{e}\mathcal{A}^\alpha(\pi)}^{\sigma\text{-w}}.$$

For this end, set $\mathcal{F} = \mathcal{A}^\alpha(\pi)^*\tilde{f}\mathcal{A}^\alpha(\pi)$ and $\mathcal{E} = \mathcal{A}^\alpha(\pi)^*\tilde{e}\mathcal{A}^\alpha(\pi)$. Since $\mathcal{A}^\alpha(\pi)$ is a right $\mathcal{Q}^{\beta_\pi}$-module, both $\mathcal{E}$ and $\mathcal{F}$ are $*$-algebras, and $\mathcal{E}$ is a two-sided ideal of $\mathcal{F}$. Thus $\overline{\mathcal{E}}^{\sigma\text{-w}}$ and $\overline{\mathcal{F}}^{\sigma\text{-w}}$ are σ-weakly closed $*$-algebras. Moreover, since $\tilde{a}\,\mathcal{A}^\alpha(\pi) \subseteq \mathcal{A}^\alpha(\pi)$ for any $a \in \mathcal{A}^\alpha$ and $f\,L^2(\mathcal{A}) = [\mathcal{A}^\alpha\, e\, L^2(\mathcal{A})]$ (= the closed subspace generated by the set $\mathcal{A}^\alpha e L^2(\mathcal{A})$), we have

$$[\overline{\mathcal{E}}^{\sigma\text{-w}}(L^2(\mathcal{A}) \otimes \mathcal{H}_\pi)] = [\overline{\mathcal{F}}^{\sigma\text{-w}}(L^2(\mathcal{A}) \otimes \mathcal{H}_\pi)].$$

This means that the projection onto this closed subspace is the identity for both $\overline{\mathcal{E}}^{\sigma\text{-w}}$ and $\overline{\mathcal{F}}^{\sigma\text{-w}}$. Since $\overline{\mathcal{E}}^{\sigma\text{-w}}$ also is a two-sided ideal of $\overline{\mathcal{F}}^{\sigma\text{-w}}$, they must coincide with each other. This completes the proof. □

As in Section 2, we set $q_\pi = 1 \otimes \lambda^\sigma(\chi_\pi^* \varphi)$.

Lemma 3.4 *Let π be in* $\mathrm{Irr}(\mathbb{K})$. *Then we have $q_{\bar{\pi}} \alpha(a) q_\iota = \alpha(E_\pi(a)) q_\iota$ for all $a \in \mathcal{A}$. Here ι denotes the trivial representation.* (*So* $q_\iota = 1 \otimes \lambda^\sigma(\varphi)$).

Proof. Let $a \in \mathcal{A}$ and $\omega \in \mathcal{M}_*$. Then, by the proof of Lemma 2.1, $(1 \otimes \lambda^\sigma(\omega))\alpha(a)$ is approximately equal to $\sum_i \alpha(a_i)(1 \otimes \lambda^\sigma((\kappa(x_i)\omega)))$ in the sense of σ-strong* topology. But we have

$$\begin{aligned} &\sum_i \alpha(a_i)(1 \otimes \lambda^\sigma((\kappa(x_i)\omega)))(1 \otimes \lambda^\sigma(\varphi)) \\ &\qquad = \sum_i \omega(\kappa(x_i))\alpha(a_i)(1 \otimes \lambda^\sigma(\varphi)) \\ &\qquad = \alpha\Big((id \otimes \omega \circ \kappa)\Big(\sum_i a_i \otimes x_i\Big)\Big)(1 \otimes \lambda^\sigma(\varphi)). \end{aligned}$$

From this, it follows that

$$(1 \otimes \lambda^\sigma(\omega))\alpha(a)(1 \otimes \lambda^\sigma(\varphi)) = \alpha((id \otimes \omega \circ \kappa) \circ \alpha(a))(1 \otimes \lambda^\sigma(\varphi)). \tag{3.5}$$

Therefore, to obtain the assertion of this lemma, one has only to put $\omega = \chi_\pi \varphi$ in the above identity. □

Lemma 3.6 *For any $\pi \in$* $\mathrm{Irr}(\mathbb{K})$, *$\mathcal{A}_\pi \neq \{0\}$ if and only if $q_{\bar{\pi}}(\mathcal{A} \rtimes_\alpha \mathbb{K}) q_\iota \neq \{0\}$.*

Proof. Let π be in $\mathrm{Irr}(\mathbb{K})$.

Suppose first that $\mathcal{A}_\pi = \{0\}$. Then, by definition, $E_\pi = 0$, so that $q_{\bar{\pi}} \alpha(a) q_\iota = 0$ for any $a \in \mathcal{A}$ by Lemma 3.4. From this, it follows that $q_{\bar{\pi}} \alpha(a)(1 \otimes \lambda^\sigma(\omega)) q_\iota = 0$ for any $a \in \mathcal{A}$ and $\omega \in \mathcal{M}_*$. Then, by Lemma 2.1, we have $q_{\bar{\pi}}(\mathcal{A} \rtimes_\alpha \mathbb{K}) q_\iota = \{0\}$.

Conversely, if there exixts a non-zero $a \in \mathcal{A}_\pi$, then, by Lemma 3.4, we have $q_{\bar{\pi}}\alpha(a)q_\iota = \alpha(a)q_\iota$. From this, we have

$$(q_{\bar{\pi}}\alpha(a)q_\iota)^*(q_{\bar{\pi}}\alpha(a)q_\iota) = q_\iota\alpha(a^*a)q_\iota = E_\alpha(a^*a)\otimes\lambda^\sigma(\varphi).$$

Since E_α is faithful, $q_{\bar{\pi}}\alpha(a)q_\iota$ must be non-zero. □

The first part of the next lemma is obtained in Remark of [S]. We, however, exhibit its proof for the sake of completeness of our argument.

Lemma 3.7 *If the central support $z(q_\iota)$ of the projection $q_\iota = 1\otimes\lambda^\sigma(\varphi)$ in the crossed product $\mathcal{A}\rtimes_\alpha\mathbb{K}$ equals* 1, *then the linear span of the set $\{(\omega\otimes id)\circ\alpha(a) : a\in\mathcal{A},\ \omega\in\mathcal{A}_*\}$ is σ-weakly dense in $\mathcal{M}$. Moreover, if this is the case, we have $\mathcal{A}^\alpha(\pi)\neq\{0\}$ for any $\pi\in\mathrm{Irr}(\mathbb{K})$.*

Proof. Let us suppose that the central support $z(q_\iota)$ is 1, and that some $\rho\in\mathcal{M}_*$ satisfies $\rho((\omega\otimes id)\circ\alpha(a)) = 0$ for all $a\in\mathcal{A}$ and $\omega\in\mathcal{A}_*$. We show in due course that $\rho = 0$. From this condition, we have $(id\otimes\rho)\circ\alpha(a) = 0$ for any $a\in\mathcal{A}$. From this and (3.5), it follows that

$$\begin{aligned}(1\otimes\lambda^\sigma(\rho\circ\kappa))\alpha(a)(1\otimes\lambda^\sigma(\varphi)) &= \alpha((id\otimes\rho)\circ\alpha(a))(1\otimes\lambda^\sigma(\varphi))\\ &= 0.\end{aligned}$$

This, together with Lemma 2.1, implies that $(1\otimes\lambda^\sigma(\rho\circ\kappa))(\mathcal{A}\rtimes_\alpha\mathbb{K})q_\iota = 0$. By assumption, we get $1\otimes\lambda^\sigma(\rho\circ\kappa) = 0$. Since the regular representation λ^σ is faithful, it follows that $\rho = 0$.

The last assertion follows from the preceding paragraph and Lemma 2.13 of [Y2]. □

The next theorem can be regarded as an extension of the result of Peligrad [Pel, Corollary 3.7].

Theorem 3.8 *The following are equivalent:*

(1) *The crossed product $\mathcal{A}\rtimes_\alpha\mathbb{K}$ is a factor;*
(2) *For each $\pi\in\mathrm{Irr}(\mathbb{K})$, $\mathcal{A}^\alpha(\pi)\neq\{0\}$ and $\mathcal{Q}^{\beta_\pi}$ is a factor;*
(3) *For each $\pi\in\mathrm{Irr}(\mathbb{K})$, $\mathcal{A}^\alpha(\pi)\neq\{0\}$ and $\mathcal{F}_\pi^c$ is a factor.*

Proof. Due to Corollary 2.10, equivalence of (2) and (3) is clear. Thus we prove below that the condition (1) is equivalent to (3).

Suppose first that the crossed product $\mathcal{A}\rtimes_\alpha\mathbb{K}$ is a factor. By Corollary 2.10, $\mathcal{F}_\pi^c$ is a factor. Moreover, since the central support $z(q_\iota)$ is 1, $\mathcal{A}^\alpha(\pi)\neq\{0\}$ for any $\pi\in\mathrm{Irr}(\mathbb{K})$ by Lemma 3.7.

Conversely, assume that the condition (3) holds. Then, by Lemma 2.6

and Proposition A.1 of [Y2], we have $\mathcal{A}_\pi \neq \{0\}$ for all $\pi \in \mathrm{Irr}(\mathbb{K})$. From this and Lemma 3.6, it follows that $q_{\bar{\pi}}(\mathcal{A} \rtimes_\alpha \mathbb{K})q_\iota \neq \{0\}$. Hence $q_{\bar{\pi}}(\mathcal{A} \rtimes_\alpha \mathbb{K})q_\iota(\mathcal{A} \rtimes_\alpha \mathbb{K})q_{\bar{\pi}}$ is a non-zero two-sided ideal of the reduced von Neumann algebra $q_{\bar{\pi}}(\mathcal{A} \rtimes_\alpha \mathbb{K})q_{\bar{\pi}}$. But, from Corollary 2.10, this reduced algebra is a factor. Accordingly, the ideal $q_{\bar{\pi}}(\mathcal{A} \rtimes_\alpha \mathbb{K})q_\iota(\mathcal{A} \rtimes_\alpha \mathbb{K})q_{\bar{\pi}}$ is σ-strongly* dense. Thus we have

$$\begin{aligned} q_{\bar{\pi}}z(q_\iota)(L^2(\mathcal{A}) \otimes L^2(\varphi)) &= [q_{\bar{\pi}}(\mathcal{A} \rtimes_\alpha \mathbb{K})q_\iota(L^2(\mathcal{A}) \otimes L^2(\varphi))] \\ &\supseteq [q_{\bar{\pi}}(\mathcal{A} \rtimes_\alpha \mathbb{K})q_\iota(\mathcal{A} \rtimes_\alpha \mathbb{K})q_{\bar{\pi}}(L^2(\mathcal{A}) \otimes L^2(\varphi))] \\ &= [q_{\bar{\pi}}(\mathcal{A} \rtimes_\alpha \mathbb{K})q_{\bar{\pi}}(L^2(\mathcal{A}) \otimes L^2(\varphi))] \\ &= q_{\bar{\pi}}(L^2(\mathcal{A}) \otimes L^2(\varphi)) \end{aligned}$$

Here, in general, $[A]$ stands for the closed linear space generated by a subset A of a Hilbert space. This shows that $q_{\bar{\pi}}z(q_\iota) \geq q_{\bar{\pi}}$, which is $q_{\bar{\pi}}z(q_\iota) = q_{\bar{\pi}}$. Since $\sum_{\pi \in \mathrm{Irr}(\mathbb{K})} q_\pi = 1$, it follows that $z(q_\iota) = 1$. This means that the induction $(\mathcal{A} \rtimes_\alpha \mathbb{K})' \longrightarrow (\mathcal{A} \rtimes_\alpha \mathbb{K})'q_\iota$ is a $*$-isomorphism. But, since

$$\begin{aligned} (\mathcal{A} \rtimes_\alpha \mathbb{K})'q_\iota &= \{q_\iota(\mathcal{A} \rtimes_\alpha \mathbb{K})q_\iota\}' \\ &= \{\mathcal{F}_\iota^c \otimes \lambda^\sigma(\varphi)\}' \, (= \{\mathcal{A}^\alpha \otimes \lambda^\sigma(\varphi)\}'), \end{aligned}$$

$(\mathcal{A} \rtimes_\alpha \mathbb{K})'q_\iota$ is a factor. Therefore, the crossed product $\mathcal{A} \rtimes_\alpha \mathbb{K}$ also is a factor. □

The corollary that follows can be considered as a generalization of Corollary 3.2 of [P] to the case of actions of compact Kac algebras. The author was informed by Dr. Sekine that he had obtained exactly the same result. But we emphasize that our argument is different from his.

Corollary 3.9 *The crossed product $\mathcal{A} \rtimes_\alpha \mathbb{K}$ is a factor if and only if the fixed-point algebra $\mathcal{A}^\alpha$ is a factor and the central support $z(q_\iota)$ of the projection q_ι is* 1.

Proof. If the crossed product is a factor, then, by Theorem 3.8, the condition (3) of the theorem holds. In particular, $\mathcal{F}_\iota^c$ is a factor. This means that $\mathcal{A}^\alpha$ is factor. Since the crossed product is a factor, $z(q_\iota)$ equals the identity.

Suppose next that $\mathcal{A}^\alpha$ is a factor and that the central support $z(q_\iota)$ is 1. In this case again, by the argument set out in the second paragraph of the proof of Theorem 3.8, the crossed product must be a factor. □

Corollary 3.10 *The action α is mimimal in the sense of* [ILP] *if and only if the fixed-point algebra $\mathcal{A}^\alpha$ has trivial relative commutant in $\mathcal{A}$ and the crossed product $\mathcal{A} \rtimes_\alpha \mathbb{K}$ is a factor.*

Proof. Suppose first that α is minimal, i.e., the fixed-point algebra $\mathcal{A}^\alpha$ has trivial relative commutant in $\mathcal{A}$ and the linear span of the set $\{(\omega \otimes id) \circ \alpha(a) : a \in \mathcal{A}, \omega \in \mathcal{A}_*\}$ is σ-weakly dense in $\mathcal{M}$. By tensoring the countably decomposable type I_∞ factor with $\mathcal{A}$ if necessary, we may and do assume that $\mathcal{A}^\alpha$ is properly infinite. Then, by [Y2] (or see [ILP, Section 4]), α is a dominant action in the sense of [Y2]. Hence it is dual in particular. So, by the Takesaki duality, the crossed product $\mathcal{A} \rtimes_\alpha \mathbb{K}$ is isomorphic to $\mathcal{A}^\alpha \bar{\otimes} \mathcal{L}(L^2(\varphi))$. Therefore, it is a factor.

The other implication follows immediately from Lemma 3.7 and Corollary 3.9. □

We shall use Theorem 3.8 in order to give a necessary and sufficient condition for the crossed product $\mathcal{A} \rtimes_\alpha \mathbb{K}$ to be a factor in terms of the Connes spectrum $\Gamma(\alpha)$. To achieve this goal, we require several preparatory results.

Let us fix a π in $\mathrm{Irr}(\mathbb{K})$. To state the next proposition, we set $\mathfrak{X}_\pi = \mathcal{A}^\alpha(\pi)$. By definition, it is easy to check that $\mathfrak{X}_\pi$ is a left $\mathcal{Q}^{\bar{\alpha}}$, right $\mathcal{Q}^{\beta_\pi}$ bimodule. Moreover, on $\mathfrak{X}_\pi$, there are a $\mathcal{Q}^{\bar{\alpha}}$-valued inner product $\langle \cdot , \cdot \rangle_{\mathcal{Q}^{\bar{\alpha}}}$ and a $\mathcal{Q}^{\beta_\pi}$-valued inner product $\langle \cdot , \cdot \rangle_{\mathcal{Q}^{\beta_\pi}}$ defined by

$$\langle X, Y \rangle_{\mathcal{Q}^{\bar{\alpha}}} = XY^*, \qquad \langle X, Y \rangle_{\mathcal{Q}^{\beta_\pi}} = X^*Y \qquad (X, Y \in \mathfrak{X}_\pi).$$

Clearly, we have

$$\langle X, Y \rangle_{\mathcal{Q}^{\bar{\alpha}}} Z = X \langle Y, Z \rangle_{\mathcal{Q}^{\beta_\pi}}$$

for any $X, Y, Z \in \mathfrak{X}_\pi$. Hence we obtain the following proposition, which we believe are well known.

Proposition 3.11 *If the sets $\mathfrak{X}_\pi \mathfrak{X}_\pi^*$ and $\mathfrak{X}_\pi^* \mathfrak{X}_\pi$ are σ-weakly dense in $\mathcal{Q}^{\bar{\alpha}}$ and $\mathcal{Q}^{\beta_\pi}$, respectively, then $\mathfrak{X}_\pi$ is a $\mathcal{Q}^{\bar{\alpha}}$-$\mathcal{Q}^{\beta_\pi}$-equivalence bimodule in the sense of* [R]. *Therefore, if this is the case, then $\mathcal{Q}^{\bar{\alpha}}$ is Morita equivalent to $\mathcal{Q}^{\beta_\pi}$.*

Proof. The first part is just the definition of an equivalence bimodule between two von Neumann algebras (see [R, Definition 7.5]). The second part follows from Theorem 7.9 of [R]. □

For the next lemma, we first make a small observation.

Let $a \in \mathcal{A}$ be in the spectral subspace $\mathcal{A}_\pi$. We set $D_a = \{(id \otimes \omega) \circ \alpha(a) : \omega \in \mathcal{M}_*\}$. Then, as shown in Appendix of [Y2], D_a is a finite-dimensional α-invariant subspace of $\mathcal{A}$ wihch is "equivalent to π." So there is a basis $\{v_i\}_{i=1}^{d(\pi)}$ for D_a such that $\alpha(v_j) = \sum_{i=1}^{d(\pi)} v_i \otimes V(\pi)_{i,j}$. With the help of this expression, it can be verified that we have

$$\alpha(a) = \sum_{i,j=1}^{d(\pi)} P^\pi_{i,j}(a) \otimes V(\pi)_{i,j}, \tag{3.12}$$

where $P^\pi_{i,j}$ is a map from $\mathcal{A}$ into itself defined by $P^\pi_{i,j} = d(\pi)(id \otimes V(\pi)^*_{i,j}\varphi) \circ \alpha$. By using the identity $P^\pi_{i,j} \circ P^{\pi'}_{k,\ell} = \delta_{\pi,\pi'}\delta_{j,k}P^\pi_{i,\ell}$, one can prove that, with $X_a = \sum_{i,j=1}^{d(\pi)} P^\pi_{j,i}(a) \otimes e^\pi_{i,j} \in \mathcal{Q} = \mathcal{A} \otimes \mathcal{L}(\mathcal{H}_\pi)$, we have $\bar{\alpha}(X_a) = (X_a)_{12}V(\pi)_{23}$. Namely, X_a thus defined belongs to the eigenspace $\mathcal{A}^\alpha(\pi)$. Conversely, if $X = \sum_{i,j=1}^{d(\pi)} X_{i,j} \otimes e^\pi_{i,j}$ is in $\mathcal{A}^\alpha(\pi)$, then, as in the proof of Lemma 2.6 of [Y2] (see [Y2, Equation (2.7)]), we have

$$\alpha(X_{j,k}) = \sum_{i=1}^{d(\pi)} X_{j,i} \otimes V(\pi)_{i,k}.$$

From this, it follows easily that $P^\pi_{m,n}(X_{j,k}) = \delta_{k,n}X_{j,m}$. With this identity, one can prove by a direct computation that, if we set $a = \sum_{j=1}^{d(\pi)} X_{j,j}$, which clearly belongs to $\mathcal{A}_\pi$, then we get $X_a = \sum_{i,j=1}^{d(\pi)} P^\pi_{j,i}(a) \otimes e^\pi_{i,j} = X$. This shows that the linear map $a \in \mathcal{A}_\pi \longmapsto X_a \in \mathcal{A}^\alpha(\pi)$ is bijective. We make use of this fact in the lemma that follows.

Lemma 3.13 *Let e be an element of the fixed-point algebra $\mathcal{A}^\alpha$. Then we have*

$$(\overline{\mathcal{A}e\mathcal{A}}^{\sigma\text{-w}})^\alpha = \overline{\mathrm{span}}^{\sigma\text{-w}}\{(id \otimes Tr)(\mathcal{A}^\alpha(\pi)(e \otimes 1)\mathcal{A}^\alpha(\pi)^*) : \pi \in \mathrm{Irr}(\mathbb{K})\}.$$

Proof. Let us denote by $\mathfrak{A}$ the set on the right-hand side of the asserted identity.

Let X, Y be in $\mathcal{A}^\alpha(\pi)$. By expressing X and Y in the form $X = \sum_{i,j} X_{i,j} \otimes e^\pi_{i,j}$, $Y = \sum_{i,j} Y_{i,j} \otimes e^\pi_{i,j}$, we obtain $(id \otimes \mathrm{Tr})(X(e \otimes 1)Y^*) = \sum_{i,j=1}^{d(\pi)} X_{i,j}eY^*_{i,j}$. Thus we have $\mathfrak{A} \subseteq \overline{\mathcal{A}e\mathcal{A}}^{\sigma\text{-w}}$. Moreover, since

$$\alpha((id \otimes \mathrm{Tr})(X(e \otimes 1)Y^*)) = (id \otimes id \otimes \mathrm{Tr}) \circ (\alpha \otimes id)(X(e \otimes 1)Y^*)$$

$$\begin{aligned}&= (id \otimes \mathrm{Tr} \otimes id) \circ \bar{\alpha}(X(e\otimes 1)Y^*)\\ &= (id \otimes \mathrm{Tr} \otimes id)(X_{12}(e\otimes 1\otimes 1)Y_{12}^*)\\ &= (id \otimes \mathrm{Tr})(X(e\otimes 1)Y^*)_{12},\end{aligned}$$

the element $(id \otimes \mathrm{Tr})(X(e\otimes 1)Y^*)$ belongs to the fixed-point algebra $\mathcal{A}^\alpha$. Hence $\mathfrak{A} \subseteq (\overline{\mathcal{A}e\mathcal{A}}^{\sigma\text{-w}})^\alpha$.

Conversely, if $a \in \mathcal{A}_\pi$ and $b \in \mathcal{A}_{\pi'}$ ($\pi, \pi' \in \mathrm{Irr}(\mathbb{K})$), then, by (3.12), we have

$$\begin{aligned}&E_\alpha(aeb^*)\\ &= (id\otimes\varphi)\circ\alpha(aeb^*)\\ &= \sum_{i,j=1}^{d(\pi)}\sum_{k,\ell=1}^{d(\pi')}(id\otimes\varphi)((P_{i,j}^\pi(a)\otimes V(\pi)_{i,j})(e\otimes 1)(P_{k,\ell}^{\pi'}(b)^*\otimes V(\pi')_{k,\ell}^*))\\ &= \sum_{i,j=1}^{d(\pi)}\sum_{k,\ell=1}^{d(\pi')}\varphi(V(\pi)_{i,j}V(\pi')_{k,\ell}^*)P_{i,j}^\pi(a)eP_{k,\ell}^{\pi'}(b)^*\\ &= \frac{\delta_{\pi,\pi'}}{d(\pi)}\sum_{i,j=1}^{d(\pi)}P_{i,j}^\pi(a)eP_{i,j}^{\pi'}(b)^*.\end{aligned}$$

Hence, with the notation introduced in the discussion preceding this lemma, we have $E_\alpha(aeb^*) = \frac{1}{d(\pi)}(id\otimes\mathrm{Tr})(X_a(e\otimes 1)Y_b^*)$ if $\pi = \pi'$. Since the subspace generated by $\{\mathcal{A}_\pi\}_\pi$ is σ-weakly dense in $\mathcal{A}$, it follows from the discussion just before this lemma that $E_\alpha(\overline{\mathcal{A}e\mathcal{A}}^{\sigma\text{-w}})$ is contained in $\mathfrak{A}$. But, since $E_\alpha(\overline{\mathcal{A}e\mathcal{A}}^{\sigma\text{-w}}) = (\overline{\mathcal{A}e\mathcal{A}}^{\sigma\text{-w}})^\alpha$, the assertion follows. □

To characterize factoriality of crossed products in terms of the Connes spectrum, we need to introduce a notion of central ergodicity of an action. For this end, we prepare the following lemma, which may be a folklore.

Lemma 3.14 *The α-invariant σ-weakly closed two-sided ideals of $\mathcal{A}$ are in natural bijective correspondence with the projections in $\mathcal{Z}(\mathcal{A}) \cap \mathcal{A}^\alpha$.*

Proof. Let $\mathcal{I}$ be an α-invariant σ-weakly closed two-sided ideal of $\mathcal{A}$. Take a unique central projection p in $\mathcal{A}$ with $\mathcal{I} = \mathcal{A}p$. From the α-invariance of $\mathcal{I}$, we have $\alpha(xp)(p\otimes 1) = \alpha(xp)$ for all $x \in \mathcal{A}$. By applying $id\otimes\varphi$ to both sides of this identity, we obtain $E_\alpha(xp)p = E_\alpha(xp)$. By taking $x = 1$ in this identity, we have $E_\alpha(p)p = E_\alpha(p)$. From this, one has $E_\alpha(p)^2 = E_\alpha(p)$, which means that $E_\alpha(p)$ is a projection satisfying $E_\alpha(p) \leq p$. Since $E_\alpha(p - E_\alpha(p)) = 0$, it follows from the faithfulness of E_α that $p = E_\alpha(p)$.

Thus p belongs to $\mathcal{A}^\alpha$. The reverse correspondence is obvious. $\square$

Definition 3.15 We say that an action γ of a compact Kac algebra on a von Neumann algebra $\mathcal{P}$ is *centrally ergodic* if $\mathcal{Z}(\mathcal{P}) \cap \mathcal{P}^\gamma = \mathbf{C}$. By Lemma 3.14, this condition is equivalent to the one that there is no non-trivial γ-invariant σ-weakly closed two-sided ideal in $\mathcal{P}$.

Lemma 3.16 *If the action α is centrally ergodic with full Connes spectrum $\Gamma(\alpha) = \mathfrak{D}(\mathbb{K})$, then the fixed-point algebra $\mathcal{A}^\alpha$ is a factor.*

Proof. Let e be a central projection in $\mathcal{A}^\alpha$. Set $\mathcal{B} = e\mathcal{A}e$. Note that $\mathcal{B}^{\alpha^e}(\pi) = (e \otimes 1)\mathcal{A}^\alpha(\pi)(e \otimes 1)$ for any $\pi \in \mathrm{Irr}(\mathbb{K})$. Since $\mathrm{Sp}(\alpha^e) = \mathfrak{D}(\mathbb{K})$ by assumption, we have

$$\begin{aligned}\overline{\tilde{e}\mathcal{A}^\alpha(\pi)^*\tilde{e}\mathcal{A}^\alpha(\pi)\tilde{e}}^{\sigma\text{-w}} &= \overline{\mathcal{B}^{\alpha^e}(\pi)^*\mathcal{B}^{\alpha^e}(\pi)}^{\sigma\text{-w}} \\ &= (\mathcal{B} \otimes \mathcal{L}(\mathcal{H}_\pi))^{\mathrm{Ad}\, V(\pi)_{23} \circ \overline{\alpha^e}} = \tilde{e}\mathcal{Q}^{\beta_\pi}\tilde{e}\end{aligned}$$

for any $\pi \in \mathrm{Irr}(\mathbb{K})$, where $\tilde{e} = e \otimes 1$. From this, it follows that

$$\overline{\mathcal{A}^\alpha(\pi)\tilde{e}\mathcal{A}^\alpha(\pi)^*\tilde{e}\mathcal{A}^\alpha(\pi)\tilde{e}\mathcal{A}^\alpha(\pi)^*}^{\sigma\text{-w}} = \overline{\mathcal{A}^\alpha(\pi)\tilde{e}\mathcal{Q}^{\beta_\pi}\tilde{e}\mathcal{A}^\alpha(\pi)^*}^{\sigma\text{-w}}.$$

Since $\mathcal{A}^\alpha(\pi)\tilde{e}\mathcal{A}^\alpha(\pi)^*$ is contained in $\mathcal{A}^\alpha \otimes \mathcal{L}(\mathcal{H}_\pi)$ as noted in the discussion preceding Proposition 3.11, it results that

$$\mathcal{A}^\alpha(\pi)\tilde{e}\mathcal{A}^\alpha(\pi)^*\tilde{e}\mathcal{A}^\alpha(\pi)\tilde{e}\mathcal{A}^\alpha(\pi)^* \subseteq \mathcal{A}^\alpha e \otimes \mathcal{L}(\mathcal{H}_\pi).$$

From this, we have $\mathcal{A}^\alpha(\pi)\tilde{e}\mathcal{Q}^{\beta_\pi}\tilde{e}\mathcal{A}^\alpha(\pi)^* \subseteq \mathcal{A}^\alpha e \otimes \mathcal{L}(\mathcal{H}_\pi)$. In particular, we have $\mathcal{A}^\alpha(\pi)\tilde{e}\mathcal{A}^\alpha(\pi)^* \subseteq \mathcal{A}^\alpha e \otimes \mathcal{L}(\mathcal{H}_\pi)$. Accordingly, we find that $(id \otimes \mathrm{Tr})(\mathcal{A}^\alpha(\pi)\tilde{e}\mathcal{A}^\alpha(\pi)^*)$ is contained in $\mathcal{A}^\alpha e$ for any $\pi \in \mathrm{Irr}(\mathbb{K})$. By Lemma 3.13, we then obtain $(\overline{\mathcal{A}e\mathcal{A}}^{\sigma\text{-w}})^\alpha \subseteq \mathcal{A}^\alpha e$. But note that $\overline{\mathcal{A}e\mathcal{A}}^{\sigma\text{-w}}$ is clearly an α-invariant σ-weakly closed two-sided ideal of $\mathcal{A}$. Hence, from the central ergodicity of α, this ideal must be $\mathcal{A}$ itself. This implies that $\mathcal{A}^\alpha e$ contains $\mathcal{A}^\alpha$, which means that $e = 1$. Therefore, the fixed-point algebra $\mathcal{A}^\alpha$ must be a factor. $\square$

Now we are in a position to prove our main theorem of this section on factoriality of crossed products.

Theorem 3.17 *The following are equivalent*:

(1) *The crossed product $\mathcal{A} \rtimes_\alpha \mathbb{K}$ is a factor*;

(2) *The action α is centrally ergodic and has full Connes spectrum $\Gamma(\alpha) = \mathfrak{D}(\mathbb{K})$.*

Proof. (1) ⇒ (2): By Corollary 3.9, $\mathcal{A}^\alpha$ is a factor. Hence α is centrally ergodic. Then, by Lemma 3.3, we have $\Gamma(\alpha) = \mathrm{Sp}(\alpha)$. For each $\pi \in \mathrm{Irr}(\mathbb{K})$, Theorem 3.8 guarantees that $\overline{\mathcal{A}^\alpha(\pi)^*\mathcal{A}^\alpha(\pi)}^{\sigma\text{-w}}$ is a non-zero σ-weakly closed two-sided ideal of a factor $\mathcal{Q}^{\beta_\pi}$. Thus we have $\overline{\mathcal{A}^\alpha(\pi)^*\mathcal{A}^\alpha(\pi)}^{\sigma\text{-w}} = \mathcal{Q}^{\beta_\pi}$. This shows that α has full Connes spectrum.

(2) ⇒ (1): By Lemma 3.16, $\mathcal{A}^\alpha$ is a factor. For any $\pi \in \mathrm{Irr}(\mathbb{K})$, we have $\overline{\mathcal{A}^\alpha(\pi)^*\mathcal{A}^\alpha(\pi)}^{\sigma\text{-w}} = \mathcal{Q}^{\beta_\pi}$. On the other hand, $\overline{\mathcal{A}^\alpha(\pi)\mathcal{A}^\alpha(\pi)^*}^{\sigma\text{-w}}$ is a non-zero σ-weakly closed two-sided ideal of a factor $\mathcal{Q}^{\bar{\alpha}} = \mathcal{A}^\alpha \otimes \mathcal{L}(\mathcal{H}_\pi)$. Hence we have $\overline{\mathcal{A}^\alpha(\pi)\mathcal{A}^\alpha(\pi)^*}^{\sigma\text{-w}} = \mathcal{Q}^{\bar{\alpha}}$. From this together with Proposition 3.11, it follows that $\mathcal{Q}^{\beta_\pi}$ is Morita equivalent to $\mathcal{Q}^{\bar{\alpha}}$. Since Morita equivalent von Neumann algebras have isomorphic centers, $\mathcal{Q}^{\beta_\pi}$ is a factor. From Theorem 3.8, the crossed product $\mathcal{A} \rtimes_\alpha \mathbb{K}$ is a factor. □

4. Examples and computation

In this section, we shall give several examples of actions of compact Kac algebras, and discuss (or compute) thier (Connes) spectrum.

A. Actions of compact groups. We first treat the case of actions of commutative compact Kac algebras, that is, actions α of compact groups G on von Neumann algebras $\mathcal{A}$. In this case, to the best of author's knowledge, there have already been two (different) definitions of a spectrum; one was introduced by Roberts in [Ro, Definition 6.3] (see [NT, p. 72] also); the other was introduced by Evans and Sund in [EvS, p. 301]. Roberts' spectrum is called the monoidal spectrum and denoted by $\mathrm{Msp}(\alpha)$. The monoidal spectrum behaves well particularly for actions with properly infinite fixed-point algebra, as seen in [Ro] and [AHKT]. It is defined as follows:

$$\mathrm{Msp}(\alpha) = \{\pi \in \widehat{G} : \mathcal{A}^\alpha(\pi) \text{ contains a unitary}\}.$$

Thus it is obvious that $\mathrm{Msp}(\alpha) \subseteq \mathrm{Sp}(\alpha)$. The spectrum $\mathrm{Sp}_{ES}(\alpha)$ of Evans-Sund [EvS, p. 307] is given, in terms of our language, by

$$\underset{ES}{\mathrm{Sp}}(\alpha) = \{\pi \in \widehat{G} : \mathcal{A}^\alpha(\pi) \neq \{0\}\}.$$

Thier Γ-spectrum $\Gamma_{ES}(\alpha)$ is then defined by $\Gamma_{ES}(\alpha) = \bigcap\{\mathrm{Sp}_{ES}(\alpha^e) :$ e is a non-zero projection in $\mathcal{A}^\alpha\}$. Clearly, we have $\mathrm{Sp}(\alpha) \subseteq \mathrm{Sp}_{ES}(\alpha)$, $\Gamma(\alpha) \subseteq \Gamma_{ES}(\alpha)$. There are results in [NT, Proposition 2.4–5, p. 72] (see also [AHKT]) which describe one relationship among these spectrums. For readers' reference, we state them in a single proposition that follows.

Proposition *Assume that $\mathcal{A}^\alpha$ is properly infinite. If there exists an ergodic subgroup of* $\mathrm{Aut}(\mathcal{A})$ *commuting with α_t, $t \in G$, then* $\mathrm{Msp}(\alpha) = \mathrm{Sp}(\alpha) = \mathrm{Sp}_{ES}(\alpha)$. *Moreover, if α is faithful, then they equal $\widehat{G}$.*

It is easily verified also that $\Gamma(\alpha)$ coincides with the original Connes spectrum when G is a compact abelian group.

B. Coactions of discrete groups. In this subsection, we are concerned with the symmetric (= cocommutative) Kac algebra $\mathbb{K}_s(G)$ canonically associated with each discrete group G (see [ES, Theorem 3.7.5] for the definition of $\mathbb{K}_s(G)$). Its underlying von Neumann algebra is the group von Neumann algebra $\mathcal{R}(G)$ generated by the left regular representation λ_G of G on $\ell^2(G)$. Note that $\mathbb{K}_s(G)$ is a compact Kac algebra, since G is discrete. Now we consider an (arbitrary) action α of $\mathbb{K}_s(G)$, in other words, a coaction α of G on a von Neumann algebra $\mathcal{A}$. It is well-known that the unitary dual $\mathfrak{D}(\mathbb{K}_s(G))$ of $\mathbb{K}_s(G)$ (or the set $\mathrm{Irr}(\mathbb{K}_s(G))$ is canonically identified with the original group G: each $s \in G$ corresponds to the one-dimensional representation $\omega \in \mathcal{R}(G)_* \longmapsto \omega(\lambda_G(s)) \in \mathbf{C}$ whose generator is $1_{\mathbf{C}} \otimes \lambda_G(s)$. Thus each eigenspace of the action α is in this case indexed by elements of G like $\mathcal{A}^\alpha(s)$, and one easily finds that it is the set defined as follows:

$$\mathcal{A}^\alpha(s) = \{a \in \mathcal{A} : \alpha(a) = a \otimes \lambda_G(s)\}. \qquad (s \in G) \tag{4.1}$$

From this observation, it follows for example that $s \in G$ is in $\mathrm{Sp}(\alpha)$ if and only if $\overline{\mathcal{A}^\alpha(s)^*\mathcal{A}^\alpha(s)}^{\sigma\text{-w}} = \mathcal{A}^\alpha$. Let us denote by $\mathrm{Sp}_N(\alpha)$ and $\Gamma_N(\alpha)$ the spectrum and the essential spectrum of α, respectively, in the sense of [N, Section 3] or [NT, Chapter IV]. Since $s \in G$ belongs to $\mathrm{Sp}_N(\alpha)$ if and only if $\mathcal{A}^\alpha(s) \neq \{0\}$ by Lemma 1.2 (iv) in Chapter IV of [NT], we have $\mathrm{Sp}(\alpha) \subseteq \mathrm{Sp}_N(\alpha)$. Hence we obtain $\Gamma(\alpha) \subseteq \Gamma_N(\alpha)$. In fact, one can show that $\Gamma(\alpha)$ equals $\Gamma_N(\alpha)$. We also remark here that, in [Q], Quigg made a systematic study on coactions of discrete groups on C^*-algebras in connection with C^*-algebraic bundles. In that paper, among others, he showed a C^*-algebraic version of our main theorem in this case [Q, Theorem 2. 10–11].

A1. *The case where α is dual.* In this case, there exists an action γ of G on a von Neumann algebra $\mathcal{B}$ ($*$-isomorphic to $\mathcal{A}^\alpha$) such that $(\mathcal{A}, \mathbb{K}_s(G), \alpha)$ is (conjugate to) the system $(\mathcal{B} \rtimes_\gamma G, \mathbb{K}_s(G), \hat{\gamma})$, where $\hat{\gamma}$ is the dual action of γ on the crossed product $\mathcal{B} \rtimes_\gamma G$. Following the convention, we denote by π_γ the embedding of $\mathcal{B}$ in $\mathcal{B} \rtimes_\gamma G$. Then we have $\mathcal{A}^\alpha = \pi_\gamma(\mathcal{B})$. Moreover, we have $\alpha(1 \otimes \lambda_G(t)) = 1 \otimes \lambda_G(t) \otimes \lambda_G(t)$ for any $t \in G$. Hence it follows that

$$\mathcal{A}^\alpha(s) = \pi_\gamma(\mathcal{B})(1 \otimes \lambda_G(s)). \qquad (s \in G)$$

Let e be a central projection in $\mathcal{A}^\alpha$. Since $\mathcal{A}^\alpha = \pi_\gamma(\mathcal{B})$, there is a unique central projection p in $\mathcal{B}$ such that $e = \pi_\gamma(p)$. With this notation, we have

$$\mathcal{A}^{\alpha^e}(s) = e\mathcal{A}^\alpha(s)e = \pi_\gamma(\mathcal{B}p\gamma_s(p))(1 \otimes \lambda_G(s)).$$

Therefore, we conclude that $s \in \mathrm{Sp}(\alpha^e)$, i.e., $\overline{\mathcal{A}^{\alpha^e}(s)^*\mathcal{A}^{\alpha^e}(s)}^{\sigma\text{-w}} = \mathcal{A}^\alpha e$ if and only if $\mathcal{B}p\gamma_{s^{-1}}(p) = \mathcal{B}p$ if and only if $p\gamma_{s^{-1}}(p) = p$. From this, it follows that $s \in G$ belongs to the Connes spectrum $\Gamma(\alpha)$ if and only if $p\gamma_{s^{-1}}(p) = p$ for any central projection p in $\mathcal{B}$, which is in turn equivalent to the condition that the restriction $\gamma_{s^{-1}}|_{\mathcal{Z}(\mathcal{B})}$ of $\gamma_{s^{-1}}$ to the center $\mathcal{Z}(\mathcal{B})$ of $\mathcal{B}$ equals the identity. So we have shown that

$$\Gamma(\alpha) = \{s \in G : \gamma_{s^{-1}}|_{\mathcal{Z}(\mathcal{B})} = id_{\mathcal{Z}(\mathcal{B})}\}.$$

By [N, Theorem 6.1] or [NT, Theorem 1.5, p. 66], we have $\Gamma_N(\alpha) = \Gamma(\alpha)$ in this case.

A2. *An example considered by Sutherland and Takesaki.* (see [ST, Examples 5.6]). In this example, let K and G be discrete groups, and $\theta : G \to \mathrm{Aut}(K)$ be a homomorphism. We form a semi-direct product group $H = K \rtimes_\theta G$ associated with θ. Also consider a 2-cocycle ω on G. We lift ω to a 2-cocycle on H and denote it by ω again. Namely, the lifted 2-cocycle ω on H is given by

$$\omega((k,s),(k',t)) = \omega(s,t) \qquad (k,k' \in K,\ s,t \in G).$$

Then we consider the twisted group von Neumann a lgebra $\mathcal{R}(H,\omega)$ generated by the twisted left regular representation λ_H^ω on $\ell^2(H)$ defined by

$$\{\lambda_H^\omega(h)\xi\}(h') = \omega(h, h^{-1}h')\xi(h^{-1}h'). \qquad (\xi \in \ell^2(H),\ h,h' \in H)$$

With $h = (k,s)$ and $h' = (k',t)$ in the above expression, the identity can be rewritten in the form:

$$\{\lambda_H^\omega(k,s)\xi\}(k',t) = \omega(s, s^{-1}t)\xi(\theta_{s^{-1}}(k)^{-1}k', s^{-1}t).$$

For readers' information, we exhibit the twisted convolution $f *_\omega g$ of two functions f and g on $H = K \rtimes_\theta G$ in the following:

$$(f *_\omega g)(k,s) = \sum_{(k',t)\in H} \omega(s, s^{-1}t) f(k',t) g(\theta_{t^{-1}}(k')^{-1}k, t^{-1}s). \quad (4.2)$$

Next we define a unitary W_θ on $\ell^2(H) \otimes \ell^2(G) = \ell^2(H \times G)$ by

$$\{W_\theta \eta\}((k,s),t) = \eta((k,s),st) \quad ((k,s)\in H,\ t\in G,\ \eta \in \ell^2(H\times G)).$$

A simple calculation shows that

$$W_\theta^*(\lambda_H^\omega(k,s)\otimes 1)W_\theta = \lambda_H^\omega(k,s)\otimes \lambda_G(s). \quad (4.3)$$

Hence the equation

$$\alpha(a) = W_\theta^*(a\otimes 1)W_\theta \qquad (a \in \mathcal{R}(H,\omega))$$

defines a coaction α of G on $\mathcal{A} = \mathcal{R}(H,\omega)$. It can be proven that the fixed-point algebra $\mathcal{A}^\alpha$ consists of all operators of the form $\lambda_H^\omega(f)$, where $f \in \ell^2(H)$ is a function with support contained in the subset $K \times \{e\}$ of H. (Note that the canonical conditional expectation E_α from $\mathcal{A}$ onto $\mathcal{A}^\alpha$ is given by $E_\alpha = (id \otimes \varphi_G)\circ\alpha$, where φ_G is the Plancherel weight (trace) of G). Moreover, one can show from (4.1) and (4.3) that

$$\mathcal{A}^\alpha(s) = \mathcal{A}^\alpha \cdot \lambda_H^\omega(e,s). \qquad (s\in G) \quad (4.4)$$

In particular, we get $\mathcal{A}^\alpha(s)^*\mathcal{A}^\alpha(s) = \mathcal{A}^\alpha$.

Let p be a central projection in $\mathcal{A}^\alpha$. With the aid of (4.2), it is verified that $p\lambda_H^\omega(e,s)p = \lambda_H^\omega(e,s)p$ for any $s\in G$. (As noted above, p is of the form $p = \lambda_H^\omega(f)$ for some $f\in \ell^2(H)$). From this together with (4.4), we have

$$\mathcal{A}^{\alpha^p}(s) = p\mathcal{A}^\alpha(s)p = \mathcal{A}^\alpha(s)p.$$

Hence

$$\overline{\mathcal{A}^{\alpha^p}(s)^*\mathcal{A}^{\alpha^p}(s)}^{\sigma\text{-w}} = \overline{p\mathcal{A}^\alpha(s)^*\mathcal{A}^\alpha(s)p}^{\sigma\text{-w}} = \mathcal{A}^\alpha p.$$

Therefore, every $s\in G$ is in $\mathrm{Sp}(\alpha^p)$. This means that this action α has full Connes spectrum $\Gamma(\alpha) = G$. In particular, we have $\Gamma(\alpha) = \Gamma_N(\alpha)$.

C. Other cases. In this subsection, we are mainly concerned with actions of not necessarily commutative or cocommutative compact Kac algebras.

According to [U], every compact Kac algebra $\mathbb{K}$ admits a minimal action α on a full factor $\mathcal{A}$ of type II_1. By Corollary 3.10, the crossed product $\mathcal{A} \rtimes_\alpha \mathbb{K}$ of $\mathcal{A}$ by this minimal action α is a factor. Hence, from Theorem 3.17, α has full Connes spectrum $\Gamma(\alpha)$.

For every compact Kac algebra $\mathbb{K}$, its coproduct Γ can be regarded as an action of $\mathbb{K}$ on $\mathcal{M}$ itself. This action is well known to be ergodic. Thus this example is at the opposite extreme of the preceeding example (i.e., minimal actions). Since the crossed product $\mathcal{M} \rtimes_\Gamma \mathbb{K}$ is $*$-isomorphic to $\mathcal{L}(L^2(\varphi))$ by the Takesaki duality, it is a factor. So, by Theorem 3.17 again, this action has full Connes spectrum. Not all ergodic actions, however, have full Connes spectrum as we can see next.

Let G_1 be a discrete group, G_2 a compact group, and $\alpha : G_1 \to \mathrm{Aut}(G_2)$ be a homomorphism. (More generally, consider a modular matched pair $(G_1, G_2, \alpha, \beta)$ of groups in the sense of Majid [M]). We continue to denote by α the action of G_1 on $L^\infty(G_2)$ induced by α. Then form the crossed product $L^\infty(G_2) \rtimes_\alpha G_1$. This algebra $\mathbb{K}$ turns out to have Kac algebra structure and is called the bicrossproduct Kac algebra associated with the system (G_1, G_2, α) (see [DeC] or [M] for more details). From [Y1], this Kac algebra $\mathbb{K}$ admits an action δ_α on $L^\infty(G_2)$ which is ergodic. In particular, the action is centrally ergodic. But, from Proposition 2.8 of [Y1], it is easily seen that the crossed product $L^\infty(G_2) \rtimes_{\delta_\alpha} \mathbb{K}$ is never a factor except for trivial cases. Hence, by Theorem 3.17, the action δ_α cannot have full Connes spectrum.

One can construct an ergodic action of a (noncommutative and noncocommutative) finite-dimensional Kac algebra on a finite type I factor without full Connes spectrum by the same spirit of [GLP, Example 5.2]. For this, one may use for example the 8 dimensional Kac algebra constructed by Kac-Paljutkin in [KP, 8.5].

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