Hokkaido Mathematical Journal Vol. 28 (1999) p. 405–407

On the diameter of closed minimal submanifolds in a real projective space

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(Received May 27, 1998)

Abstract. In this note, we prove an optimal lower bound estimate for the diameter of closed minimal submanifolds in a real projective space.

Key words: diameter, minimal submanifolds, real projective space.

1. Introduction

Many results for the *pinching problem* of closed minimal submanifolds in a rank one symmetric space have been obtained in the past years. One can find various *curvature pinching theorems* about them (cf. [2], [3], [6], [8]). In [1], Chen proved an optimal *volume pinching theorem* for the above minimal submanifolds. To author's knowledge, few is known about the *diameter pinching problem* of the same kind of minimal submanifolds. In this paper, we obtain an optimal lower bound for the diameter of closed minimal submanifolds in a real projective space.

Theorem 1 *Let M^n be an n-dimensional connected immersed closed minimal submanifold in $RP^m(1)$, the m-dimensional real projective space of curvature* 1. *Then the diameter of M^n satisfies $d(M^n) \geq \frac{\pi}{2}$ with equality holding if and only if M^n is totally geodesic.*

2. A Proof of Theorem 1

Before proving Theorem 1, we list the following

Lemma 1 [4] *An immersed closed minimal submanifold in a Riemannian manifold N of positive sectional curvature must intersect every closed totally geodesic hypersurface of N.*

Proof of Theorem 1. Let p be an arbitrary fixed point of M^n and denote by $RP_p^{m-1}(1)$ the closd totally geodesic hypersurface of $RP^m(1)$ which is

1991 Mathematics Subject Classification : 53C20, 53C42.

farthest from p, thus the distance measured in $RP^m(1)$ between p and any point $\tilde{p} \in RP_p^{m-1}(1)$ is $\frac{\pi}{2}$. From Lemma 1, we know that $M_p^n := M^n \cap RP_p^{m-1}(1)$ is nonempty. Since the extrinsic distance is less than or equal to the intrinsic one on M^n, any minimal geodesic γ in M^n from p to $\tilde{p} \in M_p^n$ has length greater than or equal to $\frac{\pi}{2}$. Thus the diameter of M^n satisfies $d(M^n) \geq \frac{\pi}{2}$. This completes the proof of the first part of Theorem 1.

If M^n is totally geodesic, it is well known that $d(M^n) = \frac{\pi}{2}$. Now we assume conversely that $d(M^n) = \frac{\pi}{2}$. In this case we know from the proof of the first part that for any pair of points $p \in M^n$ and $\tilde{p} \in M_p^n$, any minimal geodesic γ in M^n from p to $\tilde{p} \in M_p^n$ has length $\frac{\pi}{2}$ and so it is a minimal geodesic in $RP^m(1)$ and is orthogonal to $RP_p^{m-1}(1)$ at $\tilde{p} \in M_p^n$. Thus for any $p \in M^n$, the intersection of M^n with $RP_p^{m-1}(1)$ is transversal and consequently M_p^n is an $(n-1)$-dimensional closed submanifold of $RP^m(1)$. Now, for any $p \in M^n$, we denote by $\widetilde{RP_p^n}(1)$ the n-dimensioanl closed totally geodesic submanifold of $RP^m(1)$ which passes through p and has the same tangent space as M^n at this point. We claim that for any $p \in M^n$, $M_p^n = \widetilde{RP_p^n}(1) \cap RP_p^{m-1}(1)$. To see this, we fix a point $p \in M^n$ and let $q \in M_p^n$ be an arbitrary point. Then the distance between p and q measured in M^n and in $RP^m(1)$ is the same number $\frac{\pi}{2}$. Now we take a normal minimal geodesic $\gamma : [0, \frac{\pi}{2}] \to M^n$ which connects p and q. Then $\gamma'(0) \in T_pM^n = T_p\widetilde{RP_p^n}(1)$ and γ is a minimal geodesic in $RP^m(1)$ and therefore $\gamma \subset \widetilde{RP_p^n}(1)$. This implies that $q = \gamma(\frac{\pi}{2}) \in \widetilde{RP_p^n}(1) \cap RP_p^{m-1}(1)$ which in turn implies that $M_p^n \subset \widetilde{RP_p^n}(1) \cap RP_p^{m-1}(1)$ by the arbitrarity of $q \in M_p^n$. On the other hand, since both M_p^n and $\widetilde{RP_p^n}(1) \cap RP_p^{m-1}(1)$ are closed $(n-1)$-dimensional submanifolds of $RP^m(1)$ and $\widetilde{RP_p^n}(1) \cap RP_p^{m-1}(1)$ is connected, we conclude therefore that $M_p^n = \widetilde{RP_p^n}(1) \cap RP_p^{m-1}(1)$. This proves our claim. We are now in a position to prove that M^n is totally geodesic. In fact, for a fixed point $p \in M^n$, since any minimal geodesic in M^n from p to $\tilde{p} \in M_p^n$ is a minimal geodesic in $RP^m(1)$ and $M_p^n = \widetilde{RP_p^n}(1) \cap RP_p^{m-1}(1)$ is totally geodesic in $RP^m(1)$, we know that for any $q \in M_p^n$, M^n is totally geodessic at q. Now we take a point $\tilde{p} \in M_p^n$, then $p \in M_{\tilde{p}}^n$ and therefore M^n is totally geodesic at p by the arguments just made. Since p is arbitrary, M^n is totally geodesic. This completes the Proof of Theorem 1. □

Remark. By using Lemma 1, one can also show that the diameter of a closed minimal submanifold in a unit sphere is greater than $\frac{\pi}{2}$. Since the complex projective space $CP^2(\frac{4}{3})$ of complex dimension 2 and of holomorphic sectional curvature $\frac{4}{3}$ can be isometrically and minimally imbedded in a unit sphere $S^7(1)$ of dimensin 7 (see [7]) and the diameter of $CP^2(\frac{4}{3})$ is $\frac{\sqrt{3}\pi}{2}$, one can't expect that the lower bound for the diameter of closed minimal submanifolds in a unit sphere is achieved at the totally geodesic spheres. Therefore the following problem remains open.

Problem What is the optimal lower bound for the diameter of closed minimal submanifolds in a unit sphere?

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