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Oscillatory behavior of higher order nonlinear neutral difference equation

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Abstract. In this paper, we are concerned with the oscillation of the solutions of certain more general higher order nonlinear neutral difference equation, we obtained several criteria for oscillatory behavior.

Key words: nonlinear delay difference equation, higher order difference equation, oscillatory behavior.

1. Introduction

We consider the higher order nonlinear neutral difference equation

$$\Delta^m (x_n - p_n x_{n-\tau}) + \sum_{i=1}^{k} Q_i(n) f_i(x_{n-\sigma_i(n)}) = 0, \quad n = 0, 1, 2, \ldots, \tag{1}$$

where Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$, $m \geq 2$ is even, τ is a positive integer. $\{p_n\}$ is a positive real sequence. $\{Q_i(n)\}$ are nonnegative real sequences, $\{\sigma_i(n)\}$ are nonnegative integer sequences and $\lim_{n\to\infty}(n - \sigma_i(n)) = \infty$, for $i = 1, 2, \ldots, k$. Moreover, there is at least an integer j, $1 \leq j \leq k$, such that $Q_j(n) > 0$, $\sigma_j(n) > 0$ for $n = 0, 1, 2, \ldots$. $f_i(u) \in C(R, R)$ are nondecreasing functions, $uf_i(u) > 0$ for $u \neq 0$ and $i = 1, 2, \ldots, k$.

Let $\mu = \max\{\tau, \sup[\sigma_i(n) \mid 1 \leq i \leq k, n \geq 0]\}$. Then by a solution of (1), We mean a real sequence $\{x_n\}_{n=-\mu}^{\infty}$ which satisfies equation (1) for $n \geq 0$. A solution $\{x_n\}$ of (1) is said to be eventually positive if $x_n > 0$ for all large n, and eventually negative if $x_n < 0$ for all large n. It is said to be oscillatory if it is neither eventually positive nor eventually negative. We will also say that (1) is oscillatory if every of its solution is oscillatory.

For the sake of convenience, the sequence $\{z_n\}$ is defined by

$$z_n = x_n - p_n x_{n-\tau}. \tag{2}$$

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As is customary, empty sums will be taken to be zero.

Lemma 1 [1] *Let $\{y_n\}$ be a sequence of real number in $N = \{0, 1, 2, \ldots\}$, Let $y_n > 0$ and $\Delta^m y_n$ be of constant sign with $\Delta^m y_n$ not being identically zero on any subset $\{n_0, n_0 + 1, \ldots\}$. Then, there exists an integer l, $0 \le l \le m$, with $m + l$ odd for $\Delta^m y_n \le 0$, and $m + l$ even for $\Delta^m y_n \ge 0$, such that*

$$l \le m - 1 \text{ implies } (-1)^{l+k} \Delta^k y_n > 0, \quad \text{for all } n \in N, \quad l \le k \le m - 1;$$

and

$$l \ge 1 \text{ implies } \Delta^k y_n > 0, \quad \text{for all } n \in N, \quad 1 \le k \le l - 1.$$

Lemma 2 [8] *Assume that $\{p_n\}$ is a sequence of nonnegative real numbers and k is a positive integer, then either one of the following conditions*

$$\liminf_{n\to\infty} \sum_{i=n-k}^{n-1} p_i > \left(\frac{k}{k+1}\right)^{k+1}$$

or

$$a_0 = \liminf_{n\to\infty} \sum_{i=n-k}^{n-1} p_i \le \left(\frac{k}{k+1}\right)^{k+1}$$

and

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n} p_i > 1 - \frac{1 - a_0 - \sqrt{1 - 2a_0 - a_0^2}}{2}$$

implies that

(H$_1$) *The difference inequality*

$$\Delta x_{n+} p_n x_{n-k} \le 0, \quad n = 0, 1, 2, \ldots,$$

has no eventually positive solutions;

(H$_2$) *The difference inequality*

$$\Delta x_n + p_n x_{n-k} \ge 0, \quad n = 0, 1, 2, \ldots,$$

has no eventually negative solutions.

Lemma 3 *Let $0 < p_n < 1$ for $n = 0, 1, 2, \ldots$. Assume that there is at least an integer j, $1 \le j \le k$, such that $\sum_{n=n_0}^{\infty} Q_j(n) = \infty$, If $\{x_n\}$ is* *eventually positive (or negative) solution of equation* (1), *then* $\lim_{n\to\infty} z_n = 0$, *moreover,* $(-1)^s\Delta^s z_n < 0$ (*or* > 0), *for* $s = 0, 1, 2, \ldots, m$ *and all large* n.

Proof. Let $\{x_n\}$ be an eventually positive solution of Eq. (1) (the proof when $\{x_n\}$ is eventually negative is similar), and without loss of generality, assume that $x_n > 0$, $x_{n-\tau} > 0$, $x_{n-\sigma_i(n)} > 0$ for $i = 1, 2, \ldots, k$ and $n \geq n_0$. By (1) and (2), we have

$$\Delta^m z_n = -\sum_{i-1}^{k} Q_i(n) f_i(x_{n-\sigma_i(n)}) < 0, \quad \text{for } n \geq n_1. \tag{3}$$

It follows that $\Delta^s z_n$ $(s = 0, 1, 2, \ldots, m-1)$ are strictly monotone and are of constant sign eventually. Hence, we may see

$$\lim_{n\to\infty} z_n = L \ (-\infty \leq L \leq \infty).$$

If $-\infty \leq L < 0$. Then there exists a constant $C > 0$ and a $n_2 \geq n_1$, such that $z_n < -C < 0$ for $n \geq n_2$. By (2), we have

$$x_n < p_n x_{n-\tau} - C \leq x_{n-\tau} - C, \quad \text{for } n \geq n_2.$$

This implies

$$x_{n+\tau} < x_n - C, \quad \text{for } n \geq n_2,$$

so that

$$x_{n_2+h\tau} < x_{n_2} - hC, \quad \text{for } h = 1, 2, \ldots,$$

and set $h \to \infty$, we obtain

$$x_{n_2+h\tau} \to -\infty$$

This contradicts $x_n > 0$ for $n \geq n_1$. Hence, $-\infty \leq L < 0$ is impossible.

If $0 < L \leq \infty$. Then there exist a constant $C > 0$ and a $n_2 \geq n_1$, such that $z_n > C > 0$ for $n \geq n_2$. In view of $\Delta^m z_n < 0$ for $n \geq n_1$ and m is even. By Lemma 1, there exist an integer $l \in \{1, 3, \ldots, m-1\}$ and a $n_3 \geq n_2$, such that, as $n \geq n_3$, $\Delta^s z_n > 0$ for $s = 0, 1, 2, \ldots, l-1$, and $(-1)^{s+l}\Delta^s z_n > 0$ for $s = l, l+1, \ldots, m-1$. In particular, $\Delta^{m-1} z_n > 0$ for $n \geq n_3$. Observe that $z_n > C > 0$, from (2), we have

$$x_n > z_n > C > 0, \quad \text{for } n \geq n_3.$$

Therefore, we may take a $n_4 \geq n_3$, such that $x_{n-\sigma_i(n)} > z_{n-\sigma_i(n)} > C > 0$ for $n \geq n_4$ and $i = 1, 2, \ldots, k$. Since f_i is nondecreasing, from (3), we have

$$\Delta^m z_n \leq -\sum_{i=1}^{k} Q_i(n) f_i(C) \leq -b \sum_{i=1}^{k} Q_i(n) \leq -b Q_j(n), \quad \text{for } n \geq n_4, \tag{4}$$

where $b = \min_{1 \leq i \leq k}\{f_i(c)\} > 0$.

By summing (4) from n_4 to n and then set $n \to \infty$, we have $\Delta^{m-1} z_n \to -\infty$ as $n \to \infty$. This contradicts $\Delta^{m-1} z_n > 0$ for $n \geq n_3$. Hence $0 < L \leq \infty$ is impossible. So that $L = 0$ holds, that is $\lim_{n\to\infty} z_n = 0$.

Since $\lim_{n\to\infty} z_n = 0$, it is not difficult to use proof by contradiction to show that $\lim_{n\to\infty} \Delta^s z_n = 0$ for $s = 0, 1, 2, \ldots, m-1$. In view of $\Delta^m z_n < 0$ for $n \geq n_1$ and m is even, hence, it is easy to see that, for all large n, $(-1)^s \Delta^s z_n < 0$ for $s = 1, 2, \ldots, m$. The proof is complete. □

Lemma 4 *Let $1 < p_n \leq p$ for $n \geq n_0$ and some positive constant p. Assume that there is at least an integer j, $1 \leq j \leq k$ such that $\sum_{n=n_0}^{\infty} Q_j(n) = \infty$. If $\{x_n\}$ is a eventually bounded positive (or negative) solution of equation (1), then $\lim_{n\to\infty} z_n = 0$, moreover, $(-1)^s \Delta^s z_n < 0$ (or > 0) for $s = 0, 1, 2, \ldots, m$ and all large n.*

Proof. Let $\{x_n\}$ be an eventually bounded positive solution of Eq. (1) (the proof when $\{x_n\}$ is eventually bounded negative solution is similar), and without loss of generality, assume that $x_n > 0$, $x_{n-\tau} > 0$, $x_{n-\sigma_i(n)} > 0$ for $n \geq n_1 \geq n_0$ and $i = 1, 2, \ldots, k$. By (1), (2), we have

$$\Delta^m z_n = -\sum_{i=1}^{k} Q_i(n) f_i(x_{n-\sigma_i(n)}) < 0, \quad \text{for } n \geq n_1. \tag{5}$$

It follows that $\Delta^s z_n$ $(s = 0, 1, 2, \ldots, m-1)$ are strictly monotone and are of constant sign eventually. Observe that $\{x_n\}$ is bounded, $1 < p_n \leq p$ for $n \geq n_0$, by (2), $\{z_n\}$ is bounded. Hence, we may set $\lim_{n\to\infty} z_n = L$ $(-\infty < L < \infty)$.

If $-\infty < L < 0$, then there exist a constant $C > 0$ and a $n_2 \geq n_1$, such that $z_n < -C < 0$ for $n \geq n_2$. Since $\Delta^m z_n < 0$ for $n \geq n_1$ and $\{z_n\}$ is bounded, set $y_n = -z_n > 0$, then $\Delta^m y_n = -\Delta^m z_n > 0$ for $n \geq n_2$, moreover, $\{y_n\}$ is bounded. In view of m is even, it follows, by Lemma 1, that there exists a $n_3 \geq n_2$ and an integer $l = 0$, such that $(-1)^s \Delta^s y_n > 0$

for $s = 0, 1, 2, \ldots, m-1$ and $n \geq n_3$. This implies $(-1)^s \Delta^s z_n < 0$ for $s = 0, 1, 2, \ldots, m-1$ and $n \geq n_3$, in particular, $\Delta^{m-1} z_n > 0$ for $n \geq n_3$.

On the other hand, since $\{x_n\}$ is bounded, we set $\lim_{n\to\infty} \inf x_n = a$ $(0 \leq a < \infty)$. We wish to show that $a > 0$. Otherwise, if $a = 0$, then there is a sequence $\{n_i\}$, $\lim_{i\to\infty} n_i = \infty$ such that $\lim_{i\to\infty} x_{n_i} = a = 0$. By (2), we have

$$x_{n_i+\tau} = z_{n_i+\tau} + p_{n_i+\tau} x_{n_i}. \tag{6}$$

From (6), set $i \to \infty$ and observe that $1 < p_n \leq p$, we have

$$x_{n_i+\tau} \to L < 0, \quad \text{as} \ \ i \to \infty.$$

This contradicts $x_n > 0$ for $n \geq n_1$. Hence $a > 0$ holds, that is $\lim_{n\to\infty} x_n = a > 0$. Then there exist a constant $C_1 > 0$ and a $n_4 \geq n_3$ such that $x_n > C_1 > 0$, and $x_{n-\sigma_i(n)} > C_1 > 0$ for $n \geq n_4$ and $i = 1, 2, \ldots, k$. So by (5) and hypothesis on $f_i(u)$, we have

$$\Delta^m z_n \leq -\sum_{i=1}^{k} f_i(C_1) Q_i(n) \leq -b \sum_{i=1}^{k} Q_i(n) \leq -b Q_j(n), \quad \text{for} \ \ n \geq n_4 \tag{7}$$

where $b = \min_{1\leq i \leq k}\{f_i(C_1)\} > 0$.

By summing (7) from n_4 to n and then set $n \to \infty$, we have

$$\Delta^{m-1} z_n \to -\infty, \quad \text{as} \ \ n \to \infty.$$

This contradicts $\Delta^{m-1} z_n > 0$ for $n \geq n_3$, Hence $-\infty < L < 0$ is impossible.

If $0 < L < \infty$, as in the proof of Lemma 3 for case $0 < L \leq \infty$. We imply that $0 < L < \infty$ is impossible. Hence, $L = 0$ holds, that is, $\lim_{n\to\infty} z_n = 0$ holds.

The rest of the proof is similar to that of Lemma 3, we may get for all large n.

$$(-1)^s \Delta^s z_n < 0 \quad \text{for} \ \ s = 0, 1, 2, \ldots, m.$$

The proof is complete. □

Theorem 1 *Assume that*

(C_1) $0 < p_n \leq p$ *for* $n \geq n_0$ *and some positive constant* p, $0 < p \leq 1$;

(C_2) *There exists a positive constant* λ, *such that* $\liminf_{u\to 0} \frac{f_i(u)}{u} > \lambda$, *for* $i = 1, 2, \ldots, k$;

(C_3) *There exists at least an integer j, $1 \le j \le k$, such that $Q_j(n) > 0$, $\sigma_j(n) \ge \sigma_j > 0$ for $n \ge n_0$ and some positive constant σ_j.*
Moreover, there exists a positive constant k, such that

$$\frac{\lambda}{p}Q_j(n) \ge k^m \text{ and either } k\frac{\sigma_j - \tau - m}{m} > \left(\frac{\sigma_j - \tau}{\sigma_j - \tau + m}\right)^{\frac{\sigma_j-\tau+m}{m}},$$

or

$$a_0 = k\frac{\sigma_j - \tau - m}{m} \le \left(\frac{\sigma_j - \tau}{\sigma_j - \tau + m}\right)^{\frac{\sigma_j-\tau+m}{m}}$$

and

$$k\frac{\sigma_j - \tau}{m} > 1 - \frac{1 - a_0 - \sqrt{1 - 2a_0 - a_0^2}}{2}$$

Then every solution of Eq. (1) *oscillates.*

Proof. Let $\{x_n\}$ be a nonoscillatory solution of Eq. (1). Without loss of generality, assume that $x_n > 0$, $x_{n-\tau} > 0$, $x_{n-\sigma_i(n)} > 0$ $(i = 1, 2, \dots, k)$ for $n \ge n_1 \ge n_0$ (the proof for $x_n < 0$ is similar). From (C_3), we have

$$Q_j(n) \ge \frac{pk^m}{\lambda} > 0, \quad \text{for } n \ge n_0.$$

It follows that

$$\sum_{n=n_0}^{\infty} Q_j(n) = \infty.$$

By Lemma 3, we have

$$\lim_{n\to\infty} z_n = 0,$$

moreover, there exists $n_2 \ge n_1$, such that

$$(-1)^s \Delta^s z_n < 0, \quad \text{for } n \ge n_2 \text{ and } s = 0, 1, 2, \dots, m. \tag{8}$$

In particular, $z_n < 0$ for $n \ge n_2$. So by (2) we have

$$z_n > -p_n x_{n-\tau} \ge -p x_{n-\tau}, \quad \text{for } n \ge n_2.$$

This implies that

$$x_n > -\frac{1}{p} z_{n+\tau} > 0, \quad \text{for } n \ge n_2.$$

Hence, we may take a $n_3 \geq n_2$, such that

$$x_{n-\sigma_i(n)} > -\frac{1}{p} z_{n+\tau-\sigma_i(n)} > 0, \quad \text{for } n \geq n_3 \text{ and } i = 1, 2, \ldots, k. \tag{9}$$

From (1), (2) and (9), we have

$$\Delta^m z_n \leq -\sum_{i=1}^{k} Q_i(n) f_i \left[-\frac{1}{p} z_{n+\tau-\sigma_i(n)} \right], \quad \text{for } n \geq n_3. \tag{10}$$

Since $\lim_{n\to\infty} z_n = 0$, so that

$$\lim_{n\to\infty} \left[-\frac{1}{p} z_{n+\tau-\sigma_i(n)} \right] = 0, \quad \text{for } i = 1, 2, \ldots, k.$$

By (C_2), then there exists $n_4 \geq n_3$, such that, as $n \geq n_4$,

$$\frac{f_i[-\frac{1}{p} z_{n+\tau-\sigma_i(n)}]}{-\frac{1}{p} z_{n+\tau-\sigma_i(n)}} > \lambda > 0, \quad \text{for } i = 1, 2, \ldots, k. \tag{11}$$

From (8), $\Delta z_n > 0$ for $n \geq n_2$, and combining (10), (11), we get

$$\begin{aligned} \Delta^m z_n &\leq -\sum_{i=1}^{k} Q_i(n) f_i \left[-\frac{1}{p} z_{n+\tau-\sigma_i(n)} \right] \\ &\leq \frac{\lambda}{p} \sum_{i=1}^{k} Q_i(n) z_{n+\tau-\sigma_i(n)} \\ &\leq \frac{\lambda}{p} Q_j(n) z_{n+\tau-\sigma_j(n)} \leq \frac{\lambda}{p} Q_j(n) z_{n+\tau-\sigma_j}, \quad \text{for } n \geq n_4. \end{aligned} \tag{12}$$

set $w_n = \sum_{i=1}^{m} (-1)^{i-1} k^{i-1} \Delta^{m-i} z_{n-(i-1)\frac{\sigma_j-\tau}{m}}$.

By (8), $w_n > 0$ for $n \geq n_4$, so that

$$\begin{aligned} \Delta w_n + k w_{n-\frac{\sigma_j-\tau}{m}} &= \sum_{i=1}^{m} (-1)^{i-1} k^{i-1} \Delta^{m-i+1} z_{n-(i-1)\frac{\sigma_j-\tau}{m}} \\ &\quad + \sum_{i=1}^{m} (-1)^{i-1} k^{i} \Delta^{m-i} z_{n-i\frac{\sigma_j-\tau}{m}} \\ &= \Delta^m z_n + \sum_{i=2}^{m} (-1)^{i-1} k^{i-1} \Delta^{m-i+1} z_{n-(i-1)\frac{\sigma_j-\tau}{m}} \\ &\quad - \sum_{i=1}^{m-1} (-1)^{i} k^{i} \Delta^{m-i} z_{n-i\frac{\sigma_j-\tau}{m}} - k^m z_{n+\tau-\sigma_j} \end{aligned}$$

$$= \Delta^m z_n - k^m z_{n+\tau-\sigma_j}, \quad \text{for } n \geq n_4. \tag{13}$$

From (12), (13), and (C_3) we have

$$\begin{aligned} \Delta w_n + k w_{n-\frac{\sigma_j-\tau}{m}} &\leq \frac{\lambda}{p} Q_j(n) z_{n+\tau-\sigma_j} - k^m z_{n+\tau-\sigma_j} \\ &= \left(\frac{\lambda}{p} Q_j(n) - k^m\right) z_{n+\tau-\sigma_j} \leq 0, \quad \text{for } n \geq n_4. \end{aligned}$$

That is

$$\Delta w_n + k w_{n-\frac{\sigma_j-\tau}{m}} \leq 0, \quad \text{for } n \geq n_4. \tag{14}$$

By (C_3) and Lemma 2, (14) has no eventually positive solutions, this contradicts $w_n > 0$ for $n \geq n_4$, and the proof is complete. □

Theorem 2 *Let conditions* (C_1) *and* (C_2) *be satisfied. Moreover, assume that*

(C_4) *There exists at least an integer* j, $1 \leq j \leq k$, *such that* $Q_j(n) > 0$, $\sigma_j(n) \geq \sigma_j > \tau$ *for* $n \geq n_0$ *and some positive constant* σ_j, *and that*

$$\limsup_{n\to\infty} \sum_{s=n+\tau-\sigma_j}^{n} \frac{\lambda}{p} Q_j(s)(s-n+m-2)^{(m-1)} > (m-1)!,$$

where $(s-n+m-2)^{(m-1)}$ *is the usual factorial function. Then every solution of Eq.* (1) *oscillates.*

Proof. Let $\{x_n\}$ be a nonosillatory solution of Eq. (1). Without loss of generality, assume that $x_n > 0$, $x_{n-\tau} > 0$, $x_{n-\sigma_i(n)} > 0$ ($i = 1, 2, \ldots, k$) for $n \geq n_1 \geq n_0$ (the proof when $x_n < 0$, $n \geq n_1$, is similar). Observe that $\sigma_j > \tau > 0$ and $m \geq 2$, we have

$$\sum_{s=n+\tau-\sigma_j}^{n} \frac{\lambda}{p} Q_j(s)(\sigma_j - \tau)^{m-1} \geq \sum_{s=n+\tau-\sigma_j}^{n} \frac{\lambda}{p} Q_j(s)(n-s)^{m-1},$$

using (C_4), we have

$$\limsup_{n\to\infty} \sum_{s=n+\tau-\sigma_j}^{n} Q_j(s) \geq \frac{p(m-1)!}{\lambda(\sigma_j - \tau)^{m-1}} > 0.$$

So that $\sum_{s=n_0}^{\infty} Q_j(s) = \infty$. By Lemma 3, we get $\lim_{n\to\infty} z_n = 0$, and there exists $n_2 \geq n_1$, such that

$$(-1)^s \Delta^s z_n < 0, \quad \text{for } n \geq n_2 \text{ and } s = 0, 1, 2, \ldots, m. \tag{15}$$

In particular, $z_n < 0$ for $n \geq n_2$. As in the proof of Theorem 1, we get that (12) holds, that is

$$\Delta^m z_n \leq \frac{\lambda}{p} Q_j(n) z_{n+\tau-\sigma_j}, \quad \text{for } n \geq n_4. \tag{16}$$

Set $s \geq n \geq n_4$, from (16) and by discrete Taylor's formula[1], we have

$$\begin{aligned} \Delta^m z_s &\leq \frac{\lambda}{p} Q_j(s) z_{n+\tau-\sigma_j} \\ &= \frac{\lambda}{p} Q_j(s) \Bigg[\sum_{i=0}^{m-1} \frac{(s-n+i-1)^{(i)}}{i!} (-1)^i \Delta^i z_{n+\tau-\sigma_j} \\ &\quad - \frac{(-1)^{m-1}}{(m-1)!} \sum_{l=s+\tau-\sigma_j}^{n+\tau-\sigma_j-1} (l+m-1-s-\tau+\sigma_j)^{(m-1)} \Delta^m z_l \Bigg]. \end{aligned} \tag{17}$$

From (15) and (17), we have

$$\Delta^m z_s \leq \frac{-\lambda\,(s-n+m-2)^{(m-1)}}{p(m-1)!} Q_j(s) \Delta^{m-1} z_{n+\tau-\sigma_j}. \tag{18}$$

By summing (18), from $n+\tau-\sigma_j$ to n, we get

$$\begin{aligned} &\Delta^{m-1} z_{n+1} - \Delta^{m-1} z_{n+\tau-\sigma_j} \\ &\quad \leq \frac{-(s-n+m-2)^{(m-1)}}{(m-1)!} \Delta^{m-1} z_{n+\tau-\sigma_j} \sum_{s=n+\tau-\sigma_j}^{n} \frac{\lambda}{p} Q_j(s). \end{aligned} \tag{19}$$

From (15), $\Delta^{m-1} z_{n+1} > 0$ for $n \geq n_4$, It follows, from (19), that

$$\sum_{s=n+\tau-\sigma_j}^{n} \frac{\lambda}{p} Q_j(s) < \frac{(m-1)!}{(s-n+m-2)^{(m-1)}}, \quad \text{for } n \geq n_4.$$

So that

$$\limsup_{n\to\infty} \sum_{s=n+\tau-\sigma_j}^{n} \frac{\lambda}{p} Q_j(s) \leq \frac{(m-1)!}{(s-n+m-2)^{(m-1)}}.$$

This contradicts (C_4), and the proof is complete. □

Using Lemma 4 and following the proof of Theorem 1 and Theorem 2, we have the following results.

Theorem 3 *Let condition* (C_1) *in Theorem* 1 *be replaced by*

(C_5) $1 \leq p_n \leq p$ *for* $n \geq n_0$ *and some positive constant* p.

Then every bounded solution of Eq. (1) *oscillates.*

Theorem 4 *Let condition* (C_1) *in Theorem* 2 *be replaced by* (C_5). *Then every bounded solution of Eq.* (1) *oscillates.*

Example. Consider the equation

$$\triangle^5 \left(x_n - \frac{1}{2} x_{n-1} \right) + 48 x_{n-2} = 0, \quad n = 0, 1, 2, \ldots . \tag{20}$$

So that $m = 5$, $p_n = 1/2$, $\tau = 1$, $\sigma = 2$, $Q(n) = 48$ and $f(u) = u$. It is easy to verify that the conditions of Theorem 1 are satisfied. Therefore (20) has an oscillatory solution. For instance, $\{x_n\} = (-1)^n$ is such an solution.

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