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Blow-analytic SV-sufficiency does not always imply Blow-analytic sufficiency

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Abstract. Thanks to the Kuiper-Kuo-Bochnak-Lojasiewicz theorem, we have known that SV-sufficiency of jets is equivalent to C^0-sufficiency of jets in the case of functions. This fact is compatible with the Thom-Kuo principle. C^0-equivalence generically implies blow-analytic equivalence. Since sufficiency of jets is a generic property, it is natural to ask whether the Thom-Kuo principle holds also in the blow-analytic category. In this note we give a negative answer to this problem.

Key words: blow-analyticity, sufficiency of jets, Fukui's invariant.

1. Introduction

Let $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, and let $f, g : (\mathbf{K}^n, 0) \to (\mathbf{K}, 0)$ be function germs. In the real case they are C^s functions, and in the complex case they are holomorphic functions. We say that f and g are *SV-equivalent* (or $(\mathbf{K}^n, f^{-1}(0))$ and $(\mathbf{K}^n, g^{-1}(0))$ are *C^0-equivalent*), if there is a local homeomorphism $\sigma : (\mathbf{K}^n, 0) \to (\mathbf{K}^n, 0)$ such that $\sigma(f^{-1}(0)) = g^{-1}(0)$. We further say that f and g are *C^0-equivalent* (or more precisely, *R-C^0-equivalent*), if there is a local homeomorphism $\sigma : (\mathbf{K}^n, 0) \to (\mathbf{K}^n, 0)$ such that $f = g \circ \sigma$.

Observation 1.1 *Let $f_t : (\mathbf{K}^n, 0) \to (\mathbf{K}, 0)$ $(t \in I)$ be a family of analytic functions with isolated singularities. Here I is a closed interval. If the topological type of $(\mathbf{K}^n, f_t^{-1}(0))$ is constant, then is the family $\{(\mathbf{K}^n, f_t^{-1}(0))\}_{t \in I}$ topologically trivial?*

(1) *In the complex case the answer is yes for $n \neq 3$. For the constancy of topological types implies μ-constancy* (B. Teissier [19]), *and μ-constancy implies topological triviality of the family for $n \neq 3$* (Lê Dũng Tráng - C.P. Ramanujan [16]). *We have no counterexample in the case $n = 3$. In this note, μ means the Milnor number.*

(2) *In the real case the answer is no. Consider the family $f_t : (\mathbf{R}^2, 0) \to (\mathbf{R}, 0)$ $(t \in \mathbf{R})$ defined by*

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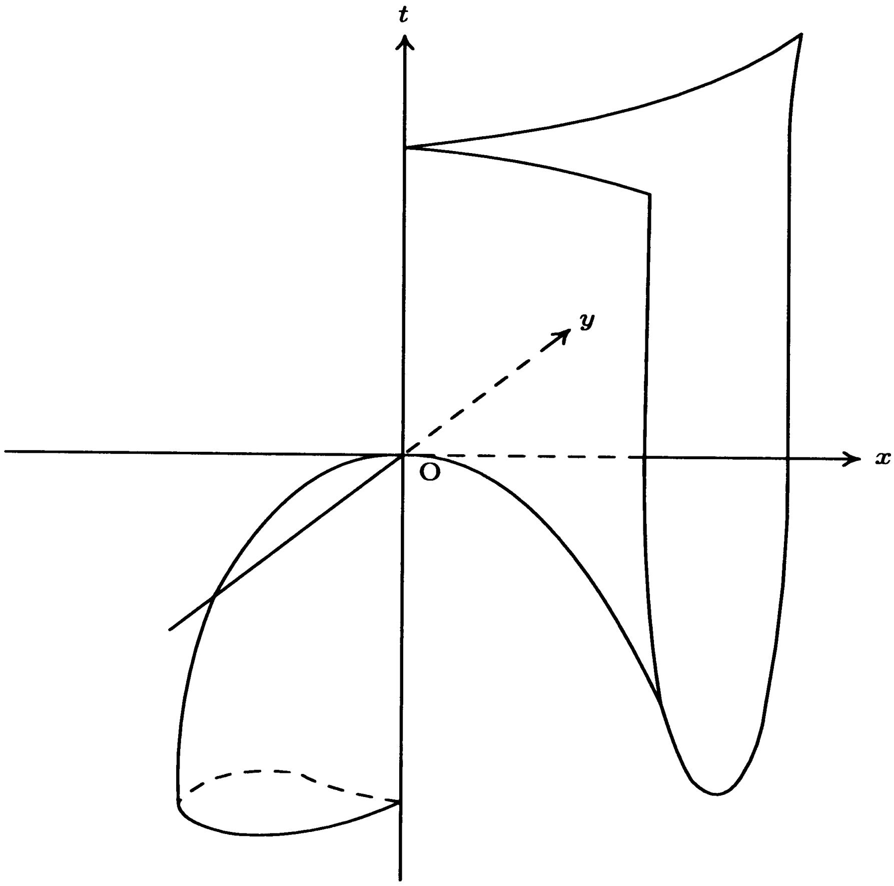
$$f_t(x,y) = y^2 - tx^3 - x^5.$$

Let $F(x,y,t) = f_t(x,y)$*. The figure of* $F^{-1}(0)$ *is the following:*



Then the topological type of $(\mathbf{R}^2, f_t^{-1}(0))$ *is constant, but the family* $\{(\mathbf{R}^2, f_t^{-1}(0))\}_{t\in\mathbf{R}}$ *is not topologically trivial at* $0 \in \mathbf{R}$.

Now we recall the notion of sufficiency of jets. We say that an r-jet $w \in J^r_{\mathbf{K}}(n,1)$ is *SV-sufficient* (resp. C^0*-sufficient*) in $\mathcal{E}$, if any two functions $f, g \in \mathcal{E}$ such that $j^r f(0) = j^r g(0) = w$ are SV-equivalent (resp. C^0-equivalent). Here $\mathcal{E}$ denotes the set of C^s-functions ($s \geq r$) in the real case and the set of holomorphic functions in the complex case. We shall identify r-jets with their polynomial representatives of degree not exceeding r. Concerning the notions above, we have

Theorem 1.2 (N. Kuiper [12], T.C. Kuo [13], J. Bochnak - S. Lojasiewicz [1]). *For an* r*-jet* $w \in J^r_{\mathbf{R}}(n,1)$*, the following conditions are equivalent.*

(1) *w is SV-sufficient in C^r (resp. C^{r+1})-functions.*
(2) *w is C^0-sufficient in C^r (resp. C^{r+1})-functions.*

Theorem 1.3 (S.H. Chang - Y.C. Lu [3], J. Bochnak - W. Kucharz [2]). *For an r-jet $w \in J^r_{\mathbf{C}}(n,1)$, the following conditions are equivalent.*
(1) *w is SV-sufficient in holomorphic functions.*
(2) *w is C^0-sufficient in holomorphic functions.*
(3) *For any holomorphic function f such that $j^r f(0) = w$, $\mu(f) = \mu(w)$.*

Remark 1.4 This theorem is also recovered in the recent paper of A. Parusiński [17].

These results are very reasonable in certain sense. René Thom had an insight into the fact that to every theorem on C^0-sufficiency of jets, there is a similar theorem on V- or SV-sufficiency of jets, in particular, their results coincide in the case of functions. See Kuo [14] for the definition of V-sufficiency of jets. Tzee-Char Kuo is the first singularitist who demonstrated the Thom's insight. Nowadays it is called the *Thom-Kuo Principle* (see [11] also). Therefore the results above are compatible with the Thom-Kuo Principle.

In Theorems 1.2 and 1.3, the implication (2) $\Rightarrow$ (1) is obvious from the definitions of SV- and C^0-sufficiencies of jets. We next consider the implication (1) $\Rightarrow$ (2). Let $f, g : (\mathbf{K}^n, 0) \to (\mathbf{K}, 0)$ be functions such that $j^r f(0) = j^r g(0) = w$, and let I be a closed disk in $\mathbf{K}$ containing the interval $[0,1]$. Define $f_t : (\mathbf{K}^n, 0) \to (\mathbf{K}, 0)$ ($t \in I$) by $f_t(x) = (1-t)f(x) + tg(x)$. Then, by the proof of this implication we have the following property:

> *SV-sufficiency of w implies the topological triviality of $\{f_t\}_{t \in I}$* as a family of functions.

This means that the constancy of topological type of zero-sets for *any* higher degree's direction implies the topological triviality of the family of functions, nevertheless the constancy of topological type of zero-sets does not always imply even topological triviality of zero-sets in the real case, as seen in Observation 1.1. (We can say that SV-sufficiency of w controls not only its realizations but also the homotopy connecting them. So this is also one of reasons why we consider the question below.)

Let $f, g : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ be analytic functions. We defined the notions of C^0-equivalence and SV-equivalence for analytic functions. If C^0-equivalence (resp. C^0-equivalence of embedded zero-sets) is attained from an analytic isomorphism via blowings-up, then we call it Blow-analytic equivalence (resp. Blow-analytic SV-equivalence). Namely, we say that f and g are *Blow-analytically equivalent* (resp. *Blow-analytically SV-equivalent*), if there are two successive blowings-up with smooth centers $\beta : \mathcal{M} \to \mathbf{R}^n$ and $\beta' : \mathcal{M}' \to \mathbf{R}^n$ and an analytic isomorphism $\Phi : \mathcal{M} \to \mathcal{M}'$ which induces a local homeomorphism $\phi : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ such that $f = g \circ \phi$ (resp. $\phi(f^{-1}(0)) = g^{-1}(0)$).

We define the notions of Blow-analytic sufficiency of jets and Blow-analytic SV-sufficiency of jets in the similar way to C^0-sufficiency of jets and SV-sufficiency of jets, respectively. Blow-analytic equivalence preserves more quantities than C^0-equivalence. But the difference is not so large. In fact, C^0-equivalence generically implies Blow-analytic equivalence in the sense that the Thom-Varčenko type's theorem holds. Thus this gives rise to the following question naturally:

Question 1.5 *Does Blow-analytic SV-sufficiency imply Blow-analytic sufficiency in the function case?*

The answer is no. In other words, the Thom-Kuo Principle does not hold in the Blow-analytic category.

Example 1.6 Let $w = x^3 + 3xy^{2k} \in J^{2k+1}_{\mathbf{R}}(2,1)$ $(k \geq 3)$. Then w is Blow-analytically SV-sufficient, but not Blow-analytically sufficient.

These jets look like the so-called Koike-Kucharz jets ([10]): $w = x^3 \pm 3xy^{2k-1} \in J^{2k}_{\mathbf{R}}(2,1)$ $(k \geq 3)$, but they are not the same.

2. Proof of Example 1.6

Here we show the case $k = 3$. Therefore let $w = x^3 + 3xy^6 \in J^7_{\mathbf{R}}(2,1)$. The other cases follow similarly. We first make the following remark.

Remark 2.1 Any analytic function $f : (\mathbf{R}^2, 0) \to (\mathbf{R}, 0)$ with $j^7 f(0) = w$ has an isolated singularity at $0 \in \mathbf{R}^2$.

This is not valid for $w = x^3 - 3xy^6 \in J^7_{\mathbf{R}}(2,1)$. For instance, $f = x^3 - 3xy^6 + 2y^9$ does not have an isolated singularity at $0 \in \mathbf{R}^2$.

We next recall some important results on Blow-analytic equivalence and

Blow-analytic SV-equivalence to show this example.

Theorem 2.2 (T. Fukui - E. Yoshinaga [4], T. Fukui - L. Paunescu [8]). *Given a system of weights* $\alpha = (\alpha_1, \alpha_2)$. *Let* $f_t : (\mathbf{R}^2, 0) \to (\mathbf{R}, 0)$ *be an analytic function for* $t \in I = [0,1]$. *Suppose that for each* $t \in I$, *the weighted initial form of* f_t *with respect to* α *is of the same weighted degree and has an isolated singularity at* $0 \in \mathbf{R}^2$. *Then* $\{f_t\}_{t \in I}$ *is Blow-analytically trivial over* I.

We can show 2.2 using the result in [4]. Fukui and Paunescu [8] proved that under the same hypothesis as the theorem above, $\{f_t\}_{t\in I}$ is blow-analytically trivial over I for general n variables. Note that blow-analytic equivalence is a different notion from Blow-analytic equivalence and the latter always implies the former. For the details on blow-analyticity, consult the survey article [6].

Here we recall the definition of blow-analytic equivalence. Let $g : U \to \mathbf{R}$, U open in $\mathbf{R}^n$, be a continuous function. We say that g is *blow-analytic*, if there exists a multi-blowing-up β such that the composition $g \circ \beta$ is analytic. Let $h : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ be a local homeomorphism. We say that h is *blow-analytic*, if the components of both h and h^{-1} are blow-analytic functions. Given $f, g : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$, we say that they are *blow-analytically equivalent*, if there exists such an h with $f = g \circ h$.

T. Fukui ([5]) gave some invariants for Blow-analytic equivalence. One of them is defined as follows:

For an analytic function $f : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$, set

$$A(f) = \{O(f \circ \lambda) \in \mathbf{N} \cup \{\infty\} \mid \lambda : (\mathbf{R}, 0) \to (\mathbf{R}^n, 0)\ C^\omega\}.$$

Then we have

Theorem 2.3 (Fukui's invariant). *Suppose that analytic functions* $f, g : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ *are Blow-analytically equivalent* (*or blow-analytically equivalent*). *Then* $A(f) = A(g)$.

On the other hand, concerning Blow-analytic SV-equivalence, we have

Theorem 2.4 (M. Kobayashi - T.C. Kuo [9]). *Let* $(\mathbf{R}^2, \{f = 0\})$ *and* $(\mathbf{R}^2, \{g = 0\})$ *be germs of analytic curves such that their complexifications have only one branch. Then they are Blow-analytically equivalent, namely,* f *and* g *are Blow-analytically* SV*-equivalent.*

Before starting the proof of the example, we make one more remark.

Remark 2.5 In the two variables case, Blow-analytic equivalence and Blow-analytic SV-equivalence are equivalence relations. It's a hard problem to know whether they are equivalence relations or not in general. On the other hand, blow-analytic equivalence and blow-analytic SV-equivalence are equivalence relations in the general case. For the details, see [6] and [15].

We first show w is Blow-analytically SV-sufficient. Let $f : (\mathbf{R}^2, 0) \to (\mathbf{R}, 0)$ be an analytic function such that $j^7 f(0) = x^3 + 3xy^6$. Consider the Taylor expansion of f:

$$f(x, y) = x^3 + 3xy^6 + \sum_{i=0}^{8} a_i x^{8-i} y^i + \sum_{i=0}^{9} b_i x^{9-i} y^i + \cdots .$$

In the case $a_8 \neq 0$, $x^3 + a_8 y^8$ is the weighted initial form of f with respect to the system of weights $(\frac{1}{3}, \frac{1}{8})$ and has an isolated singularity. By Theorem 2.2, f is Blow-analytically equivalent to $x^3 + a_8 y^8$. On the other hand, $x^3 + a_8 y^8$ is linearly equivalent to $x^3 + y^8$ (resp. $x^3 - y^8$) in the case $a_8 > 0$ (resp. $a_8 < 0$). Since $x^3 + y^8$ and $x^3 - y^8$ are RL-linearly equivalent, they are Blow-analytically SV-equivalent. It follows from Remark 2.5 that f is Blow-analytically SV-equivalent to $x^3 + y^8$.

In the case $a_8 = 0$, it follows from Remark 2.1 and Theorem 2.2 that f is Blow-analytically equivalent to $x^3 + 3xy^6$. As set-germs, $(\mathbf{R}^2, \{x^3 + 3xy^6 = 0\}) = (\mathbf{R}^2, \{x = 0\})$. Therefore $(\mathbf{R}^2, \{f = 0\})$ is Blow-analytically equivalent to $(\mathbf{R}^2, \{x = 0\})$.

By Theorem 2.4, it is easy to see that $(\mathbf{R}^2, \{x^3 + y^8 = 0\})$ is Blow-analytically equivalent to $(\mathbf{R}^2, \{x = 0\})$. Therefore it follows from Remark 2.5 that $w = x^3 + 3xy^6 \in J^7_{\mathbf{R}}(2, 1)$ is Blow-analytically SV-sufficient.

We next consider the Fukui's invariants of $x^3 + y^8$ and $x^3 + 3xy^6$. Then we have

$$A(x^3 + y^8) = \{3, 6, 8, 9, 12, 15, 16, 18, 21, 24, 25, 26, 27, \dots, \infty\}$$
$$A(x^3 + 3xy^6) = \{3, 6, 9, 10, 11, 12, \dots, \infty\}.$$

The former contains 8, but the latter does not contain 8. By Theorem 2.3, $x^3 + y^8$ is not Blow-analytically equivalent to $x^3 + 3xy^6$. On the other hand, as seen in the above, $x^3 + y^8$ is Blow-analytically equivalent to $x^3 + 3xy^6 + y^8$. Therefore it follows from Remark 2.5 that $w = x^3 + 3xy^6 \in$

$J^7_{\mathbf{R}}(2,1)$ is not Blow-analytically sufficient.

Remark 2.6 (1) It follows from the proof above that in Example 1.5 we can replace "Blow-analytically" by "blow-analytically."

(2) We say that an analytic function $f : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ is a Nash function if the graph of f is semialgebraic in $\mathbf{R}^n \times \mathbf{R}$. We can define the notions of Blow-Nash sufficiency and Blow-Nash SV-sufficiency in Nash functions.

As a matter of course, the above $w = x^3 + 3xy^6 \in J^7_{\mathbf{R}}(2,1)$ is Blow-analytically SV-sufficient in Nash functions. Then we can approximate the Blow-analytic SV-equivalence between any two Nash realizations of w by a Blow-Nash SV-equivalence, using the similar arguments in [7]. Therefore in Example 1.5, we can replace also "Blow-analytically" by "Blow-Nash."

Remark 2.7 Let s be the number of elements of the quotient set of

$$\{f : (\mathbf{R}^2, 0) \to (\mathbf{R}, 0)\ C^\omega \mid j^7 f(0) = x^3 + 3xy^6\}$$

by Blow-analytic equivalence. In the proof above, we have shown that s is equal to the number of elements of the quotient set of

$$\{x^3 + 3xy^6, x^3 + y^8, x^3 - y^8\}$$

by Blow-analytic equivalence and $s \geq 2$. We can easily see that $x^3 + y^8$ is $R - C^0$-equivalent to $x^3 - y^8$ and $A(x^3 + y^8) = A(x^3 - y^8)$. In the sense of the Blow-analytic type, we cannot distinguish $x^3 + y^8$ from $x^3 - y^8$, using only $A(f)$. But by the invariant on the graph introduced by Fukui [5] we see that $x^3 + y^8$ is not Blow-analytically equivalent to $x^3 - y^8$. Therefore $s = 3$.

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