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A note on the Schiffer conjecture

Robert DALMASSO

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Abstract. A domain $\Omega\subset\mathbb{R}^{n}$ $(n\geq 2)$ with smooth connected boundary is said to have the Schiffer property if there is no $\lambda>0$ such that the overdetermined boundary value problem $\Delta u+\lambda u+1=0$ in Ω, $u=\frac{\partial u}{\partial\nu}=0$ on $\partial\Omega$ where ν is the exterior normal to $\partial\Omega$, has a solution. We prove integral identities for the exterior normal to the boundary of a domain Ω lacking the Schiffer property.

Key words: Schiffer conjecture, Pompeiu problem.

1. Introduction

The Schiffer conjecture (cf. Yau [15, problem 80]) is that balls are the only domains $\Omega\subset\mathbb{R}^{n}$ $(n\geq 2)$ with smooth connected boundaries such that the overdetermined boundary value problem

$$\Delta u+\lambda u+1=0 \quad \text{in} \quad \Omega,\ \lambda>0, \tag{1.1}$$

$$u=\frac{\partial u}{\partial\nu}=0 \quad \text{on} \quad \partial\Omega, \tag{1.2}$$

where ν is the exterior normal to $\partial\Omega$, has a solution. Ω is said to have the Schiffer property if there is no $\lambda>0$ such that (1.1)–(1.2) has a solution. It is well known that balls do not have the Schiffer property. Indeed let J_{μ} denote the μ-th Bessel function and let $\lambda>0$ be such that $J_{n/2}(\sqrt{\lambda})=0$. Then the function

$$u(x)=\frac{1}{\lambda}\left(\frac{J_{\frac{n}{2}-1}(\sqrt{\lambda}|x|)}{J_{\frac{n}{2}-1}(\sqrt{\lambda})|x|^{\frac{n}{2}-1}}-1\right), \quad x\in\Omega$$

satisfies (1.1)–(1.2) when Ω is the unit ball. The Schiffer problem consists of deciding which sets Ω (with smooth connected boundaries) have the Schiffer property.

The Schiffer property is related to the Pompeiu property. A nonempty bounded open set $\Omega\subset\mathbb{R}^{n}$ $(n\geq 2)$ is said to have the Pompeiu property if

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and only if the only continuous function f on $\mathbb{R}^n$ for which the integral of f over $\sigma(\Omega)$ is zero for all rigid motions σ of $\mathbb{R}^n$ is $f \equiv 0$. The Pompeiu problem asks: which sets Ω have the Pompeiu property?

Now we assume that Ω is a nonempty bounded open set with Lipschitz boundary $\partial\Omega$, and that the complement of $\overline{\Omega}$ is connected. When $\overline{\Omega}$ is rotationally symmetric this implies that $\overline{\Omega} = \overline{B}(a, R)$, the closed ball of center a and radius R for some $a \in \mathbb{R}^n$, $0 < R < \infty$. Then Williams [13] proved that the possession of the Pompeiu property is equivalent to the possession of the Schiffer property (see also Berenstein [1] when $\partial\Omega \in C^{2+\varepsilon}$). Williams in [14] proved that the existence of a solution to (1.1), (1.2) implies that $\partial\Omega$ is real analytic.

References and information about various aspects of the Pompeiu problem can be found in the surveys by Zalcman [16], [17] and in the paper of Berenstein [2]. Let us mention also the remarkable results proved by Garofalo and Segala [9–11] and Ebenfelt [6–8] in the 2-dimensional case.

To explain our result we first recall a theorem obtained in [5].

Theorem 1.1 [5] *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a nonempty bounded open set such that $\partial\Omega \in C^2$. Assume that problem* (1.1), (1.2) *has a solution $u \in C^2(\overline{\Omega})$. Then, for any $y \in \mathbb{R}^n$, we have*

$$\int_{\partial\Omega} \nu_j^2(x)(x-y).\nu(x)\,d\sigma = \int_{\partial\Omega} \nu_k^2(x)(x-y).\nu(x)\,d\sigma, \quad j,k \in \{1,\ldots,n\}, \tag{1.3}$$

and

$$\int_{\partial\Omega} \nu_j(x)\nu_k(x)(x-y).\nu(x)\,d\sigma = 0, \quad j \neq k, \tag{1.4}$$

where $\nu = (\nu_1, \ldots, \nu_n)$ is the exterior normal to $\partial\Omega$.

From Theorem 1.1 we deduced using elementary calculations that ellipsoids and certain solid tori in $\mathbb{R}^n$ have the Pompeiu property. We also gave examples of domains in $\mathbb{R}^n$ ($n \geq 2$) having the Pompeiu property.

Very little was known before about the Pompeiu problem in $\mathbb{R}^n$ for $n \geq 3$. It was proved in [13] that proper ellipsoids have the Pompeiu property (see [4] when $n = 2$ and also Johnsson [12] when $n \geq 2$). Finally Berenstein and Khavinson [3] proved that certain tori in $\mathbb{R}^4$ have the Pompeiu property.

In this note we first prove the following theorem in Section 2.

Theorem 1.2 *Let $\Omega \subset \boldsymbol{R}^n$, $n \geq 2$, be a nonempty bounded open set such that $\partial\Omega \in C^2$. Assume that problem* (1.1), (1.2) *has a solution $u \in C^2(\overline{\Omega})$. Then, for any $y \in \mathbb{R}^n$, we have*

$$\int_{\partial\Omega} \nu_p(x)\nu_q(x)((x_j - y_j)\nu_k(x) - (x_k - y_k)\nu_j(x))\, d\sigma = 0, \tag{1.5}$$

for $p, q \in \{1, \ldots, n\}$ and $j \neq k$, where $\nu = (\nu_1, \ldots, \nu_n)$ is the exterior normal to $\partial\Omega$.

Then in Sections 3 and 4 we show that ellipsoids and certain tori in $\mathbb{R}^n$ have the Schiffer property. The proofs are simpler than in [5]. Finally in Section 5 we examine the case of planar domains: We show that conditions (1.3)–(1.4) are equivalent to condition (1.5).

2. Proof of Theorem 1.2

Let $u \in C^2(\overline{\Omega})$ be a solution of the overdetermined problem (1.1), (1.2). Since $u = \partial u/\partial\nu = 0$ on $\partial\Omega$, we can write[1]

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{\partial^2 u}{\partial \nu^2}\nu_i\nu_j \quad \text{on } \partial\Omega \quad \text{for } i, j \in \{1, \ldots, n\}.$$

The following functions

$$z = \frac{\partial^2 u}{\partial x_p \partial x_q} \quad \text{and} \quad v = (x_k - y_k)\frac{\partial u}{\partial x_j} - (x_j - y_j)\frac{\partial u}{\partial x_k}$$

satisfy

$$\Delta z + \lambda z = 0 \quad \text{in } \Omega, \quad z = -\nu_p\nu_q \quad \text{on } \partial\Omega,$$

and

$$\begin{aligned} &\Delta v + \lambda v = 0 \quad \text{in } \Omega, \quad v = 0, \\ &\frac{\partial v}{\partial \nu} = (x_j - y_j)\nu_k - (x_k - y_k)\nu_j \quad \text{on } \partial\Omega. \end{aligned}$$

Using Green's formula we can write

$$\lambda \int_\Omega vz\, dx = -\int_\Omega v\Delta z\, dx$$

[1]See Pucci P. and Serrin J.: A general variational identity, Indiana University Mathematics Journal, 35-3 (1986), p. 699.

$$= -\int_\Omega z\Delta v\,dx - \int_{\partial\Omega}\left(v\frac{\partial z}{\partial\nu} - z\frac{\partial v}{\partial\nu}\right)d\sigma$$
$$= \lambda\int_\Omega zv\,dx - \int_{\partial\Omega}\nu_p(x)\nu_q(x)((x_j-y_j)\nu_k(x) - (x_k-y_k)\nu_j(x))\,d\sigma$$

which gives (1.5).

3. Ellipsoids in $\boldsymbol{R}^n$ ($n \geq 2$)

Theorem 3.1 *Let $a_j > 0$, $j = 1,\dots,n$ and assume that $a_j \neq a_k$ for some $j \neq k$. Then the ellipsoid*

$$\Omega = \left\{x \in \boldsymbol{R}^n;\ \sum_{j=1}^n \frac{x_j^2}{a_j^2} < 1\right\}$$

has the Schiffer property.

Proof. Using Theorem 1.2 it is enough to show that (1.5) does not hold. Let $a_r = \min\{a_j;\ j = 1,\dots,n\}$ and $a_s = \max\{a_j;\ j = 1,\dots,n\}$. Our assumption implies that $r \neq s$.

We denote by $\mu = (\mu_1,\dots,\mu_n)$ the exterior normal to $\partial B(0,R)$. Using polar coordinates we can write

$$\begin{aligned}
\mu_1 &= \cos\theta_1\cos\theta_2\cdots\cos\theta_{n-2}\cos\theta_{n-1}\\
\mu_2 &= \cos\theta_1\cos\theta_2\cdots\cos\theta_{n-2}\sin\theta_{n-1}\\
\mu_3 &= \cos\theta_1\cos\theta_2\cdots\cos\theta_{n-3}\sin\theta_{n-2}\\
&\ \vdots\\
\mu_{n-1} &= \cos\theta_1\sin\theta_2\\
\mu_n &= \sin\theta_1
\end{aligned}$$

where $-\frac{\pi}{2} \leq \theta_1,\dots,\theta_{n-2} \leq \frac{\pi}{2}$ (if $n \geq 3$) and $-\pi \leq \theta_{n-1} < \pi$. We can parametrize $\partial\Omega$ by

$$x_1 = a_1\mu_1,\dots,x_n = a_n\mu_n.$$

Then the exterior normal to $\partial\Omega$ is given by $\nu = (\nu_1,\dots,\nu_n)$:

$$\nu_j = \frac{a_1\cdots a_{j-1}a_{j+1}\cdots a_n\mu_j}{(a_2^2\cdots a_n^2\mu_1^2 + \cdots + a_1^2\cdots a_{n-1}^2\mu_n^2)^{1/2}},$$

for $j = 1, \dots, n$. Let

$$I = \int_{\partial\Omega} \nu_r(x)\nu_s(x)(x_r\nu_s(x) - x_s\nu_r(x))\, d\sigma.$$

We have

$$\begin{aligned} d\sigma &= (a_2^2\mu_1^2 + a_1^2\mu_2^2)^{1/2} d\theta_1 \quad \text{if} \;\; n = 2, \\ d\sigma &= \cos^{n-2}\theta_1 \cdots \cos\theta_{n-2} \\ &\qquad \times (a_2^2 \cdots a_n^2\mu_1^2 + \cdots + a_1^2 \cdots a_{n-1}^2\mu_n^2)^{1/2} d\theta_1 \cdots d\theta_{n-1} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if} \;\; n \geq 3. \end{aligned}$$

When $n = 2$ we have

$$I = a_r a_s(a_r^2 - a_s^2) \int_{-\pi}^{\pi} \frac{\cos^2\theta \sin^2\theta}{a_2^2\cos^2\theta + a_1^2\sin^2\theta}\, d\theta \neq 0.$$

If $n \geq 3$ we can assume that $r < s$ and we easily obtain

$$\begin{aligned} I &= a_1^3 \cdots a_{r-1}^3 a_r a_{r+1}^3 \cdots a_{s-1}^3 a_s a_{s+1}^3 \cdots a_n^3(a_r^2 - a_s^2) \int_{-\pi}^{\pi} d\theta_{n-1} \\ &\int_{-\pi/2}^{\pi/2} d\theta_{n-2} \cdots \int_{-\pi/2}^{\pi/2} d\theta_1 \frac{\mu_r^2\mu_s^2 \cos^{n-2}\theta_1 \cdots \cos\theta_{n-2}}{a_2^2 \cdots a_n^2\mu_1^2 + \cdots + a_1^2 \cdots a_{n-1}^2\mu_n^2} \neq 0. \end{aligned}$$

The proof of the theorem is complete. □

4. Solid Tori in $\mathbb{R}^n$

We consider a special kind of tori in $\mathbb{R}^n$, $n \geq 3$. Let $a > R > 0$ and let $D(a, R)$ denote the disk of center $(a, 0, \dots, 0)$ and radius R in the plane $x_2 = \cdots = x_{n-1} = 0$ of $\mathbb{R}^n$. By rotating this disk about the x_n-axis in $\mathbb{R}^n$ we obtain a torus Ω of equation

$$\left(\sqrt{x_1^2 + \cdots + x_{n-1}^2} - a\right)^2 + x_n^2 < R^2. \tag{4.1}$$

Theorem 4.1 *Let $a > R > 0$ and let Ω be the solid torus in $\mathbb{R}^n$ defined by* (4.1), *then Ω has the Schiffer property.*

Proof. We can parametrize $\partial\Omega$ by

$$x_1 = (a + R\cos\theta_{n-1})\cos\theta_1 \cdots \cos\theta_{n-3}\cos\theta_{n-2}$$

$$\begin{aligned}
x_2 &= (a + R\cos\theta_{n-1})\cos\theta_1 \cdots \cos\theta_{n-3}\sin\theta_{n-2} \\
&\ \vdots \\
x_{n-2} &= (a + R\cos\theta_{n-1})\cos\theta_1 \sin\theta_2 \\
x_{n-1} &= (a + R\cos\theta_{n-1})\sin\theta_1 \\
x_n &= R\sin\theta_{n-1}
\end{aligned}$$

where $-\frac{\pi}{2} < \theta_1, \dots, \theta_{n-3} < \frac{\pi}{2}$ (if $n \geq 4$) and $-\pi \leq \theta_{n-2}, \theta_{n-1} < \pi$. Then the exterior normal to $\partial\Omega$ is given by $\nu = (\nu_1, \dots, \nu_n)$:

$$\begin{aligned}
\nu_1 &= \cos\theta_{n-1}\cos\theta_1\cos\theta_2 \cdots \cos\theta_{n-3}\cos\theta_{n-2} \\
\nu_2 &= \cos\theta_{n-1}\cos\theta_1\cos\theta_2 \cdots \cos\theta_{n-3}\sin\theta_{n-2} \\
&\ \vdots \\
\nu_{n-2} &= \cos\theta_{n-1}\cos\theta_1\sin\theta_2 \\
\nu_{n-1} &= \cos\theta_{n-1}\sin\theta_1 \\
\nu_n &= \sin\theta_{n-1}
\end{aligned}$$

Using Theorem 1.2 it is enough to show that

$$J_n = \int_{\partial\Omega} \nu_1(x)\nu_n(x)(x_1\nu_n(x) - x_n\nu_1(x))\, d\sigma \neq 0.$$

We have

$$\begin{aligned}
d\sigma &= R(a + R\cos\theta_2)d\theta_1 d\theta_2 \quad \text{if} \ \ n = 3, \\
d\sigma &= R(a + R\cos\theta_{n-1})^{n-2}\cos^{n-3}\theta_1 \cdots \cos\theta_{n-3} d\theta_1 \cdots d\theta_{n-1} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if} \ \ n \geq 4.
\end{aligned}$$

Clearly

$$\begin{aligned}
J_3 &= aR\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}(a + R\cos\theta_2)\cos^2\theta_1\cos\theta_2\sin^2\theta_2\, d\theta_1 d\theta_2 \\
&= aR^2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\cos^2\theta_1\cos^2\theta_2\sin^2\theta_2\, d\theta_1 d\theta_2 \\
&= \frac{aR^2\pi^2}{4} \neq 0.
\end{aligned}$$

Now if $n \geq 4$, we write

$$\begin{aligned} J_n &= aR \int_{-\pi}^{\pi} d\theta_{n-1} \int_{-\pi}^{\pi} d\theta_{n-2} \int_{-\pi/2}^{\pi/2} d\theta_{n-3} \cdots \\ &\quad \int_{-\pi/2}^{\pi/2} d\theta_1 [(a + R\cos\theta_{n-1})^{n-2} \\ &\qquad \times \cos\theta_{n-1} \sin^2\theta_{n-1} \cos^{n-1}\theta_1 \cdots \cos^2\theta_{n-2}] \\ &= A_n \int_{-\pi}^{\pi} (a + R\cos\theta)^{n-2} \cos\theta \sin^2\theta \, d\theta, \end{aligned}$$

where $A_n > 0$. We have

$$J_n = \sum_{0 < 2j+1 \leq n-2} \binom{n-2}{2j+1} a^{n-3-2j} R^{2j+1} \int_{-\pi}^{\pi} \cos^{2j+2}\theta \sin^2\theta \, d\theta > 0.$$

The proof of the theorem is complete. □

Remark 1. Notice that, as in [5, Section 6], we can extend the example of the present Section to domains bounded by hypersurfaces of revolution.

5. Planar domains

In this Section we consider the case of planar domains. We shall prove that conditions (1.3)–(1.4) are equivalent to condition (1.5).

Let $x = x(s) = (x_1(s), x_2(s))$, $s \in [0, L]$, be a parametrization of $\partial\Omega$ by arc length. Now we denote by $\nu = \nu(s) = (\nu_1(s), \nu_2(s))$, $s \in [0, L]$ the outward normal to $\partial\Omega$ at $x(s)$. We have

$$\nu_1(s) = x_2'(s), \quad \nu_2(s) = -x_1'(s),$$

and the Frenet formulas

$$x''(s) = -\kappa(s)\nu(s), \quad \nu'(s) = \kappa(s)x'(s)$$

where $\kappa = \kappa(s)$ denotes the curvature.

1) Assume first that (1.3) and (1.4) hold for all $y \in \mathbb{R}^2$. Taking $y = 0$ we obtain

$$\int_0^L {x_1'}^2 (x_1 x_2' - x_2 x_1') \, ds = \int_0^L {x_2'}^2 (x_1 x_2' - x_2 x_1') \, ds, \tag{5.1}$$

and

$$\int_0^L x_1' x_2' (x_1 x_2' - x_2 x_1')\, ds = 0. \tag{5.2}$$

Therefore, for any $y \in \mathbb{R}^2$ we have

$$\int_0^L {x_1'}^2 (y.\nu)\, ds = \int_0^L {x_2'}^2 (y.\nu)\, ds, \tag{5.3}$$

and

$$\int_0^L x_1' x_2' (y.\nu)\, ds = 0. \tag{5.4}$$

Taking $y = (1,0)$ and $y = (0,1)$ in (5.4) we get

$$\int_0^L {x_1'}^2 x_2'\, ds = \int_0^L x_1' {x_2'}^2\, ds = 0. \tag{5.5}$$

Then, taking $y = (1,0)$ and $y = (0,1)$ in (5.3) and using (5.5) we obtain

$$\int_0^L {x_1'}^3\, ds = \int_0^L {x_2'}^3\, ds = 0. \tag{5.6}$$

From (5.5) and (5.6) we deduce that for all $y \in \mathbb{R}^2$ we have

$$\int_0^L x_p' x_q' ((x_1 - y_1) x_1' + (x_2 - y_2) x_2')\, ds = \int_0^L x_p' x_q' (x_1 x_1' + x_2 x_2')\, ds, \tag{5.7}$$

for $p, q \in \{1, 2\}$. Then it is enough to show that (5.1) and (5.2) imply that the right hand side in (5.7) is equal to zero.

Lemma 5.1 *We have:*

$$\int_0^L x_p' x_q' (x_1 x_1' + x_2 x_2')\, ds = 0 \quad \textit{for} \quad p, q \in \{1, 2\}$$

is equivalent to

$$\int_0^L \kappa (x_1^2 + x_2^2)({x_1'}^2 - {x_2'}^2)\, ds = 0 \tag{5.8}$$

and

$$\int_0^L \kappa (x_1^2 + x_2^2) x_1' x_2'\, ds = 0. \tag{5.9}$$

Proof. Integrating by parts we can write

$$0 = \int_0^L x_p' x_q' (x_1 x_1' + x_2 x_2')\, ds = \int_0^L (x_1^2 + x_2^2)(x_p'' x_q' + x_p' x_q'')\, ds$$

for $p, q \in \{1, 2\}$. Using the Frenet formulas and taking $p = q = 1$ and $p = 1$, $q = 2$, the lemma follows.

Now from (5.2) we deduce that

$$\begin{aligned} 0 &= \int_0^L ((x_1^2)' {x_2'}^2 - (x_2^2)' {x_1'}^2)\, ds = \int_0^L (x_1^2 x_2' x_2'' - x_2^2 x_1' x_1'')\, ds \\ &= \int_0^L \kappa (x_1^2 + x_2^2) x_1' x_2'\, ds \end{aligned}$$

which proves (5.9). Writing

$$\begin{aligned} &\int_0^L {x_1'}^2 (x_1 x_2' - x_2 x_1')\, ds \\ &\quad = \int_0^L \left(\frac{1}{2} (x_1^2)' x_1' x_2' - x_2 {x_1'}^3 \right) ds \\ &\quad = -\frac{1}{2} \int_0^L x_1^2 (x_1'' x_2' + x_1' x_2'')\, ds - \int_0^L x_2 {x_1'}^3\, ds \\ &\quad = -\frac{1}{2} \int_0^L \kappa x_1^2 ({x_1'}^2 - {x_2'}^2)\, ds - \int_0^L x_2 {x_1'}^3\, ds, \end{aligned}$$

and in the same way

$$\int_0^L {x_2'}^2 (x_1 x_2' - x_2 x_1')\, ds = \frac{1}{2} \int_0^L \kappa x_2^2 ({x_1'}^2 - {x_2'}^2)\, ds + \int_0^L x_1 {x_2'}^3\, ds,$$

and then using (5.1) we get

$$\frac{1}{2} \int_0^L \kappa (x_1^2 + x_2^2)({x_1'}^2 - {x_2'}^2)\, ds + \int_0^L (x_1 {x_2'}^3 + x_2 {x_1'}^3)\, ds = 0. \tag{5.10}$$

We have

$$\begin{aligned} \int_0^L x_1 {x_2'}^3\, ds &= -\int_0^L x_2 (x_1' {x_2'}^2 + 2 x_1 x_2' x_2'')\, ds \\ &= -\int_0^L x_2 (x_1' {x_2'}^2 + 2\kappa x_1 x_1' x_2')\, ds, \end{aligned}$$

and in the same way

$$\int_0^L x_2 {x'_1}^3 \, ds = -\int_0^L x_1 (x'_2 {x'_1}^2 - 2\kappa x_2 x'_1 x'_2) \, ds.$$

Therefore we have

$$\begin{aligned}\int_0^L (x_1 {x'_2}^3 + x_2 {x'_1}^3) \, ds &= -\int_0^L (x_2 x'_1 {x'_2}^2 + x_1 x'_2 {x'_1}^2) \, ds \\ &= -\int_0^L x'_1 x'_2 (x_1 x'_1 + x_2 x'_2) \, ds \\ &= \frac{1}{2}\int_0^L (x_1^2 + x_2^2)(x''_1 x'_2 + x'_1 x''_2) \, ds \\ &= \frac{1}{2}\int_0^L \kappa (x_1^2 + x_2^2)({x'_1}^2 - {x'_2}^2) \, ds.\end{aligned}$$

Now using (5.10) we deduce (5.8). Then by Lemma 5.1 we have proved that (1.3) and (1.4) imply (1.5).

2) To prove that (1.5) implies (1.3)–(1.4), the arguments are analogous. □

Remark 2. We refer the reader to [5, Section 5] for a detailed discussion of the case of planar domains.

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Laboratoire LMC-IMAG - Equipe EDP
Tour IRMA-BP 53
F-38041 Grenoble Cedex 9, France
E-mail: Robert.Dalmasso@imag.fr