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Some operators on Lorentz spaces

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Abstract. It is shown that the spaces $A(p,q)$ and $M(p,q)$ defined by Chen and Lai [1] coincide for $1<p<2$ and $1<q<\infty$. Also the Banach algebraic properties of Lorentz-improving operators are investigated.

Key words: Lorentz space, Fourier-Stietjes transform, operating function.

1. Introduction

Let G be a locally compact abelian group (LCA group), $dx=dm$ the Haar measure of G, and Γ the dual group. Also the space of bounded regular Borel measures on G will be denoted by $M(G)$, and $L^p(G)$ the L^p space with the norm $\|\cdot\|_p$ on G.

In this paper, we study the properties of some bounded linear operators on Lorentz spaces $L(p,q)$ $(=L(p,q)(G))$ $(1\le p,\ q\le\infty)$.

First we recall some definitions and basic properties of Lorentz spaces.

Definition 1.1 Let f be a complex-valued measurable function on G which is finite m a.e. The distribution function of f is defined by

$$m_f(y)=m\{x\in G \mid |f(x)|>y\} \quad (y\ge 0).$$

The non-increasing rearrangement of f is the function f^* defined by

$$f^*(t)=\inf\{y>0 \mid m_f(y)\le t\} \quad (t\ge 0).$$

The Lorentz space $L(p,q)$ is defined as the set of equivalence classes of functions f as above such that $\|f\|_{pq}^*<\infty$, where

$$\|f\|_{pq}^*=\begin{cases}\left(\dfrac{q}{p}\displaystyle\int_0^\infty (t^{1/p}f^*(t))^q\frac{dt}{t}\right)^{1/q} & \text{if } 1\le p,\ q<\infty\\ \displaystyle\sup_{t\in(0,\infty)} t^{1/p}f^*(t) & \text{if } 1\le p\le\infty,\ q=\infty.\end{cases}$$

Since f^* and f have the same distribution function, it follows that

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$\|f\|_{pp}^* = \|f\|_p$, so the Lorentz space $L(p,p)$ is equal to L^p.

The function $\|\cdot\|_{pq}^*$ is a quasi-norm, but is not in general a norm. For this reason it is useful to define the function f^{**} by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds \quad (t > 0),$$

and then set

$$\|f\|_{(p,q)} = \begin{cases} \left(\int_0^\infty (t^{1/p}f^{**}(t))^q \frac{dt}{t}\right)^{1/q} & \text{if } 1 \le p,\ q < \infty \\ \sup\limits_{t\in(0,\infty)} t^{1/p}f^{**}(t) & \text{if } 1 \le p \le \infty,\ q = \infty. \end{cases}$$

If $1 < p,\ q < \infty$ or $p = q \in \{1, \infty\}$, then $L(p,q)$ is a Banach space with the norm $\|\cdot\|_{(p,q)}$. Also we can prove the inequalities

$$(p/q)^{1/q}\|f\|_{pq}^* \le \|f\|_{(p,q)} \le p/(p-1)(p/q)^{1/q}\|f\|_{pq}^*,$$

where $(p/q)^{1/q} = 1$ if $q = \infty$. Also we remark that any element of $M(G)$ can be considered a bounded linear operator on $L(p,q)$ by convolution (cf. [2]).

Now in §2, we study the problem posed by Chen and Lai [1; p. 255, Remark]. They define the spaces $A(p,q)$ and $M(p,q)$ concerning with Lorentz space.

Definition 1.2 Let $1 < p < \infty$ and $1 \le q < \infty$. Put

$$\begin{aligned} A(p,q) &= \{f \in L^1(G) \mid \widehat{f} \in L(p,q)(\Gamma)\}, \\ M(p,q) &= \{\mu \in M(G) \mid \widehat{\mu} \in L(p,q)(\Gamma)\}, \end{aligned}$$

where $\widehat{f}$ (resp. $\widehat{\mu}$) is the Fourier transform (resp. the Fourier-Stieltjes transform) of $f \in L^1(G)$ (resp. $\mu \in M(G)$). For every $f \in A(p,q)$ (resp. $\mu \in M(G)$) we supply a norm by

$$\|f\|_{A(p,q)} = \max\{\|f\|_1, \|\widehat{f}\|_{(p,q)}\}$$
$$(\text{resp. } \|\mu\|_{M(p,q)} = \max\{\|\mu\|, \|\widehat{\mu}\|_{(p,q)}\}),$$

where $\|\mu\|$ is the total variation norm of μ.

Then Chen and Lai [1] proposed the following problem:

If $1 < p < q \le 2$, is $M(p,q)$ equal to $A(p,q)$?

In this section, we show the equality.

In §3, we study the algebra which concerns with Lorentz-improving measures. The measures, which act by convolution map L^p to $L^{p+\epsilon}$ for some $\epsilon = \epsilon(p) > 0$ and $1 < p < \infty$, are called L^p-improving measures and have been investigated in a number of recent papers (cf. [4] and the papers cited therein). Also Grinnell and Hare [2] developed the study of L^p-improving measures, and characterized the class of Lorentz-improving measures on the Lorentz spaces. We will give a definition of Lorentz-improving operators which generalizes Lorentz-improving measures, and investigate the properties of Lorentz-improving operators.

In this paper, for $1 \leq p \leq \infty$ we denote p' by $1/p + 1/p' = 1$, $T(G)$ by all trigonometric polynomials, and C_j $(j = 1, 2, \ldots)$ by appropriate positive constants.

2. Chen and Lai's problem

Throughout this section, let G be a nondiscrete LCA group. Chen and Lai [1] gives a problem with respect to some operators concerning with Lorentz spaces. In this section, we consider this problem.

Now Chen and Lai [1; Theorem 3.6 (i)] show the following:

Theorem 2.1 *If $1 \leq q \leq p \leq 2$, then $M(p,q) = A(p,q)$.*

But they say that we do not know what happens for the case $1 < p < q \leq 2$ ([1; p. 255 Remark]). We prove the following result for this problem.

Theorem 2.2 *If $1 < p < 2$ and $1 < q < \infty$, then $M(p,q) = A(p,q)$.*

It is easy to see that the Theorem 2.2 follows from Theorems 2.7 and 2.8. Then we will show those theorems.

To proof of Theorems 2.7 and 2.8, we prepare some lemmas.

Lemma 2.3 *For $1 < p, q < \infty$, there exists $\{f_\alpha\} \subset L^1(G)$ an approximate identity of $L(p,q)$ such that f_α is a nonnegative function with $\|f_\alpha\|_1 = 1$, $\operatorname{supp} \widehat{f_\alpha}$ a compact set, $\widehat{f_\alpha} \to 1$ $(\alpha \to \infty)$ on any compact set of Γ, and $\int_{W^c} f_\alpha(x)dx \to 0$ $(\alpha \to \infty)$ for any neighborhood of unit W.*

Proof. By Hewitt and Ross [5], there exists a net $\{f_\alpha\} \subset L^1(G)$ such that f_α is a nonnegative function with $\|f_\alpha\|_1 = 1$, $\operatorname{supp} \widehat{f_\alpha}$ a compact set, $\widehat{f_\alpha} \to 1$ $(\alpha \to \infty)$ on any compact set of Γ, and $\int_{W^c} f_\alpha(x)dx \to 0$ $(\alpha \to \infty)$ for any neighborhood of unit W. Then it is easy to show that f_α is in $L(p,q)$.

(cf. [8]) Also in the same way as [1; Lemma 3.3], we can show that $\{f_\alpha\}$ is an approximate identity of $L(p,q)$. We omit the details. □

Lemma 2.4 *For $1 < p,\ q < \infty$, we define*

$$\mathcal{F} = \{f \in L^1(G) \mid \operatorname{supp} \widehat{f} \text{ is a compact set}\}.$$

Then $\mathcal{F}$ is dense in $L(p,q)$ $(= L(p,q)(G))$.

Proof. It is easy to see $\mathcal{F} \subset L(p,q)$. Let $\{f_\alpha\}$ be in Lemma 2.3, and $f \in C_c(G)$. Then by Lemma 2.3, we have

$$f * f_\alpha \in \mathcal{F}, \quad \text{and} \quad \|f - f * f_\alpha\|_{(p,q)} \to 0 \ (\alpha \to \infty).$$

So we can show that $\mathcal{F}$ is dense in $L(p,q)$. □

It is easy to see the next lemma (cf. [1]).

Lemma 2.5 *For $1 < p,\ q < \infty$, we define*

$$\widehat{\mathcal{F}} = \{\widehat{f} \mid f \in \mathcal{F}\}.$$

Then $\widehat{\mathcal{F}}$ is dense in $L(p,q)(\Gamma)$.

The next definition was suggested by Saeki and Thome [10].

Definition 2.6 Let μ be in $M(G)$. μ is called in $A(p,q)^\sim$ if there exists a net $\{\mu_\alpha\} \subset A(p,q)$ such that $\{\mu_\alpha\}$ is bounded in $A(p,q)$, and $\mu_\alpha \to \mu$ (as $\alpha \to \infty$) in the w^*-topology (i.e. $\sigma(M(G), C_0(G))$).

Now we can show the following:

Theorem 2.7 *If $1 < p,\ q < \infty$, then $A(p,q)^\sim = M(p,q)$.*

Proof. Let μ is an $A(p,q)^\sim$. By the definition, there exists a net $\{\mu_\alpha\} \subset A(p,q)$ such that $\mu_\alpha \to \mu$ in $\sigma(M(G), C_0(G))$ and $\|\mu_\alpha\|_{A(p,q)} \leq C_1$. Let $\mathcal{F}$ be in Lemma 2.4. For $f \in \mathcal{F}$, we can show

$$\int f d\mu_\alpha = \int \widehat{f}(-\gamma)\widehat{\mu}_\alpha(\gamma)d\gamma,$$

where $d\gamma$ is the Haar measure of Γ. By the assumption and the duality (cf. [6]), it follows that

$$\left|\int f d\mu_\alpha\right| = \left|\int \widehat{f}(\gamma)\widehat{\mu}_\alpha(-\gamma)d\gamma\right|$$

$$\leq \|\widehat{f}\|_{(p',q')}\|\widehat{\mu}_\alpha\|_{(p,q)}$$
$$\leq C_2\|\widehat{f}\|_{(p',q')},$$

and $|\int f d\mu| \leq C_2\|\widehat{f}\|_{(p',q')}$ $(f \in \mathcal{F})$. On the other hand, let $\widehat{\mathcal{F}}$ be in Lemma 2.5. Since $\widehat{\mathcal{F}}$ is dense in $L(p',q')(\Gamma)$, by the duality [5] it follows that $\widehat{\mu} \in L(p,q)(\Gamma)$, $\|\widehat{\mu}\|_{(p,q)} \leq C_3$, and $\mu \in M(p,q)$.

Conversely, let μ be in $M(p,q)$, and $\{f_\alpha\}$ in Lemma 2.3. Putting $\mu_\alpha = f_\alpha * \mu$, it follows that $\|\mu_\alpha\|_1 \leq \|\mu\|$, $\widehat{\mu}_\alpha = \widehat{f}_\alpha\widehat{\mu}$, and $\|\widehat{\mu}_\alpha\|_{(p,q)} \leq C_4\|\widehat{\mu}\|_{(p,q)}$. Moreover, for $f \in \mathcal{F}$ ($\mathcal{F}$ in Lemma 2.4) it follows that

$$\int f d\mu_\alpha = \int \widehat{f}(-\gamma)\widehat{f}_\alpha(\gamma)\widehat{\mu}d\gamma \to$$
$$\int \widehat{f}(-\gamma)\widehat{\mu}(\gamma)d\gamma = \int f d\mu \ (\alpha \to \infty)$$

by Lemma 2.3. So we have

$$\int f d\mu_\alpha \to \int f d\mu \ (\alpha \to \infty) \quad (f \in \mathcal{F}).$$

Also let g be in $C_0(G)$. Then for any $\epsilon > 0$, there exists $f \in \mathcal{F}$ with $\|f - g\|_\infty < \epsilon$. In fact, $\mathcal{F}$ is a subalgebra of $C_0(G)$, closed under complex conjugation, and separetes points of G. Therefore, $\mathcal{F}$ is dense in $C_0(G)$ by the Stone-Weierstrass theorem.

Now by the above results, it follows that

$$\left|\int g d\mu_\alpha - \int g d\mu\right| \leq \|g - f\|_\infty\|\mu_\alpha\| + \left|\int f d\mu_\alpha - \int f d\mu\right| + \|f - g\|_\infty\|\mu\|,$$

and

$$\int g d\mu_\alpha \to \int g d\mu \ (\alpha \to \infty).$$

Therefore μ is in $A(p,q)^\sim$. □

Theorem 2.8 *If* $1 < p < 2$ *and* $1 < q < \infty$, *then* $A(p,q)^\sim = A(p,q)$.

Proof. Let μ be in $A(p,q)^\sim$. Then there exist a net $\{\mu_\alpha\} \subset A(p,q)$ and $C_5 > 0$ such that $\|\widehat{\mu}_\alpha\|_{(p,q)} \leq C_5$ and $\mu_\alpha \to \mu$ $(\alpha \to \infty)$ in $\sigma(M(G), C_0(G))$. Then in the same way as the first paragraph of the proof of Theorem 2.7, it follows that

$$\left|\int f d\mu\right| \leq C_6 \|\widehat{f}\|_{(p',q')} \quad (f \in \mathcal{F}).$$

Here, by Hausdorff-Young's inequality (cf. [2]) and Calderon-Hunt's interpolation theorem (cf. [8]), we can show that

$$\|\widehat{f}\|_{(p',q')} \leq C_7 \|f\|_{(p,q')} \quad (f \in \mathcal{F}),$$

and

$$\left|\int f d\mu\right| \leq C_7 \|f\|_{(p,q')} \quad (f \in \mathcal{F}).$$

On the other hand, $\mathcal{F}$ is dense in $L(p,q')$ by Lemma 2.4. So by the duality of $L(p,q')$ (cf. [6]) and Lemma 2.3, it follows that there exists $g \in L(p',q)(G)$ such that

$$\int f d\mu = \int f g dx (f \in \mathcal{F}), \ \mu = g dx, \ \text{ and } \ \mu \in L^1(G).$$

□

3. Lorentz-improving operators

Throughout this section, let G be an infinite compact abelian group. In this section, we define Lorentz-improving operators, and characterize them. Also we give some equivalent conditions of $\Lambda_2(2,q)$-set. Following Grinnell-Hare [2], we will show it.

Definition 3.1 An operator T is called a Lorentz-improving operator (LI operator) if there exist p, q, r $(1 < p < \infty,\ 1 \leq q < r \leq \infty)$, and $\phi \in l^\infty(\Gamma)$ such that $\widehat{Tf} = \phi \widehat{f}$ $(f \in T(G))$ and T has a bounded extension from $L(p,r)$ with the norm $\|\cdot\|_{(p,r)}$ to $L(p,q)$ with the norm $\|\cdot\|_{(p,q)}$. Then we put $\widehat{T} = \phi$. Also we denote by $M_p(r,q)$ the set of all T above, and $M_p(r,q)^\wedge = \{\widehat{T} \in l^\infty(\Gamma) \mid T \in M_p(r,q)\}$.

Here, we remark that $M_p(r,q)$ is a commutative Banach algebra without unit by Yap [12].

Remark 3.2 There exists an LI operator which is not in $M(G)$. (cf. [4])

Definition 3.3 ([2]) Let $1 < p < \infty$, $1 \leq q < \infty$, and $E \subset \Gamma$. E is called

$\Lambda_2(2,q)$-set if there is some $r > q$ such that

$$\{f \in L(p,q) \mid \widehat{f} = 0 \text{ on } E^c\} = \{f \in L(p,r) \mid \widehat{f} = 0 \text{ on } E^c\}.$$

For $1 \leq q < 2$, we define

$$\Lambda_2(2,q;E) = \text{supp}\{\|f\|_{(2,q)} \mid f \in L^2,\ \|f\|_2 \leq 1,\ \widehat{f} = 0 \text{ on } E^c\}.$$

Then we have the following:

Theorem 3.4 *The following are equivalent*:
(i) *T is an LI operator*;
(ii) *There exist $1 \leq q < 2$ and $\alpha \geq 1$ such that for any $\epsilon > 0$ and $E(\epsilon) = \{\gamma \in \Gamma \mid |\widehat{T}(\gamma)| > \epsilon\}$, $E(\epsilon)$ is $\Lambda_2(2,q)$-set with $\Lambda_2(2,q;E(\epsilon)) = O(\epsilon^{-\alpha})$*;
(iii) *There exist $1 \leq q < 2$ and a natural number n such that $T^n : L(2,q') \to L^2$ is an LI operator.*

The proof of Theorem 3.4 is similar to [2; Theorem 3.4]. We omit the details.

Theorem 3.5 (cf. [11]) *Let $E \subset \Gamma$, and $1 < q < 2$. The following are equivalent*:
(i) *E is $\Lambda_2(2,q)$-set*;
(ii) *There exists a positive constant C such that for any $g \in L(2,q')$ there exists $h \in L^2$ such that $\widehat{h} = 0$ on E^c, $\widehat{g}|_E = \widehat{h}|_E$, and $\|h\|_2 \leq C\|g\|_{(2,q')}$*;
(iii) *$\xi_E \in M_2(r,q)^\wedge$ for some r $(q < r \leq \infty)$, where ξ_E is the characteristic function of E*;
(iv) *$M_2(r,q)^\wedge|_E = l^\infty(E)$ for some r $(q < r \leq \infty)$*;
(v) *There exist r $(q < r \leq \infty)$ and $T \in M_2(r,q)$ such that*

$$\inf\{|\widehat{T}(\gamma)| \mid \gamma \in E\} > 0.$$

Proof. By [2; Theorem 3.3], (i) is equivalent to (ii).

(ii $\Rightarrow$ iii) We define $\widehat{T} = \xi_E$, and have $\|Tf\|_2 \leq C_9\|f\|_{(2,q')}$. By $q > 1$ and the duality, $\|Tf\|_{(2,q)} \leq C_9\|f\|_2$. Hence, we may put $r = 2$.

(iii $\Rightarrow$ iv) Since $M_2(r,q)^\wedge|_E \subset l^\infty(E)$ by the definition, it is sufficient that we show the converse.

Case 1: $q < 2 \leq r$. For any $\phi \in l^\infty(E)$, let $\Psi(\gamma)$ be $\Psi(\gamma) = \xi_E(\gamma)\phi(\gamma)$. Then by (iii) it follows that $\xi_E \in M_2(r,q)^\wedge \subset M_2(2,q)^\wedge$, and $\xi_E \in M_2(2,q)^\wedge$.

So for any $f \in T(G)$, it follows that

$$\begin{aligned}\left\|\sum \widehat{f}(\gamma)\Psi(\gamma)\xi_E(\gamma)\gamma\right\|_{L(2,q)} &\leq C_{10}\left\|\sum \widehat{f}(\gamma)\Psi(\gamma)\gamma\right\|_2 \\ &\leq C_{10}\|\Psi\|_\infty\left\|\sum \widehat{f}(\gamma)\xi_E(\gamma)\gamma\right\|_2 \\ &\leq C_{11}\|\phi\|_\infty\|f\|_{L(2,\gamma)}.\end{aligned}$$

Then it follows that $\Psi \in M_2(r,q)^\wedge$.

Case 2: $q < r \leq 2$. By the assumption, $\widehat{T} = \xi_E \in M_2(r,q)^\wedge = M_2(q',r')^\wedge$.

By the interpolation ([6], cf. [2]), there exists a natural number N such that $T^N \in M_2(q',2)$. Let $\phi \in l^\infty(E)$. Then it follows that

$$\begin{aligned}\left\|\sum \phi(\gamma)\xi_E(\gamma)\widehat{f}(\gamma)\gamma\right\|_2 &\leq \|\phi\|_\infty\left\|\sum \xi_E\widehat{f}(\gamma)\gamma\right\|_2 \\ &= \|\phi\|_\infty\|T^N f\|_2 \\ &\leq C_{12}\|\phi\|_\infty\|f\|_{(2,q')}.\end{aligned}$$

Hence, for $\widehat{S} = \phi\xi_E$, S is in $M_2(q',2) = M_2(2,q)$, and $S \in M_2(r,q)$. Therefore, ϕ is in $M_2(r,q)^\wedge|_E$.

(iv $\Rightarrow$ v) It is clear.

(v $\Rightarrow$ i) First we give the next lemma.

Lemma (cf. [4]) *Let $T \in M_2(p,2)$ $(2 < p < \infty)$, and for any $\epsilon > 0$ let $E(\epsilon)=\{\gamma \in \Gamma \mid |\widehat{T}(\gamma)| > \epsilon\}$. Then $E(\epsilon)$ is $\Lambda_2(2,p')$-set with $\Lambda_2(2,p';E(\epsilon))= O(\epsilon^{-1})$, and $\xi_{E(\epsilon)} \in M_2(p,2)^\wedge$.*

The proof of this lemma is similar to [4; Theorem 1.5] by applying [2; Theorem 3.3].

Now let $\epsilon_0 = \inf\{|\widehat{T}(\gamma)| \mid \gamma \in E\}$ (> 0), and $E(T,\eta) = \{\gamma \in \Gamma \mid |\widehat{T}(\gamma)| \geq \eta/2\}$ for any $\eta > 0$. Since $T \in M_2(r,q)$, we may assume $r < \infty$. Since $q > 1$ and $T \in M_2(q',r')$, by Theorem 3.4 there exists a natural number N such that $T^N \in M_2(q',2)$. Then since $\{\gamma \in \Gamma \mid |\widehat{T^N}(\gamma)| > \epsilon_0^N\} = \{\gamma \in \Gamma \mid |\widehat{T}(\gamma)| > \epsilon_0\}$, $E(T^N,\epsilon_0^N) \supset E$. By $T^N \in M_2(q',2)$ and Lemma, $E(T^N,\epsilon_0^N)$ is $\Lambda_2(2,q)$-set, and E is $\Lambda_2(2,q)$-set. □

Remark 3.6 For $E \subset \Gamma$ and $1 \leq q < 2$, we can prove the same result as Theorem 3.5 in the same method as [2; Theorem 3.3].

Next we study operating functions and spectra of the Banach algebra $M_2(r,q)$ $(1 < q < r < \infty)$.

Definition 3.7 Let Φ be a complex-valued function on $[-1,1]$. Φ is called an operating function on $M_2(r,q)$ if for any $T \in M_2(r,q)$ with $\widehat{T}(\Gamma) \subset [-1,1]$ there exists $S \in M_2(r,q)$ with $\Phi(\widehat{T}) = \widehat{S}$.

Then the next result is proved:

Theorem 3.8 *Let Φ_0 be a bounded function on $[-1,1]$.*

(i) *Suppose $2 \leq q < r < \infty$ (resp. $1 < q < r < 2$). Let $\beta_0 = (1/2 - 1/q)/(1/q - 1/r)$ (resp. $\beta_0 = (1/r - 1/2)/(1/q - 1/r)$), and n_0 be the smallest integer such that $n_0 \geq \beta_0$. Then for any constants $\alpha_1, \alpha_2, \ldots, \alpha_{n_0}$*

$$\Phi(t) = \alpha_1 t + \cdots + \alpha_{n_0} t^{n_0} + |t|^{\beta_0+1}\Phi_0(t)$$

operates on $M_2(r,q)$.

(ii) *Suppose $1 < q < 2 < r < \infty$. Let $\beta_1 = \min\{(1/q - 1/2)/(1/2 - 1/r), (1/2 - 1/r)/(1/q - 1/2)\}$. Then for any constant α*

$$\Phi(t) = \alpha t + |t|^{\beta_1+1}\Phi_0(t)$$

operates on $M_2(r,q)$.

The proof of Theorem 3.8 is similar to [7; Theorem 1] by some interpolations (cf. [2]). We omit the details.

In particular, we can characterize the operating function of $M_2(2,q)$ by [3; Proposition 2] and Theorem 3.8.

Corollary 3.9 *Let Φ be a complex-valued function on $[-1,1]$, and $1 < q < 2$. Then Φ is an operating function of $M_2(2,q)$ if and only if*

$$|\Phi(t)| \leq C_{13}|t|.$$

Corollary 3.10 *Let T be in $M_2(r,q)$ $(1 < q < r < \infty)$. Then $sp(T, M_2(r,q)) = \overline{\widehat{T}(\Gamma)}$, where $\overline{\widehat{T}(\Gamma)}$ is the closure of $\widehat{T}(\Gamma)$.*

Proof. Let $\Phi_0(z)$ be a bounded function on the complex plane. By [7; Remark] and Theorem 3.8, $z^N\Phi_0(z)$ operates on $M_2(r,q)$ for sufficiently large natural number N. Then by [7; Theorem 2]

$$sp(T, M_2(r,q)) = \overline{\widehat{T}(\Gamma)}.$$

□

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