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A note on comparison principles for viscosity solutions of fully nonlinear second order partial differential equations

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Abstract. This note contains two comparison principles from the author's thesis [4] for viscosity sub- and supersolutions of fully nonlinear, second order, partial differential equations. One of these comparison principles is based on a result by R. Jensen [7]. The other one is an application of an idea, mentioned in [11]. A new kind of structure condition is introduced to prove the comparison result based on Theorem 3.1 in [7]. It allows us to compare viscosity sub- and supersolutions of the equation $F(u, Du, D^2u) = 0$ in $\Omega \subset \mathbb{R}^N$, where F does not satisfy the usual monotonicity conditions as in [3], [5], [6], [7], [9] or [11].

Key words: viscosity solutions, comparison principles, ω-ellipticity.

1. Introduction

In this note we look at fully nonlinear second order elliptic partial differential equations with Dirichlet boundary data of type

$$\left.\begin{aligned} F(u(x), Du(x), D^2u(x)) = 0 \qquad & \text{in } \Omega \\ u(x) = f(x) \qquad & \text{on } \partial\Omega \end{aligned}\right\} \tag{1}$$

for some given function $F \in C(\mathbb{R} \times \mathbb{R}^N \times \mathcal{S}(N))$ and a given function $f \in C(\partial\Omega)$. Problems of this kind have also been investigated in [7]. Here and in the sequel I will use the notation $\mathcal{S}(N)$ for the space of symmetric $N \times N$ matrices.

The goal of this note is to compare viscosity sub- and supersolutions under new and weaker conditions than before. Various comparison principles for viscosity solutions of fully nonlinear second order partial differential equations have been introduced for Dirichlet problems of type (1). To quote some famous results, let me mention [3], [6], [7] and [9]. In [3], [6] and [9]

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the authors even gave a comparison result for Dirichlet problems of type

$$\left.\begin{aligned} F(x,u(x),Du(x),D^2u(x)) = 0 \qquad & \text{in } \Omega \\ u(x) = f(x) \quad & \text{on } \partial\Omega, \end{aligned}\right\} \tag{2}$$

where the functions F and f are as in (1). All of these comparison results for equations or systems of partial differential equations (see for example [3], [5], [6], [9] or [11]) require the existence of a positive constant γ such that for $r,s \in \mathbb{R}$ with $r \geq s$

$$\gamma(r-s) \leq F(r,p,X) - F(s,p,X). \tag{3}$$

In general operators satisfying inequality (3) are called decreasing. The only ones I know, who did not use this condition up to now were N. Kutev and B. Kawohl in [10] and R. Jensen, who presented in [7] the following comparison principle:

Theorem 1 [7] *Let $u, v \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$. Assume u is a viscosity supersolution of* (1) *and v is a viscosity subsolution of* (1). *If either* (i) *or* (ii) *holds*:

(i) *$F(r,p,X)$ is degenerate elliptic and decreasing, or*

(ii) *For all $r,s \in \mathbb{R}$ with $s < r$ and all pairs $(p,X) \in \mathbb{R}^N \times \mathcal{S}(N)$ the inequality $F(s,p,X) \leq F(r,p,X)$ holds and there exist constants c_0 and c_1, such that*

$$F(r,p,Y) - F(r,q,X) \leq c_0 \cdot trace(Y-X) + c_1 \cdot |p-q|$$

then

$$\sup_{\partial\Omega}(u-v)^+ \geq \sup_{\Omega}(u-v)^+.$$

Here and in the following the operator F is called degenerate elliptic, if for all $X, Y \in \mathcal{S}(N)$ with $X \leq Y$, the inequality

$$F(r,p,Y) \leq F(r,p,X) \tag{4}$$

holds. The inequality $X \leq Y$ means $\langle X\xi,\xi\rangle \leq \langle Y\xi,\xi\rangle$ for all $\xi \in \mathbb{R}^N$, with $\langle\cdot,\cdot\rangle$ denoting the scalar product in $\mathbb{R}^N$. Operators satisfying

$$F(s,p,X) \leq F(r,p,X), \quad \text{whenever} \quad s \leq r, \tag{5}$$

are said to be proper. In [7] Jensen did not look at generalized viscosity solutions of the Dirichlet problem as they are defined for example in [3], but it is possible to relax his conditions and to prove a similar theorem for generalized viscosity solutions. There has been a remark about this fact made by Jensen et al. in [9]. So one goal is to introduce a comparison principle for generalized viscosity sub- and supersolutions of problem (1) with an operator F, which is not necessarily decreasing in the sense of (3). This is possible, as it will be shown in the following, by imposing assumptions on the ellipticity of the operator F.

But let me explain the motivation why this note deals with problems of type (1) and not of type (2). If we look at the comparison results for generalized problems of type (2), we notice that one of the assumptions usually made on the operator F (see for example [3] and [9]) is the following:

There exists a monotone increasing function $\omega : [0,\infty] \to [0,\infty]$ with $\omega(0+) = 0$, such that for $x, y \in \overline{\Omega}$, $r \in \mathbb{R}$, $p \in \mathbb{R}^N$ and matrices $X, Y \in \mathcal{S}(N)$ satisfying

$$-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

(with a constant $\alpha > 0$), the inequality

$$F(y,r,p,Y) - F(x,r,p,X) \leq \omega(\alpha|x-y|^2 + |x-y|(|p|+1)) \tag{6}$$

holds.

As it is also mentioned in [9] (see remark on assumption (8) in [9]), this condition is quite restrictive, which can be seen by looking at the following example.

Example 1. For $\Omega \subset \mathbb{R}^N$ we look at the equation $-\Delta u(x) - c(x, u(x)) = 0$, with a function $c \in C(\Omega \times \mathbb{R})$.

The operator F satisfies condition (6) only for special $c(x, u(x))$. For a function $c(x, u(x)) = g(x) \cdot u(x)$ with nonconstant $g \in C(\Omega)$ this condition is generally not fulfilled.

Furthermore in [1] the authors show that even for operators of the form

$$F(x, u(x), Du(x), D^2u(x)) = G(u(x), Du(x), D^2u(x)) - f(x), \tag{7}$$

with a Hölder-continuous function f uniqueness cannot be expected and that comparison must fail. Therefore we look at the following example.

Example 2 [1]. Let $B(0,R) \subset \mathbb{R}^N$ denote the ball with radius $R > 0$. The Dirichlet problem

$$\begin{aligned} -\Delta u(x) + |Du(x)|^m = \nu |x|^{\frac{m}{1-m}} \quad & \text{in } \ B(0,R) \\ u(x) = const \quad & \text{on } \ \partial B(0,R) \end{aligned} \tag{8}$$

with $\nu > 0$ and $0 < m < 1$, has more than one solution, whose differences are not equal to a constant. Thus, as mentioned above, comparison fails in such a case.

This motivates not to look at problems of type (2), but to face again problems of type (1) and to try to weaken the assumptions on the operator F made so far.

Since the whole paper will only deal with viscosity solutions, from now on these solutions will just be called solutions. So let us start with some basic tools.

2. Basic information

For the reader's convenience I repeat the definitions of a generalized viscosity sub- and supersolution of problem (1) and the maximum principle for semicontinuous functions introduced by Crandall and Ishii in [2].

Definition 1 Let $\Omega \subset \mathbb{R}^N$ be open and bounded and let V denote a neighbourhood of $\partial\Omega$ with respect to $\overline{\Omega}$. The function $u \in USC(\overline{\Omega})$ is called a generalized viscosity subsolution of (1), if

$$\begin{cases} \forall \ (x,\varphi) \in \Omega \times C^2(\mathbb{R}^N) \ \text{ with } \ u \le \varphi \ \text{ in } \ \Omega \ \text{ and } \ u(x) = \varphi(x) : \\ \qquad F(u(x), D\varphi(x), D^2\varphi(x)) \le 0 \\ \forall \ (x,\varphi) \in V \times C^2(\mathbb{R}^N) \ \text{ with } \ u \le \varphi \ \text{ in } \ V \\ \text{ and } \ u(x) = \varphi(x) \ \text{ for } \ x \in \partial\Omega : \\ \qquad \min\{u(x) - f(x), F(u(x), D\varphi(x), D^2\varphi(x))\} \le 0 \end{cases}$$

The function $v \in LSC(\overline{\Omega})$ is called a generalized viscosity supersolution of (1), if

$$\begin{cases} \forall\ (x,\psi) \in \Omega \times C^2(\mathbb{R}^N) \text{ with } v \geq \psi \text{ in } \Omega \text{ and } v(x) = \psi(x) : \\ \qquad F(v(x), D\psi(x), D^2\psi(x)) \geq 0 \\ \forall\ (x,\psi) \in V \times C^2(\mathbb{R}^N) \text{ with } v \geq \psi \text{ in } V \\ \text{and } v(x) = \psi(x) \text{ for } x \in \partial\Omega : \\ \qquad \max\{v(x) - f(x), F(v(x), D\psi(x), D^2\psi(x))\} \geq 0 \end{cases}$$

A function $u \in C(\overline{\Omega})$ is called a generalized viscosity solution of (1), if u is a generalized viscosity sub- and supersolution.

The function φ (resp. ψ) with the above properties is called testfunction from above (resp. from below).

Many authors working with viscosity solutions define these solutions by using the so-called super- and subjets. Statements about the equivalence of this definition and the one given here, can be found for example in [8]. Readers who are familiar with the theory of viscosity solutions know that proofs of comparison principles for these kind of viscosity solutions are mostly based on the so-called **maximum principle for semicontinuous functions** which is described very precisely in [2] and [3]. Jensen did not use this principle, when he presented his comparison result in [7], thus he assumed the viscosity sub- and supersolution to be in $C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$, but it is mentioned in "*A uniqueness result for viscosity solutions of second order fully nonlinear partial differential equations*" [9] that it is possible to extend the results of [7] to only continuous viscosity solutions. However, the following Theorem from [2] is necessary, for this extension:

Theorem 2 [2] *Let $\mathcal{O}_1, \mathcal{O}_2 \subseteq \mathbb{R}^N$ be local compact and $\mathcal{O} = \mathcal{O}_1 \times \mathcal{O}_2$. Let $v \in LSC(\mathcal{O}_1)$ and $u \in USC(\mathcal{O}_2)$. The function φ shall be twice differentiable in a neighbourhood of $\mathcal{O}$. The function w is defined as $w(x) := u(x_1) - v(x_2)$ with $x = (x_1, x_2) \in \mathcal{O}$. Let $\hat{x} = (\hat{x}_1, \hat{x}_2) \in \mathcal{O}$ be a local maximum of $w - \varphi$ in $\mathcal{O}$. Then for all $\varepsilon > 0$ there exists $X_i \in \mathcal{S}(N)$, $(i = 1, 2)$, such that the set of testfunctions for u and v is not empty. To be more precise, $F(\hat{x}_1, u(\hat{x}_1), D_{x_1}\varphi(\hat{x}), X_1) \leq 0$, $F(\hat{x}_2, -v(\hat{x}_2), D_{x_2}\varphi(\hat{x}), X_2) \leq 0$ and for X_1, X_2 the inequality*

$$-\left(\frac{1}{\varepsilon} + ||A||\right) I \leq \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \leq A + \varepsilon A^2,$$

where $A = D^2\varphi(\hat{x}) \in \mathcal{S}(2N)$ *and*

$$||A|| = \sup\{|\lambda| \mid \lambda \text{ is eigenvalue of } A\} = \sup\{|\langle A\xi, \xi\rangle| \mid |\xi| \leq 1\},$$

is satisfied.

Instead of repeating the proof of Theorem 2 given in [2] I refer the reader to [2] or [3]. Now having the necessary tools in hand, we can turn to the comparison results promised in the first section.

3. Comparison principles for the generalized Dirichlet problem

First we look at a kind of generalization of the comparison principle presented by Crandall, Ishii and Lions in [3] applied to problems of type (1).

Instead of a decreasing operator in the sense of (3) we assume here that monotone increasing functions $\omega_1 : [0,\infty] \to [0,\infty]$ and $\omega_2 : [0,\infty] \to [0,\infty]$ with $\omega_i(0+) = 0$ $(i = 1,2)$ exist, such that for all $r, s \in \mathbb{R}$ with $r > s$, $p, q \in \mathbb{R}^N$ and $X \in \mathcal{S}(N)$:

$$0 < \omega_1(r-s) \leq F(r,p,X) - F(s,p,X) \tag{9}$$

and the inequality

$$|F(r,p,X) - F(r,q,X)| \leq \omega_2(|p-q|) \tag{10}$$

holds. Now we can formulate the following Theorem.

Theorem 3 *Let* $\overline{\Omega}$ *be a compact* C^1*-submanifold with boundary of* $\mathbb{R}^N$ *and let* $F \in C(\mathbb{R}\times\mathbb{R}^N\times\mathcal{S}(N))$ *be degenerate elliptic and satisfy assumptions* (9) *and* (10) *mentioned above. Let* $u, v \in C(\overline{\Omega})$ *be a viscosity sub- respectively supersolution of the generalized Dirichlet problem* (1), *then* $u \leq v$ *on* $\overline{\Omega}$.

To prove this Theorem I adapt an idea from S. Koike [11]. Therefore I need the following Lemma:

Lemma 1 [11] *Let* K *be a compact subset of* $\mathbb{R}^N$ *and* $H : K \to \mathbb{R}$ *be an upper semicontinuous function. Then, for almost all* $q \in \mathbb{R}^N$ *the mapping* $x \mapsto H(x) + \langle q, x\rangle$ *has a unique maximum point in* K.

A proof of this Lemma can be found by the reader in "*Viscosity Solutions of a System of Nonlinear Second-Order Elliptic PDE's Arising in Switching Games*" by H. Ishii and S. Koike [5]. So let us turn to the proof of Theorem 3.

Proof of Theorem 3. Let us suppose that $\max_{x\in\overline{\Omega}}(u(x) - v(x)) = M > 0$. We choose a small $\delta \in (0, M/(4|x|^+))$, where $|x|^+ := \max\{|x| \mid x \in \overline{\Omega}\}$. Following Lemma 1 there exists a number $q = q(\delta) \in \mathbb{R}^N$ with $|q| < \delta$, such that $u(x) - v(x) + \langle q, x\rangle$ attends its maximum at the unique point $z = z(\delta) \in \overline{\Omega}$. Now set

$$M_q := \max_{\overline{\Omega}}(u(x) - v(x) + \langle q, x\rangle) \geq \frac{3M}{4},$$

exactly as in [11] and consider the following two cases.

Case 1: $z \in \partial\Omega$.

I will only prove the case that $v(z) < f(z)$ since the other case can be treated in an analogous way. Therefore we look for $\alpha > 1$ and $0 < \varepsilon < 1$ at the function

$$h(x, y) = u(x) - v(y) - |\alpha(x - y) + \varepsilon n(z)|^2 + \langle q, y\rangle,$$

where $n(z)$ will denote the outward normal vector to $\partial\Omega$ in z. Let $(x_\alpha, y_\alpha) \in \overline{\Omega} \times \overline{\Omega}$ be the maximum point of h. For sufficiently large α one can assume that $z - (\varepsilon/\alpha)n(z) \in \Omega$. The inequality $h(x_\alpha, y_\alpha) \geq h(z - (\varepsilon/\alpha)n(z), z)$ implies:

$$\begin{aligned} |\alpha(x_\alpha - y_\alpha) + \varepsilon n(z)|^2 \leq\ & u(x_\alpha) - v(y_\alpha) \\ & - u(z - (\varepsilon/\alpha)n(z)) + v(z) + \langle q, y_\alpha - z\rangle. \end{aligned}$$

For $0 < \varepsilon < 1$ arbitrary, but fixed, we have $x_\alpha, y_\alpha \to z$ and $\alpha(x_\alpha - y_\alpha) + \varepsilon n(z) \to 0$ as $\alpha \to \infty$. This can easily be verified. Since $\alpha(x_\alpha - y_\alpha)$ remains bounded as $\alpha \to \infty$ we can conclude by the assumption that $\alpha(x_\alpha - y_\alpha) \to w$ and for subsequences $x_\alpha, y_\alpha \to \tilde{z}$ as $\alpha \to \infty$, that $u(z) - v(z) + \langle q, z\rangle \leq u(\tilde{z}) - v(\tilde{z}) + \langle q, \tilde{z}\rangle$, which implies $\tilde{z} = z$ by Lemma 1, exactly as $x_\alpha = y_\alpha - (\varepsilon n(z) + o(1))/\alpha \in \Omega$ and $|\alpha(x_\alpha - y_\alpha) + \varepsilon n(z)| \to 0$ as $\alpha \to \infty$. The inequality $h(x_\alpha, y_\alpha) \geq h(z, z)$ also implies

$$u(x_\alpha) - v(y_\alpha) \geq M_q - \varepsilon^2 - \frac{M}{4} \geq \frac{M}{2} - \varepsilon^2.$$

Now using Theorem 2 for $\mathcal{O} := \overline{\Omega} \times \overline{\Omega}$ and $\phi(x, y) = |\alpha(x-y)+\varepsilon n(z)|^2 - \langle q, y\rangle$ one calculates

$$F(u(x_\alpha), 2\alpha(\alpha(x_\alpha - y_\alpha) + \varepsilon n(z)), X) \leq 0$$

$$F(v(y_\alpha), 2\alpha(\alpha(x_\alpha - y_\alpha) + \varepsilon n(z)) + q, Y) \geq 0,$$

where

$$-6\alpha^2 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 6\alpha^2 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

with

$$\begin{aligned} D\varphi(x_\alpha) &= 2\alpha(\alpha(x_\alpha - y_\alpha) + \varepsilon n(z)), \\ -D\psi(y_\alpha) &= 2\alpha(\alpha(x_\alpha - y_\alpha) + \varepsilon n(z)) + q, \\ D^2\varphi(x_\alpha) &= X \quad \text{and} \quad D^2\psi(y_\alpha) = Y \end{aligned}$$

for the testfunctions φ and ψ, whose existence is guaranteed by Theorem 2. Now calculation preserves:

$$\begin{aligned} 0 &\leq F(v(y_\alpha), 2\alpha^2(x_\alpha - y_\alpha) + \alpha\varepsilon n(z) + q, Y) \\ &\quad - F(u(x_\alpha), 2\alpha^2(x_\alpha - y_\alpha) + \alpha\varepsilon n(z), X) \\ &\leq -\omega_1(u(x_\alpha) - v(y_\alpha)) + \omega_2(|q|) \\ &\leq -\omega_1\left(\frac{M}{2} - \varepsilon^2\right) + \omega_2(|\delta|) \end{aligned}$$

Sending $\delta \to 0$ and $\varepsilon \to 0$ one gets a contradiction:

$$\begin{aligned} 0 &\leq F(v(y_\alpha), -D_y\phi(x_\alpha, y_\alpha), Y) - F(u(x_\alpha), D_x\phi(x_\alpha, y_\alpha), X) \\ &\leq -\omega_1\left(\frac{M}{2} - \varepsilon^2\right) + \omega_2(|\delta|) \to -\omega_1\left(\frac{M}{2}\right) < 0 \end{aligned}$$

The case that $u(z) > f(z)$ is proved in an analogous way by looking at the function

$$\tilde{h}(x, y) = u(x) - v(y) - |\alpha(x - y) - \varepsilon n(z)|^2 + \langle q, x \rangle.$$

Case 2: z lies in the interior of Ω.

For $\alpha > 1$ we now look at the function

$$k(x, y) = u(x) - v(y) - \frac{\alpha}{2}|x - y|^2 + \langle q, x \rangle.$$

Similarly to the first case we can show for the maximum point (x_α, y_α) of $k(x, y)$ that $\alpha|x_\alpha - y_\alpha|^2 \to 0$ and $x_\alpha, y_\alpha \to z$ as $\alpha \to \infty$. We also know that

$$u(x_\alpha) - v(y_\alpha) \geq M_q - (M/4) \geq M/2.$$

For $\phi(x,y) = \frac{\alpha}{2}|x-y|^2 - \langle q, x\rangle$ calculation shows that: $D_x\phi(x,y) = \alpha(x - y) - q$ and $-D_y\phi(x,y) = \alpha(x-y)$. Using again Theorem 2 we know that there are testfunctions φ and ψ in x_α respectively y_α for the viscosity sub- respectively viscosity supersolution, which fulfill the required inequalities from Theorem 2. Thus we have:

$$\begin{aligned} 0 &\leq F(v(y_\alpha), -D_y\phi(x_\alpha,y_\alpha), Y) - F(u(x_\alpha), D_x\phi(x_\alpha,y_\alpha), X) \\ &\leq F(u(x_\alpha), -D_y\phi(x_\alpha,y_\alpha), Y) - F(u(x_\alpha), -D_y\phi(x_\alpha,y_\alpha), X) \\ &\quad - \omega_1(u(x_\alpha) - v(y_\alpha)) + \omega_2(|q|) \\ &\leq -\omega_1\left(\frac{M}{2}\right) + \omega_2(|\delta|) \end{aligned}$$

Sending $\delta \to 0$ we obtain a contradiction, which proves together with the first case that

$$u \leq v \quad \text{on} \quad \overline{\Omega}.$$

□

Remark 1. It seems, that it is not possible to prove a comparison result by using assumption (5) with strict inequalities instead of assuming (9). Thus the assumptions made in Theorem 3 seem to be the weakest without strengthening the ellipticity assumptions on F.

Next I present a comparison result which requires a different assumption on the operator F than previously used. Let us assume that there exist a strict monotone increasing function $\omega : [0,\infty) \to [0,\infty]$ satisfying $\omega(0+) = 0$ and constants $c_0 > 0$ and $c_1 \geq 0$, such that for all $(r,p) \in \mathbb{R} \times \mathbb{R}^N$ and $X, Y \in \mathcal{S}(N)$ with $Y > X$ the following (11) and (12) holds:

$$0 < \omega(c_0 \cdot trace(Y - X)) \leq F(r,p,X) - F(r,p,Y), \tag{11}$$

as well as for all $(r,X) \in \mathbb{R} \times \mathcal{S}(N)$ and $p,q \in \mathbb{R}^N$

$$|F(r,p,X) - F(r,q,X)| \leq \omega(c_1 \cdot |p-q|). \tag{12}$$

This structure condition is stronger than assuming F to be uniformly elliptic as defined for example in [3]. Operators which have this property will be called **ω-elliptic**. This new structure condition is quite restrictive, but it is necessary for the comparison result presented below.

Theorem 4 *Let $\overline{\Omega}$ be a compact, strictly convex C^1-submanifold with* *boundary of $\mathbb{R}^N$. Let $F \in C(\mathbb{R} \times \mathbb{R}^N \times \mathcal{S}(N))$ be proper and ω-elliptic. If $u, v \in C(\overline{\Omega})$ are a viscosity sub- respectively a viscosity supersolution of* (1) *then $u(x) \le v(x)$ on $\overline{\Omega}$.*

Proof of Theorem 4. Let us assume that there exists a point $z \in \overline{\Omega}$ such that $u(z) - v(z) > 0$.

Case 1: There is a point $z \in \partial\Omega$, such that

$$u(z) - v(z) = \max_{x \in \overline{\Omega}} (u(x) - v(x)) = M > 0.$$

We fix such a point $z \in \partial\Omega$ and look at the case $u(z) > f(z)$ (The case $v(z) < f(z)$ is treated similarly.). By translation we can guarantee that $0 \notin \overline{\Omega}$. Since $\overline{\Omega}$ is strictly convex, a ball $B(0, R)$ exists, such that

$$\bullet\ \overline{\Omega} \subset B(0,R) \qquad \text{and} \qquad \bullet\ \partial B(0,R) \cap \overline{\Omega} = \{z\}$$

For $\alpha > 1$ and $0 < \varepsilon < 1$ we look at the function

$$\begin{aligned} h(x,y) &= u(x) - v(y) - \frac{1}{2}|\alpha(x-y) - \varepsilon n(z)|^2 \\ &\quad + [e^{(\gamma|x|^2)/2} - e^{(\gamma|R|^2)/2}] \end{aligned}$$

on $\overline{\Omega} \times \overline{\Omega}$. For $r(x) = [e^{(\gamma|x|^2)/2} - e^{(\gamma|R|^2)/2}]$ we have

$$\begin{cases} r(x) > 0 & \text{for } x \notin B(0,R) \\ r(x) = 0 & \text{for } x \in \partial B(0,R) \\ r(x) < 0 & \text{for } x \in B(0,R) \end{cases}$$

for all $\gamma > 0$. Now let $(x_\alpha, y_\alpha) \in \overline{\Omega} \times \overline{\Omega}$ be a maximum point of $h(x,y)$. It is possible to show that $x_\alpha, y_\alpha \to z$ and $y_\alpha = x_\alpha - (\varepsilon n(z) + o(1))/\alpha \in \Omega$ as $\alpha \to \infty$. This can be proved very briefly. For sufficiently large α the point $z - (\varepsilon/\alpha)n(z)$ lies in the interior of Ω. Now the inequality $h(x_\alpha, y_\alpha) \ge h(z, z - (\varepsilon/\alpha)n(z))$ implies:

$$\begin{aligned} &\frac{1}{2}|\alpha(x_\alpha - y_\alpha) - \varepsilon n(z)|^2 - \left[e^{\frac{\gamma|x_\alpha|^2}{2}} - e^{\frac{\gamma|R|^2}{2}}\right] \\ &\qquad \le u(x_\alpha) - v(y_\alpha) - u(z) + v\left(z - \frac{\varepsilon n(z)}{\alpha}\right) \end{aligned}$$

Since $\alpha(x_\alpha - y_\alpha)$ remains bounded as $\alpha \to \infty$, we get under the assumption that $\alpha(x_\alpha - y_\alpha) \to w$ and (for subsequences) $x_\alpha, y_\alpha \to \hat{z}$ as $\alpha \to \infty$:

$$0 \le |w - \varepsilon n(z)|^2 - [e^{(\gamma|\hat{z}|^2)/2} - e^{(\gamma|R|^2)/2}] \le 0 \quad \text{for} \quad \alpha \to \infty.$$

This implies $\hat{z} = z$ and $y_\alpha = x_\alpha - (\varepsilon n(z) + o(1))/\alpha \in \Omega$.

By the continuity of the subsolution we can assume that $u(x_\alpha) > f(x_\alpha)$. Let us now choose

$$\gamma > \gamma_0 := \max\{1, c_1|x|^+/c_0(|x|^-)^2\}$$

arbitrary, but fixed, where $|x|^+ := \max\{|x| \mid x \in \overline{\Omega}\}$ $(< \infty)$ as in the proof of Theorem 3 and $|x|^- := \min\{|x| \mid x \in \overline{\Omega}\}$ (> 0, remember $0 \notin \overline{\Omega}$).

Now applying the *maximum principle for semicontinuous functions* for $\mathcal{O} = \overline{\Omega} \times \overline{\Omega}$ and $\phi(x,y) = \frac{1}{2}|\alpha(x-y) - \varepsilon n(z)|^2 - [e^{(\gamma|x|^2)/2} - e^{(\gamma|R|^2)/2}]$ we get:

$$\begin{aligned} D_x\phi(x_\alpha, y_\alpha) &= \alpha^2(x_\alpha - y_\alpha) - \alpha\varepsilon n(z) - \gamma x_\alpha e^{(\gamma|x_\alpha|^2)/2}, \\ -D_y\phi(x_\alpha, y_\alpha) &= \alpha^2(x_\alpha - y_\alpha) - \alpha\varepsilon n(z), \end{aligned}$$

and the existence of $X, Y \in \mathcal{S}(N)$ such that the following inequalities hold:

$$\begin{aligned} F(u(x_\alpha), \alpha^2(x_\alpha - y_\alpha) - \alpha\varepsilon n(z) - \gamma x_\alpha e^{(\gamma|x_\alpha|^2)/2}, X) &\le 0, \\ F(v(y_\alpha), \alpha^2(x_\alpha - y_\alpha) - \alpha\varepsilon n(z), Y) &\ge 0 \end{aligned}$$

and

$$\begin{aligned} \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le 3\alpha^2 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \begin{pmatrix} -2\alpha^2 I & \alpha^2 I \\ \alpha^2 I & 0 \end{pmatrix} \\ + \begin{pmatrix} D_x^2\phi(x_\alpha, y_\alpha) + \frac{1}{\alpha^2}(D_x^2\phi(x_\alpha, y_\alpha))^2 & -D_x^2\phi(x_\alpha, y_\alpha) \\ -D_x^2\phi(x_\alpha, y_\alpha) & 0 \end{pmatrix}. \end{aligned}$$

By multiplying this inequality from the left and from the right with a vector $\begin{pmatrix} \xi \\ \xi \end{pmatrix} \in \mathbb{R}^{2N}$, we derive

$$\left\langle \left(X - \left(\frac{1}{\alpha^2}(D_x^2\phi(x_\alpha, y_\alpha))^2 - D_x^2\phi(x_\alpha, y_\alpha) \right) \right) \xi, \xi \right\rangle \le \langle Y\xi, \xi \rangle.$$

Now we have to look at $(D_{ij}^\alpha) := \frac{1}{\alpha^2}(D_x^2\phi(x_\alpha, y_\alpha))^2 - D_x^2\phi(x_\alpha, y_\alpha)$. This matrix has the following structure:

$$(D_{ij}^\alpha) = \begin{cases} D_{ii}^\alpha = -\gamma^2 {x_{\alpha}}_i^2 e^{(\gamma|x_\alpha|^2)/2} - \gamma e^{(\gamma|x_\alpha|^2)/2} + (S_{ii}/\alpha^2) \\ D_{ij}^\alpha = -\gamma^2 x_{\alpha i} x_{\alpha j} e^{(\gamma|x_\alpha|^2)/2} + (S_{ij}/\alpha^2) \quad \text{for} \quad i \neq j \end{cases}$$

where

$$S_{ii} = \gamma^2 e^{\gamma|x_\alpha|^2}(1 + 2\gamma {x_{\alpha}}_i^2 + \gamma^2 {x_{\alpha}}_i^2 |x_\alpha|^2) \quad \text{and}$$
$$S_{ij} = \gamma^3 x_{\alpha i} x_{\alpha j} e^{\gamma|x_\alpha|^2}(2 + \gamma|x_\alpha|^2).$$

Since $(S_{ii}/\alpha^2), (S_{ij}/\alpha^2) \to 0$ as $\alpha \to \infty$ calculation shows that for sufficiently large α the matrix (D_{ij}^α) is negative definite. So we calculate:

$$\begin{aligned} 0 &\le F(v(y_\alpha), \alpha^2(x_\alpha - y_\alpha) - \alpha\varepsilon n(z), Y) \\ &\quad - F\Big(u(x_\alpha), \alpha^2(x_\alpha - y_\alpha) - \alpha\varepsilon n(z) - \gamma x_\alpha e^{\frac{\gamma|x_\alpha|^2}{2}}, X\Big) \\ &\le \omega(c_1 \cdot |\gamma x_\alpha e^{(\gamma|x_\alpha|^2)/2}|) \\ &\quad - \omega\left(c_0 \cdot trace\left(D_x^2\phi(x_\alpha, y_\alpha) - \frac{1}{\alpha^2}(D_x^2\phi(x_\alpha, y_\alpha))^2\right)\right) \\ &= -\omega\left(c_0 \gamma e^{\frac{\gamma|x_\alpha|^2}{2}}\left(\gamma|x_\alpha|^2 + N - \frac{\gamma e^{\frac{\gamma|x_\alpha|^2}{2}}(N + 2\gamma|x_\alpha|^2 + \gamma^2|x_\alpha|^4)}{\alpha^2}\right)\right) \\ &\quad + \omega(c_1 \gamma e^{(\gamma|x_\alpha|^2)/2}|x_\alpha|) \end{aligned}$$

Sending $\alpha \to \infty$ we get a contradiction by the strict monotonicity of ω, since

$$\frac{\gamma e^{\frac{\gamma|x_\alpha|^2}{2}}(N + 2\gamma|x_\alpha|^2 + \gamma^2|x_\alpha|^4)}{\alpha^2} \to 0$$

and

$$\omega(c_1 \gamma e^{(\gamma|x_\alpha|^2)/2}|x_\alpha|) - \omega(c_0 \gamma e^{(\gamma|x_\alpha|^2)/2}(\gamma|x_\alpha|^2 + N)) < 0$$

by the choice of γ.

The case $v(z) < f(z)$ is treated similarly by replacing the function h by

$$\begin{aligned} \tilde{h}(x, y) \;=\; & u(x) - v(y) - \frac{1}{2}|\alpha(x - y) + \varepsilon n(z)|^2 \\ & + [e^{(\gamma|y|^2)/2} - e^{(\gamma|R|^2)/2}]. \end{aligned}$$

Then we get the analogous results.

Case 2: There exists a point $z \in \Omega$, such that for $M > 0$,

$$u(z) - v(z) = \max_{x\in\Omega}(u(x) - v(x)) = M > \max_{x\in\partial\Omega}(u(x) - v(x)).$$

Let us look at the function

$$k(x,y) = u(x) - v(y) - \frac{\alpha}{2}|x-y|^2 + \varrho\, e^{(\gamma|x|^2)/2}$$

where we fix $\gamma > \gamma_0$ (with γ_0 from Case 1) first and then choose $0 < \varrho$ sufficiently small and fixed, such that the maximum of $u(x)-v(x)+\varrho e^{\gamma|x|^2/2}$ remains in an interior point of $\overline{\Omega}$, where the difference of u and v is positive. The constant ϱ then depends on the fixed values $M, \gamma, |x|^+$ and $|x|^-$. For the maximum point $(x_\alpha, y_\alpha) \in \Omega\times\Omega$ of $k(x,y)$ we have that $(\alpha/2)|x_\alpha - y_\alpha|^2 \to 0$ as $\alpha \to \infty$. For proving this claim let $\hat{z} \in \Omega$ be the positive maximum of

$$g(x) = u(x) - v(x) + \varrho\, e^{(\gamma|x|^2)/2}$$

and set

$$\kappa(x) = u(x) + \varrho\, e^{(\gamma|x|^2)/2}.$$

The inequality $k(x_\alpha, y_\alpha) \geq k(\hat{z}, \hat{z})$ implies:

$$\kappa(\hat{z}) - v(\hat{z}) \leq \kappa(\hat{z}) - v(\hat{z}) + \frac{\alpha}{2}|x_\alpha - y_\alpha|^2 \leq \kappa(x_\alpha) - v(y_\alpha)$$

Since κ and $-v$ are bounded from above we get $x_\alpha, y_\alpha \to \tilde{z}$ and $(\alpha/2)|x_\alpha - y_\alpha|^2 \to 0$ as $\alpha \to \infty$. By this choice of ϱ it is guaranteed that $\lim_{\alpha\to\infty} \sup_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} k(x,y)$ approximates a point in Ω, where the difference of the sub- and the supersolution is strictly positive.

Using the well known strategy and applying Theorem 2 one gets

$$F(u(x_\alpha), \alpha(x_\alpha - y_\alpha) - \gamma\varrho\, x_\alpha e^{(\gamma|x_\alpha|^2)/2}, X) \leq 0,$$
$$F(v(y_\alpha), \alpha(x_\alpha - y_\alpha), Y) \geq 0$$

and the inequalities

$$-\left(\frac{1}{\delta} + ||D^2\phi(x_\alpha, y_\alpha)||\right)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix},$$

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \begin{pmatrix} D_x^2\phi(x_\alpha, y_\alpha) & -\alpha I \\ -\alpha I & \alpha I \end{pmatrix}$$

$$+ \delta \begin{pmatrix} (D_x^2\phi(x_\alpha, y_\alpha))^2 + \alpha^2 I & -\alpha D_x^2\phi(x_\alpha, y_\alpha) - \alpha^2 I \\ -\alpha D_x^2\phi(x_\alpha, y_\alpha) - \alpha^2 I & 2\alpha^2 I \end{pmatrix}.$$

Setting $\delta = \frac{1}{\alpha}$, the second inequality implies:

$$\left\langle \left(X - \left(\frac{1}{\alpha}(D_x^2\phi(x_\alpha, y_\alpha))^2 - D_x^2\phi(x_\alpha, y_\alpha) \right) \right) \xi, \xi \right\rangle \le \langle Y\xi, \xi \rangle.$$

For sufficiently large α the matrix $\frac{1}{\alpha}(D_x^2\phi(x_\alpha, y_\alpha))^2 - D_x^2\phi(x_\alpha, y_\alpha)$ is negative definite (as can be proved similarly to the first case). This leads to the following calculations:

$$\begin{aligned} 0 \le\; & F(v(y_\alpha), \alpha(x_\alpha - y_\alpha), Y) \\ & - F(u(x_\alpha), \alpha(x_\alpha - y_\alpha) - \gamma\varrho\, x_\alpha e^{(\gamma|x_\alpha|^2)/2}, X) \\ \le\; & \omega(c_1\gamma\varrho\, e^{(\gamma|x_\alpha|^2)/2}|x_\alpha|) \\ & - \omega\left(c_0\varrho\, \gamma e^{(\gamma|x_\alpha|^2)/2} \left(\gamma|x_\alpha|^2 + N - \frac{S}{\alpha} \right) \right) < 0. \end{aligned}$$

Sending $\alpha \to \infty$ we reach a contradiction, since $S/\alpha \to 0$ as $\alpha \to \infty$, where

$$S := \varrho\, \gamma e^{\frac{\gamma|x_\alpha|^2}{2}} (N + 2\gamma|x_\alpha|^2 + \gamma^2|x_\alpha|^4).$$

This contradicts the assumption that there exists a positive maximum of the difference between the sub- and the supersolution in the interior of Ω. This completes the proof of our Theorem. □

As a direct consequence of Theorem 4 we can formulate the following Corollary.

Corollary 1 *Let $\Omega \subset \mathbb{R}^N$ be open and bounded and let $F \in C(\mathbb{R} \times \mathbb{R}^N \times \mathcal{S}(N))$ be ω-elliptic and proper. If $u \in USC(\overline{\Omega})$ is a viscosity subsolution and $v \in LSC(\overline{\Omega})$ a viscosity supersolution of $F = 0$ in Ω with $u \le v$ on $\partial\Omega$, then $u \le v$ in $\overline{\Omega}$.*

Remark 2. Corollary 1 is a generalization of case (ii) of Theorem 1 in [7].

Remark 3. For the proof of Theorem 4 it is essential, that matrices $X, Y, W \in \mathcal{S}(N)$ exist, such that $X + W \le Y$ and $0 < W$.

This is guaranteed by setting $\phi(x,y) = \frac{1}{2}|\alpha(x-y) - \varepsilon n(z)|^2 - [e^{(\gamma|x|^2)/2} - e^{(\gamma|R|^2)/2}]$ respectively $\phi(x,y) = \frac{\alpha}{2}|x-y|^2 - \varrho e^{(\gamma|x|^2)/2}$, but it is not guaranteed when using $\phi(x,y) = \frac{1}{2}|\alpha(x-y) - \varepsilon n(z)|^2 + \varepsilon|x-z|^2$ as it has been done in [3].

4. Counterexamples

I will give three counterexamples where comparison fails and where the operator F does not fulfill the ω−ellipticity structure condition.

Example 3.

a) First we look at the problem

$$-u''(x) - (1 + (u'(x))^2)^{3/2} = 0 \text{ in } (-1, 1) \tag{13}$$

$$u(-1) = 0 \text{ and } u(1) = 0. \tag{14}$$

This problem has the viscosity solution $u(x) = \sqrt{1 - x^2}$. Every translate $u_c = u + c$ with $c \geq 0$ gives us a new generalized viscosity solution of (13) and (14). To see this we notice that u_c is a solution of (13) and a supersolution of the boundary value problem since $u_c(-1) = u_c(1) > 0$. But u_c is also a subsolution, since there exist no testfunctions for u_c at the boundary of $[-1, 1]$.

b) Next we look at

$$\frac{-u''(x)}{\sqrt{1 + (u''(x))^2}} + f(u'(x)) = 0 \text{ in } (0, 1)$$
$$u(0) = u(1) = 0,$$

where f is defined as

$$f(p) = \begin{cases} \sqrt{\dfrac{2 - p^2}{3 - 3p^2 + p^4}} & |p| \leq 1 \\ 1 & |p| > 1. \end{cases}$$

This problem has the solution $u(x) = \frac{2}{3}(x^{3/2} + (1 - x)^{3/2}) - \frac{2}{3}$. Again every translate $u_c(x) = u(x) - c$ $(c > 0)$ is a generalized viscosity solution, since

$$F(u_c(x), u_c'(x), u_c''(x)) = 0 \text{ in } (0, 1)$$
$$u_c(0) = u_c(1) = -c < 0,$$

and for every testfunction ψ from below $\psi'(0) < -1$ and $\psi'(1) > 1$, such that $f(\psi'(0)) = f(\psi'(1)) = 1$. Since we have

$$-\psi''(0)/\sqrt{1 + (\psi''(0))^2} \geq -1 \text{ and } -\psi''(1)/\sqrt{1 + (\psi''(1))^2} \geq -1,$$

one gets the two following inequalities

$$\max\{u_c(0), F(u_c(0), \psi'(0), \psi''(0))\} \geq 0$$

and

$$\max\{u_c(1), F(u_c(1), \psi'(1), \psi''(1)\} \geq 0.$$

So every translate u_c is a generalized viscosity solution. Thus comparison must fail again.

c) For the last example I refer once more to [1] where the following problem is given:

$$\begin{aligned} -\Delta u(x) + |Du(x)|^m &= 0 \quad \text{in} \quad B(0,R) \\ u(x) &= c \quad \text{on} \quad \partial B(0,R) \end{aligned}$$

where $R > 0$, $0 < m < 1$ and $c \in \mathbb{R}$. This problem has the two classical solutions $u_1(x) \equiv c$ and $u_2(x) = k(k+N-2)^{1/(m-1)}(R^k - |x|^k) + c$ for $x \in B(0,R)$ with $k = (2-m)/(1-m)$. Thus comparison fails, but as the two operators in a) and b) above the given operator is not ω-elliptic.

Remark 4. The examples a) and b) were given to me by N. Kutev. Using a generalization of the structural conditions from Theorem 4 of the present manuscript, he and B. Kawohl derived related comparison principles in [10].

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