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The subobject classifier of the category of functional bisimulations

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Abstract. We show the existence of subobject classifier in the category of nondeterministic dynamical systems and functional bisimulations.

Key words: nondeterministic dynamical system, functional bisimulation, coalgebra, subobject classifier, dense.

1. Introduction

In [8], we studied the category $\mathcal{NDyn}$ of nondeterministic dynamical systems whose morphisms are functional bisimulations.

A nondeterministic dynamical system is a labelled transition system whose label set has only one element. A functional bisimulation is a map between transition systems. The main results of [8] are the following.

- The category $\mathcal{NDyn}$ is an autonomous category, i.e., monoidal closed.
- There exists a subobject classifier.

The monoidal closedness was shown by constructing $\mathcal{NDyn}$ objects via the presheaves over the category $\mathcal{Tree}$, where the $\mathcal{Tree}$ is a small, dense subcategory of $\mathcal{NDyn}$. On the other hand the existence of the subobject classifier was proved by using the theory of hypersets.

In this paper, we prove the existence of subobject classifier in $\mathcal{NDyn}$ by using the construction via presheaves over $\mathcal{Tree}$ in the same way as the proof of monoidal closedness in [8]. The proof uses a general lemma about presheaf categories, which is given in [10].

As we remark later, $\mathcal{NDyn}$ is a category of coalgebras for finite powerset functor without empty set. We can brush up the technique which is used in this paper, to an existence theorem [7] of subobject classifiers in categories of coalgebras by using accessible category theory, which led to another existential proof [3] in the context of topos theory.

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It is generally difficult to describe explicitly the structure of the subobject classifiers for coalgebra categories even if they exist. However there are a few exceptional cases. One is in [9, 8], where the truth-value object of $\mathcal{ND}yn$ was given as a universe of hereditarily finite hypersets. Another example was given in [3] for the categories of coalgebras of finite powerset functor.

The significance and applications of the existence of subobject classifier in $\mathcal{ND}yn$ have not been fully considered yet. But we can show it implies the regularity of $\mathcal{ND}yn$ [7, 3], and hence we can define category of relations over $\mathcal{ND}yn$ which is the category of nondeterministic dynamical systems and bisimulations.

We proceed as follows. First we recall the definitions of nondeterministic dynamical systems and the category $\mathcal{ND}yn$ of them in Section 2. We recall some of the basic facts of $\mathcal{ND}yn$, for example, the existence of terminal object, characterization of monic arrows in $\mathcal{ND}yn$ and cocompleteness. We give the concrete construction of coproduct and coequalizer.

In Section 3, we show the existence of small, dense subcategory $\mathcal{T}ree$ in $\mathcal{ND}yn$. Then the category $\mathcal{ND}yn$ turns out to be a reflective subcategory of $\mathbf{Set}^{\mathcal{T}ree^{\mathrm{op}}}$, and hence complete.

We apply the criterion given in [10], and show the existence of subobject classifier in Section 4.

2. The category $\mathcal{ND}yn$

2.1. Definitions

First of all, we recall the definitions of nondeterministic dynamical systems and the category $\mathcal{ND}yn$ they form.

Definition 2.1 A **nondeterministic dynamical system** $D = (|D|, \tau_D)$ consists of a set $|D|$ with a binary relation $\tau_D \subseteq |D| \times |D|$. Elements of $|D|$ are called **states** and τ_D the **transitions**. When $(x, y) \in \tau_D$, we denote $x \to_D y$ and call y a **child** of x. For each $x \in |D|$, define

$$\mathrm{child}_D(x) = \{y \in |D| \mid x \to_D y\}.$$

We call a finite sequence $x, z_1, z_2, \ldots, z_n$ of $|D|$ a **path** from x to z_n if $x \to_D z_1$ and $z_i \to_D z_{i+1}$ for each $1 \leq i \leq n-1$.

We introduce a notion of morphism between nondeterministic dynamical systems. Let $D = (|D|, \tau_D)$, $D' = (|D'|, \tau_{D'})$ be nondeterministic dynamical systems. A map $\varphi : |D| \to |D'|$ is called a **functional bisimulation** from D to D' if

$$\varphi(\mathrm{child}_D(x)) = \mathrm{child}_{D'}(\varphi(x))$$

for all $x \in |D|$.

Next we give the definition of subsystems based on the functional bisimulations. We also give some examples of subsystems induced by them. A nondeterministic dynamical system $D_0 = (|D_0|, \tau_{D_0})$ is called a **subsystem** of $D = (|D|, \tau_D)$ if $|D_0| \subseteq |D|$ and $\mathrm{child}_{D_0}(x) = \mathrm{child}_D(x)$ for all $x \in |D_0|$. Then the inclusion map $|D_0| \hookrightarrow |D|$ is a functional bisimulation.

Let $f : D \to D'$ be a functional bisimulation.

The **image** $\mathrm{Im}(f)$ of f is defined by $\mathrm{Im}(f) = (f(|D|), \tau_{D'}|_{f(|D|)\times f(|D|)})$, where $\tau_{D'}|_{f(|D|)\times f(|D|)}$is a restriction of $\tau_{D'}$ on $f(|D|) \times f(|D|)$. Obviously $\mathrm{Im}(f)$ is a subsystem of D'.

The **inverse image** $f^{-1}(V)$ of a subsystem $V = (|V|, \tau_V)$ of D' is defined by $|f^{-1}(V)| = f^{-1}(|V|) = \{x \in |D| \mid f(x) \in |V|\}$ and $x \to_{f^{-1}(V)} y$ if $x \to_D y$. Obviously $f^{-1}(V)$ is a subsystem of D.

Now we define the category $\mathcal{NDyn}$ of nondeterministic dynamical systems.

Definition 2.2 The category $\mathcal{NDyn}$ is defined as follows. An object of $\mathcal{NDyn}$ is a nondeterministic dynamical systems $D = (|D|, \tau_D)$ which satisfies,

- $\mathrm{child}_D(x) \neq \emptyset$ for all $x \in |D|$,
- $|\mathrm{child}_D(x)| < \infty$ for all $x \in |D|$.

An arrow $\varphi : D \to D'$ from $\mathcal{NDyn}$ object D to D' is a functional bisimulation $\varphi : |D| \to |D'|$.

Remark 2.3 The category $\mathcal{NDyn}$ can also be described as the category of coalgebras for endofunctor $\mathbf{pow_o}$ on **Set**, which is defined by

- $\mathbf{pow_o}(A)$ is the set of all the nonempty finite subsets of A,
- For each map $f : A \to B$, $\mathbf{pow_o}(f) : \mathbf{pow_o}(A) \to \mathbf{pow_o}(B)$ maps a nonempty finite set $A_0 \subseteq A$ to its image $\{f(x) \mid x \in A_0\} \subset B$, which is obviously nonempty and finite.

We recall some of the basic properties of $\mathcal{NDyn}$ [8].

2.2. The terminal object

The terminal object $1 = (\{*\}, \{(*, *)\})$ exists in $\mathcal{NDyn}$.

2.3. Monic arrows

The monic arrows in $\mathcal{NDyn}$ have the following properties.

Proposition 2.4 *A $\mathcal{NDyn}$ arrow is monic if and only if the underlying map is injective.*

Proof. Let $m : D \to D'$ be a monic arrow in $\mathcal{NDyn}$. Suppose $m(x_1) = m(x_2)$ for $x_1, x_2 \in |D|$. Let $D_0 = (|D_0|, \tau_{D_0})$ be the object of $\mathcal{NDyn}$ defined by

$$|D_0| = \{(y_1, y_2) \in |D| \times |D| \mid m(y_1) = m(y_2)\},$$

and

$$(y_1, y_2) \to_{D_0} (z_1, z_2) \text{ iff } y_1 \to_D z_1 \text{ and } y_2 \to_D z_2.$$

Define $\pi_1, \pi_2 : D_0 \to D$ by $\pi_i(y_1, y_2) = y_i$ for $i = 1, 2$. These maps are arrows in $\mathcal{NDyn}$. Now $m \circ \pi_1 = m \circ \pi_2$ by construction of π_i's. Thus $\pi_1 = \pi_2$ by the monicity of m. Hence we have $x_1 = \pi_1(x_1, x_2) = \pi_2(x_1, x_2) = x_2$, and so m is injective.

The reverse implication is obvious. □

Corollary 2.5 *A subobject is represented by a uniquely determined subsystem.*

Proof. Let r be any subobject of object $D \in \mathcal{NDyn}$. Then there is a subsystem $\mathrm{Im}(r)$ of D, and r is equivalent to the inclusion $\mathrm{Im}(r) \hookrightarrow D$. □

2.4. Cocompleteness

As we noticed in Remark 2.3, the category $\mathcal{NDyn}$ is a category of coalgebras for endofunctor on **Set**, which implies the following property by using [1, Proposition 1.1 or 2, Proposition 2.1].

Proposition 2.6 *The category $\mathcal{NDyn}$ is cocomplete.*

The colimit of each diagram in $\mathcal{NDyn}$ can be constructed by using coproducts and coequalizer. We need to construct explicitly colimits of diagram in $\mathcal{NDyn}$ later, so we review the construction of coproduct and coequalizer.

Let $\{D_k : k \in K\}$ be a family of objects in $\mathcal{NDyn}$ indexed by a set K. Its coproduct $D = (|D|, \tau_D)$ is defined by

$$|D| = \coprod_{k \in K} |D_k|$$

and

$$x \to_D y \quad \text{iff} \quad \text{there exists some } k \in K \text{ such that} \\ x, y \in |D_k| \text{ and } x \to_{D_k} y$$

and universal cocone consists of the inclusion functional bisimulations $|D_k| \hookrightarrow |D|$.

Let $f, g : D_1 \to D_2$ be functional bisimulations. The coequalizer of f and g is given by a functional bisimulation $q : D_2 \to D$, which is constructed as follows: The $D = (|D|, \tau_D)$ is given by

$$|D| = |D_2|/R$$

where R is the smallest equivalence relation on $|D_2|$ generated by

$$\{(f(x), g(x)) \mid x \in D_1\},$$

and τ_D is defined by

$$[x] \to_D [y] \quad \text{if} \quad x \to_{D_2} y,$$

where $[x]$ is the equivalence class of $x \in |D_2|$. The q is given by the quotient map $q : |D_2| \to |D_2|/R$.

3. Small and dense subcategory of $\mathcal{NDyn}$

3.1. The category $\mathcal{Tree}$

Definition 3.1 Let $\mathbf{N}$ be the set of natural numbers and $\mathbf{N}^*$ be the set of finite words over $\mathbf{N}$. Let T be a subset of $\mathbf{N}^*$. We say

- T is **prefix closed** if $w \in T$ implies $v \in T$ for all prefixes v of w.
- T is **infinite** if for all $v \in T$ there is at least one $i \in \mathbf{N}$ with $v.i \in T$.
- T is **locally finite** if for each $v \in T$ the set $\{i \in \mathbf{N} \mid v.i \in T\}$ is a finite set of the form $\{1, 2, \ldots, n_v\}$.

Observe that every prefix closed subset $T \subset \mathbf{N}^*$ contains the empty word ϵ. Each prefix closed, infinite, locally finite subset $T \subset \mathbf{N}^*$ determines a $\mathcal{NDyn}$ object (T, τ_T) defined by $v \to_T w$ when $w = v.i (i \in \mathbf{N})$; which is called a **finitely branching tree**. Finitely branching trees and their functional bisimulations define a category $\mathcal{T}ree$, which is a full subcategory of $\mathcal{NDyn}$. We denote the inclusion functor by $i : \mathcal{T}ree \hookrightarrow \mathcal{NDyn}$. By the construction, the category $\mathcal{T}ree$ is small.

Let $D = (|D|, \tau_D)$ be a $\mathcal{NDyn}$ object. A **numbering** α on D is a family of bijections $\alpha_x : \mathrm{child}_D(x) \to \{1, 2, \ldots, |\mathrm{child}_D(x)|\}$ $(x \in |D|)$. Given a numbering α on D, define $\mathrm{Path}_\alpha(x) \subset \mathbf{N}^*$ for $x \in |D|$ by

$$\begin{aligned} \mathrm{Path}_\alpha(x) := \{\epsilon\} \cup \{ & \alpha_x(z_1).\alpha_{z_1}(z_2)\ldots\alpha_{z_{n-1}}(z_n) \mid \\ & x, z_1, z_2, \ldots, z_n \text{ is a path in } D\}. \end{aligned}$$

Then $\mathrm{Path}_\alpha(x)$ is a prefix closed, infinite, locally finite subset of $\mathbf{N}^*$. Hence it determines an object of $\mathcal{T}ree$, which is also denoted by $\mathrm{Path}_\alpha(x)$. There is a canonical $\mathcal{NDyn}$ arrow $\gamma_x : i(\mathrm{Path}_\alpha(x)) \to D$ in $\mathcal{NDyn}$ defined inductively by $\gamma_x(\epsilon) = x$, and $\gamma_x(v.i) = {\alpha_{\gamma_x(v)}}^{-1}(i)$, for $v.i \in \mathrm{Path}_\alpha(x)$ with $i \in \mathbf{N}$.

Lemma 3.2 *Let D be an object of $\mathcal{NDyn}$ and let α be a numbering on it. For each object $T \in \mathcal{T}ree$ and an arrow $f : i(T) \to D$ in $\mathcal{NDyn}$, there exists $x \in |D|$ and an arrow $\bar{f} : T \to \mathrm{Path}_\alpha(x)$ in $\mathcal{T}ree$ such that the following diagram commutes.*

$$\begin{array}{ccc} i(T) & \xrightarrow{\;i(\bar{f})\;} & i(\mathrm{Path}_\alpha(x)) \\ & \searrow^{f} & \downarrow{\scriptstyle \gamma_x} \\ & & D \end{array}$$

Proof. First put $x = f(\epsilon_T)$, where ϵ_T is the empty word in T. Define a map $\bar{f} : T \to \mathrm{Path}_\alpha(x)$ inductively by $\bar{f}(\epsilon_T) = \epsilon_{\mathrm{Path}_\alpha(x)}$, and for the word $w = v.i \in T$ with $i \in \mathbf{N}$, $\bar{f}(w) = \bar{f}(v).\alpha_{f(v)}(f(w))$. Then this map $\bar{f}$ turns out to be a functional bisimulation, and makes the above diagram commute. □

3.2. Density

In this subsection, we show that the category $\mathcal{T}ree$ is dense in $\mathcal{NDyn}$.

Let $D = (|D|, \tau_D)$ be any $\mathcal{NDyn}$ object and α be a numbering on it. Now let $\Gamma(D)$ be the free category generated by the graph $(|D|, \tau_D)$. Define a graph map $E : D^{\mathrm{op}} \to \mathcal{Tree}$ by $E(x) = \mathrm{Path}_\alpha(x)$ for $x \in |D|$, and $E(x \to_D y)(w) = \alpha_x(y).w$, and extending it a functor $\Gamma(D)^{\mathrm{op}} \to \mathcal{Tree}$, denoted also by E.

Lemma 3.3

$$\mathrm{Colim}(i \circ E) \cong D$$

Proof. Put $D' = \mathrm{Colim}(i \circ E)$. Then the set of states $|D'|$ is given by

$$|D'| = \coprod_{x \in |D|} |i(\mathrm{Path}_\alpha(x))| / \simeq,$$

where $\simeq$ is the smallest equivalence relation generated by

$$w \simeq i \circ E(x \to_D y)(w)$$

for $w \in |i(\mathrm{Path}_\alpha(y))|$ and $x \to_D y$. It is easily seen that

$$|i(\mathrm{Path}_\alpha(x))| \ni v \mapsto \gamma_x(v)$$

induce a bijection $|D'| \to |D|$ which is obviously functional bisimulation. Hence we have $\mathrm{Colim}\, i \circ E \cong D$. □

Proposition 3.4 *The category $\mathcal{Tree}$ is dense in $\mathcal{NDyn}$.*

Proof. In order to show the density of $\mathcal{Tree}$ in $\mathcal{NDyn}$, we have to show each object $D \in \mathcal{NDyn}$ is isomorphic to the colimit of its canonical diagram

$$i(-)/D \xrightarrow{\partial} \mathcal{Tree} \xhookrightarrow{i} \mathcal{NDyn},$$

where $i(-)/D$ is the comma category and $\partial : i(-)/D \to \mathcal{Tree}$ is the projection functor. Fix an object $D \in \mathcal{NDyn}$ and a numbering α on D. Let $G : \Gamma(D)^{\mathrm{op}} \to i(-)/D$ be the functor defined by $G(x) = \gamma_x$ and $G(x \to_D z) = E(x \to_D z)$ for $x \in \Gamma(D)^{\mathrm{op}}$ and an edge $x \to_D z$ of $\Gamma(D)$. It follows from Lemma 3.2 that

$$\begin{aligned}&\mathrm{Colim}(i(-)/D \xrightarrow{\partial} \mathcal{Tree} \xhookrightarrow{i} \mathcal{NDyn})\\ &\quad\cong \mathrm{Colim}(\Gamma(D)^{\mathrm{op}} \xrightarrow{G} i(-)/D \xrightarrow{\partial} \mathcal{Tree} \xhookrightarrow{i} \mathcal{NDyn}).\end{aligned}$$

Because the diagram $i \circ \partial \circ G = i \circ E$, we have

$$\mathrm{Colim}(i(-)/D \xrightarrow{\partial} \mathcal{T}ree \overset{i}{\hookrightarrow} \mathcal{ND}yn) \cong \mathrm{Colim}(i \circ E) \cong D$$

by Lemma 3.3. Since D was arbitrary, we have shown that the inclusion functor $i : \mathcal{T}ree \hookrightarrow \mathcal{ND}yn$ is dense. □

3.3. Reflective subcategory

As a consequence of the existence of small dense subcategory $\mathcal{T}ree$ of the cocomplete $\mathcal{ND}yn$, the category $\mathcal{ND}yn$ turns out to be a reflective subcategory of $\mathbf{Set}^{\mathcal{T}ree^{\mathrm{op}}}$ [10, Propositon 2.4]. The full and faithful right adjoint functor $R : \mathcal{ND}yn \to \mathbf{Set}^{\mathcal{T}ree^{\mathrm{op}}}$ is given for each object $D \in \mathcal{ND}yn$ by

$$R(D) = \mathcal{ND}yn(i(-), D).$$

The left adjoint functor $L : \mathbf{Set}^{\mathcal{T}ree^{\mathrm{op}}} \to \mathcal{ND}yn$ is given for each presheaf P by

$$LP = \mathrm{Colim}\Big(\int P \xrightarrow{\pi_P} \mathcal{T}ree \overset{i}{\hookrightarrow} \mathcal{ND}yn\Big).$$

Here $\int P$ is the category of elements of a presheaf P defined as follows:

- Its object is a pair (T, p) of an object $T \in \mathcal{T}ree$ and $p \in P(T)$.
- And an arrow $u : (T, p) \to (T', p')$ is a $\mathcal{T}ree$ arrow $u : T \to T'$ such that $p' \cdot u = p$, where $p' \cdot u := P(u)(p')$.

The functor $\pi_P : \int P \to \mathcal{T}ree$ is the projection: $(T, p) \mapsto T$. The composition of functors

$$\int P \xrightarrow{\pi_P} \mathcal{T}ree \overset{i}{\hookrightarrow} \mathcal{ND}yn$$

is a diagram in $\mathcal{ND}yn$ with the indexing category $\int P$, denoted simply by $i \circ \pi_P$.

Remark 3.5 The canonical diagram of each $D \in \mathcal{ND}yn$ is nothing but the diagram $i \circ \pi_{R(D)} : \int R(D) \to \mathcal{ND}yn$. Because the functor $i : \mathcal{T}ree \hookrightarrow \mathcal{ND}yn$ is dense, we have, for each object $D \in \mathcal{ND}yn$,

$$\begin{aligned} LR(D) &= \mathrm{Colim}\Big(\int R(D) \xrightarrow{\pi_{R(D)}} \mathcal{T}ree \overset{i}{\hookrightarrow} \mathcal{ND}yn\Big) \\ &= \mathrm{Colim}(i(-)/D \xrightarrow{\partial} \mathcal{T}ree \overset{i}{\hookrightarrow} \mathcal{ND}yn) \cong D. \end{aligned}$$

According to the construction in Section 2.4, the nondeterministic dynamical system LP for a presheaf P can be given concretely as follows: The set of states $|LP|$ is given by

$$|LP| = \coprod_{T \in \mathcal{T}ree} |i(T)| \times P(T) / \simeq_{LP}$$

where $\simeq_{LP}$ is the smallest equivalence relation on $\coprod_{T \in \mathcal{T}ree} |i(T)| \times P(T)$ generated by

$$(t, p' \cdot u) \simeq_{LP} (i(u)(t), p')$$

for $t \in |i(T)|$, $u : T \to T'$, $p' \in P(T')$, $T, T' \in \mathcal{T}ree$. The transition relation $\to_{LP}$ is given by

$$[(t_1, p)] \to_{LP} [(t_2, p)] \quad \text{if} \quad t_1 \to_{i(T)} t_2, \ p \in P(T), \ T \in \mathcal{T}ree.$$

According to the notation [10, Section 4.1], we denote the universal cocone $i \circ \pi_P \to LP$ for each presheaf P by

$$\left\{ \kappa_p^{LP} : i(T) \to LP \mid (T, p) \in \int P \right\}. \tag{1}$$

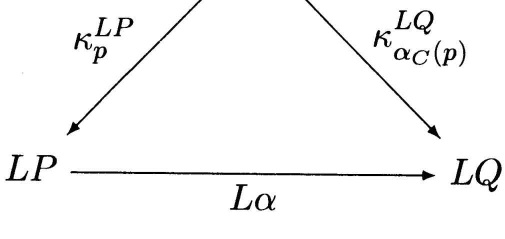
Then the element $\kappa_p^{LP} : i(T) \to LP$ is given by

$$\kappa_p^{LP}(t) = [(t, p)] \quad \text{for} \quad t \in |i(T)| \tag{2}$$

from the construction of LP.

The following results are used in Section 4.

Lemma 3.6 ([10, Lemma 4.2]) *Let $\alpha : P \to Q$ be a $\mathbf{Set}^{\mathcal{T}ree^{\mathrm{op}}}$ arrow. For each $p \in P(T)$ with $T \in \mathcal{T}ree$, the following diagram commutes in $\mathcal{ND}yn$.*



3.4. Completeness

Since $\mathcal{ND}yn$ is a reflective subcategory of presheaf category $\mathbf{Set}^{\mathcal{T}ree^{\mathrm{op}}}$

and the presheaf category is complete, we obtain by using [2, Proposition 3.5.3]:

Proposition 3.7 *The category $\mathcal{NDyn}$ is complete.*

Hence all pullbacks exist in $\mathcal{NDyn}$. In particular, the pullbacks of monic arrows exist in $\mathcal{NDyn}$. Consequently there exists a subobject functor $\mathrm{Sub} : \mathcal{NDyn}^{\mathrm{op}} \to \mathbf{Set}$, and we have the following property for the pullbacks of monic arrows.

Lemma 3.8 *Let $f : D \to D'$ be a $\mathcal{NDyn}$ arrow and $r \in \mathrm{Sub}(D')$, then the following diagram is a pullback in $\mathcal{NDyn}$*

$$\begin{array}{ccc} f^{-1}(\mathrm{Im}(r)) & \xrightarrow{f^*} & \mathrm{Im}(r) \\ \cap\downarrow & & \downarrow\cap \\ D & \xrightarrow[f]{} & D', \end{array}$$

where f^* is the restriction of f to $|f^{-1}(\mathrm{Im}(r))| \subseteq |D|$.

4. The existence of a subobject classifier in $\mathcal{NDyn}$

Now applying the criterion of [10], we show the existence of subobject classifier in the category $\mathcal{NDyn}$.

Define $\mathbf{Set}^{\mathcal{T}ree^{\mathrm{op}}}$ arrow $\xi : R(1) \to \mathrm{Sub}(i(-))$, a natural transformation, by

$$\xi_T(!_{i(T)}) = \mathrm{id}_{i(T)}$$

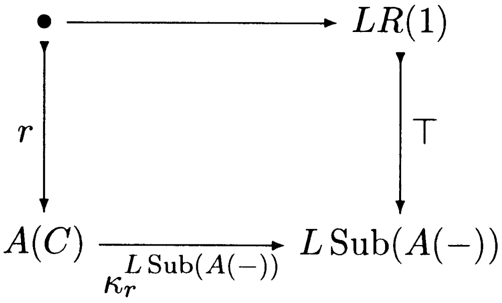
for $!_{i(T)} \in R(1)(T)$ and $T \in \mathcal{T}ree$. Set $\top = L\xi : LR(1) \to L\,\mathrm{Sub}(i(-))$. Then $\top$ is a monic $\mathcal{NDyn}$ arrow since $LR(1) \cong 1$ from Remark 3.5.

We recall the criterion for the existence of a subobject classifier.

Criterion for the existence of a subobject classifier ([10, Corollary 4.6]) *Let the universal cocone of the diagram $i \circ \pi_{\mathrm{Sub}(i(-))}$ be given by the collection*

$$\left\{\kappa_r^{L\,\mathrm{Sub}(i(-))} : i(T) \to L\,\mathrm{Sub}(i(-)) \mid (T, r) \in \int \mathrm{Sub}(i(-))\right\}. \quad (3)$$

If the following diagram is a pullback for each $(T, r) \in \int \mathrm{Sub}(i(-))$ then there exists a subobject classifier in $\mathcal{NDyn}$:



(4)

Theorem 4.1 *There exists a subobject classifier in the category $\mathcal{NDyn}$.*

Proof. Now we fix $(T, r) \in \int \mathrm{Sub}(i(-))$, and show that the diagram (4) is pullback. To show this, by Lemma 3.8, it suffices to show

$$\kappa_r^{L\,\mathrm{Sub}(i(-))^{-1}}(|\,\mathrm{Im}(\top)|) = |\,\mathrm{Im}(r)| \tag{5}$$

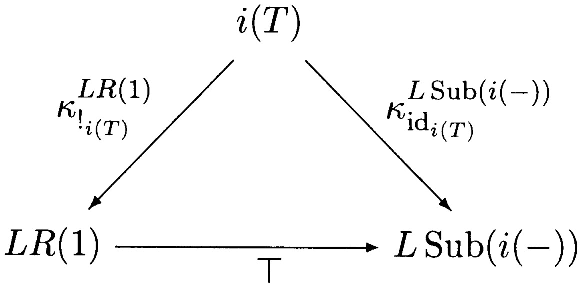
as a map.

Now we start the verification of (5). Since i is dense, the object $LR(1) \in \mathcal{NDyn}$ is isomorphic to a terminal object in $\mathcal{NDyn}$, so we denote $LR(1) = (\{*\}, \{(*, *)\})$. Put

$$\mathtt{true} = \top(*) \in |L\,\mathrm{Sub}(i(-))|.$$

Then $\mathrm{child}_{L\,\mathrm{Sub}(i(-))}(\mathtt{true}) = \{\mathtt{true}\}$ holds in $L\,\mathrm{Sub}(i(-))$, since $\top$ is a functional bisimulation.

By applying Lemma 3.6 for $\mathbf{Set}^{\mathcal{T}ree^{\mathrm{op}}}$ arrow ξ, we have the following commutative diagram since $\xi_T(!_{i(T)}) = \mathrm{id}_{i(T)}$.



Hence we have

$$\kappa_{\mathrm{id}_{i(T)}}^{L\,\mathrm{Sub}(i(-))}(t) = \top \circ \kappa_{!_{i(T)}}^{LR(1)}(t) \tag{6}$$

for any $t \in |i(T)|$. Since the left hand side of (6) is

$$\kappa_{\mathrm{id}_{i(T)}}^{L\,\mathrm{Sub}(i(-))}(t) = [(t, \mathrm{id}_{i(T)})]_{L\,\mathrm{Sub}(i(-))},$$

and since the right hand side is

$$\top\left(\kappa_{!_{i(T)}}^{LR(1)}(t)\right) = \top(*) = \mathtt{true},$$

we obtain

Lemma 4.2

$$[(t, \mathrm{id}_{i(T)})]_{L\,\mathrm{Sub}(i(-))} = \mathtt{true} \quad \textit{in } L\,\mathrm{Sub}(A(-)) \textit{ for each } t \in |i(T)|,\ T \in \mathcal{T}ree.$$

Let $t \in |i(T)|$ and $t' \in |i(T')|$ with $T, T' \in \mathcal{T}ree$.

Lemma 4.3 *If* $(t, \mathrm{id}_{i(T)}) \simeq_{L\,\mathrm{Sub}(i(-))} (t', r)$, *then* $t' \in \mathrm{Im}(r)$.

Proof. Suppose

$$(t, \mathrm{id}_{i(T)}) \simeq_{L\,\mathrm{Sub}(i(-))} (t', r). \tag{7}$$

Then there exists $\int \mathrm{Sub}(i(-))$ diagram

$$\begin{aligned}(T_1, r_1) \overset{f_1}{\to} (T_2, r_2) \overset{f_2}{\leftarrow} (T_3, r_3) \overset{f_3}{\to} \cdots \overset{f_{n-3}}{\leftarrow} (T_{n-2}, r_{n-2})\\ \overset{f_{n-2}}{\to} (T_{n-1}, r_{n-1}) \overset{f_{n-1}}{\leftarrow} (T_n, r_n)\end{aligned} \tag{8}$$

with $(T_1, r_1) = (T, \mathrm{id}_{i(T)})$ and $(T_n, r_n) = (T', r)$, which gives the equivalence (7). Then

$$\begin{aligned} i(f_1)(t) &\in |\,\mathrm{Im}(r_2)| \subseteq |T_2| && \text{by } i(f_1)^{-1}(r_2) = r_1 = \mathrm{id}_{i(T)}\\ i(f_2)^{-1}(i(f_1)(t)) &\subseteq |\,\mathrm{Im}(r_3)| && \text{by } i(f_2)^{-1}(r_2) = r_3\\ i(f_3)(i(f_2)^{-1}(i(f_1)(t))) &\subseteq |\,\mathrm{Im}(r_4)| && \text{by } i(f_3)^{-1}(r_3) = r_4. \end{aligned}$$

By induction, we can show

$$\begin{aligned} &i(f_{n-1})^{-1}(i(f_{n-2})(\cdots i(f_3)(i(f_2)^{-1}(i(f_1)(t)))\cdots))\\ &\quad \subseteq |\,\mathrm{Im}(r_n)| = |\,\mathrm{Im}(r)|. \end{aligned}$$

Since the equivalence (7) is given by $\int \mathrm{Sub}(i(-))$ arrows (8), we have

$$t' \in i(f_{n-1})^{-1}(i(f_{n-2})(\cdots i(f_3)(i(f_2)^{-1}(i(f_1)(t)))\cdots)).$$

Hence we obtain $t' \in |\operatorname{Im}(r)|$, whence the lemma. □

By using Lemma 4.2 and Lemma 4.3, we have the following property.

Lemma 4.4 *Suppose* $t \in |i(T)|$ *and* $r \in \operatorname{Sub}(i(T))$ *with* $T \in \mathcal{T}ree$. *Then* $t \in |\operatorname{Im}(r)|$ *if and only if* $[(t, r)]_{L\operatorname{Sub}(i(-))} = \mathtt{true}$.

Proof. Fix an object $T \in \mathcal{T}ree$ and $r \in \operatorname{Sub}(i(T))$, and suppose $t \in |\operatorname{Im}(r)| \subseteq |i(T)|$, then t is a finite word over $\mathbf{N}$. Here we identify the set $|i(T)|$ with the set T. Define a subset $T' \subset \mathbf{N}^*$ by

$$T' = \{v \in \mathbf{N}^* \mid t.v \in T\}.$$

The set T' is a prefix closed, infinite, locally finite subset of $\mathbf{N}^*$, hence T' determines a $\mathcal{T}ree$ object, denoted also by T'. There is a canonical injective map $m : T' \to T$ defined by $m(v) = t.v$ for each $v \in T'$, which turns out to be a functional bisimulation. Hence $m : T' \to T$ is a monic $\mathcal{T}ree$ arrow, which satisfies $i(m)(\epsilon_{i(T')}) = t$ for empty word $\epsilon_{i(T')} \in |i(T')|$.

Now we have $i(m) \subseteq r$ as a subobject since $i(m)(\epsilon_{i(T')}) = t \in \operatorname{Im}(r)$. Then the following diagram is a pullback in $\mathcal{NDyn}$:

$$\begin{array}{ccc} i(T') & \longrightarrow & \bullet \\ {\scriptstyle \mathrm{id}_{i(T')}}\big\downarrow & & \big\downarrow{\scriptstyle r} \\ i(T') & \xrightarrow[i(m)]{} & i(T) \end{array}$$

Consequently we have

$$\begin{aligned} (t, r) &= (i(m)(\epsilon_{i(T')}), r) \\ &\simeq_{L\operatorname{Sub}(i(-))} (\epsilon_{i(T')}, \mathrm{id}_{i(T')}) && \text{by the definition of } \simeq_{L\operatorname{Sub}(i(-))} \\ &= \mathtt{true} && \text{by Lemma 4.2.} \end{aligned}$$

Conversely suppose $[(t, r)]_{L\operatorname{Sub}(i(-))} = \mathtt{true}$. Then $(t, r) \simeq_{L\operatorname{Sub}(i(-))} (t'', \mathrm{id}_{i(T'')})$ for some $t'' \in |i(T'')|$, $T'' \in \mathcal{T}ree$ by Lemma 4.2. Hence we obtain $t \in |\operatorname{Im}(r)|$ by using the Lemma 4.3. □

From Lemma 4.4, we have

$$\kappa_r^{L\operatorname{Sub}(i(-))}(t) = \mathtt{true} \quad \text{if and only if} \quad t \in |\operatorname{Im}(r)|$$

for $t \in |i(T)|$ and $r \in \mathrm{Sub}(i(T))$, which is the equation (5). Hence the diagram (4) is pullback for each $(T, r) \in \int \mathrm{Sub}(i(-))$, and the criterion is satisfied. Thereby we conclude that the category $\mathcal{ND}yn$ has a subobject classifier. □

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