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A singular integral operator related to block spaces

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Abstract. Let $h(t)$ be an L^{∞} function on $(0,\infty)$, $\Omega(y')$ be a $B_q^{0,0}$ function on the unit sphere satisfying the mean zero property (1.1) and $P_N(t)$ be a real polynomial on $\mathbf{R}$ of degree N satisfying $P_N(0)=0$. We prove that the singular integral operator

$$\left(T_{P_N,h}f\right)(x)=p.v.\int_{\mathbf{R}^n} h(|y|)\Omega(y')|y|^{-n}f(x-P_N(|y|)y')dy$$

is bounded in $L^p(\mathbf{R}^n)$ for $1<p<\infty$, and the bound is independent of the coefficients of $P_N(t)$.

Key words: singular integral, rough kernel, block spaces.

1. Introduction

Let $\mathbf{R}^n$, $n\geqslant 2$, be the n-dimensional Euclidean space and $\mathbf{S}^{n-1}$ be the unit sphere in $\mathbf{R}^n$ equipped with the normalized Lebesque measure $d\sigma=d\sigma(x')$. Let $\Omega(x)$ be a homogenous function of degree zero, with $\Omega\in L^1(S^{n-1})$ and

$$\int_{\mathbf{S}^{n-1}}\Omega(x')d\sigma(x')=0, \tag{1.1}$$

where $x'=\frac{x}{|x|}$ for any $x\neq 0$.

Suppose that $h(t)\in L^{\infty}(0,\infty)$. Let $P_N(t)$ be a polynomial of degree N satisfying $P_N(0)=0$.

The singular integral operator $T_{P_N,h}f$ is defined by

$$\left(T_{P_N,h}f\right)(x)=p.v.\int_{\mathbf{R}^n}K(y)f(x-P_N(|y|)y')dy$$

where $y'=\frac{y}{|y|}\in\mathbf{S}^{n-1}$, $K(y)=h(|y|)\Omega(y')|y|^{-n}$ and $f\in S(R^n)$.

We denote $T_{P_N,h}$ by $T_{I,h}$ if $P_N(t)=t$; and we denote $T_{P_N,h}$ by T_I if $P_N(t)=t$ and $h(t)\equiv 1$.

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The investigation of such operators began with Calderón-Zygmund's pioneering papers [2] [3]. By introducing the rotation method, Calderón and Zygmund proved that the operator T_I is bounded in $L^p(\mathbf{R}^n)$, $1 < p < \infty$, provided $\Omega \in L\mathrm{Log}^+L$. However, Calderón-Zygmund's method fails in studying the general operator $T_{I,h}$, whose kernel has the additional roughness in the radial direction due to the presence of h. This phenomenon was first observed and studied by R. Fefferman [6] and subsequently was studied by many other authors ([1], [4], [5], [7], [8], [9], [10], [14], [17] et al.). Readers may see these references for backgrounds and further extensions. Among these papers, the following theorem can be found in [4] (see also [1] or [14]).

Theorem A *Suppose $n \geq 2$, that Ω is a homogeneous function of degree zero satisfying* (1.1), *h satisfies $\sup_{R>0} \frac{1}{R} \int_0^R |h(t)|^2 dt \leq C$. Then the operator $T_{I,h}$ is bounded in $L^p(\mathbf{R}^n)$, $1 < p < \infty$, provided $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q > 1$.*

In order to weaken the condition $\Omega \in L^q(\mathbf{S}^{n-1})$, Yiang and Lu introduced certain block spaces $B_q^{\mu,\nu}$ on $\mathbf{S}^{n-1}$. Below we briefly review the definition of the spaces $B_q^{\mu,\nu}$. More details can be found in [11].

A q-block on $\mathbf{S}^{n-1}$ is an L^q $(1 < q \leq \infty)$ function $b(.)$ that satisfies

$$\mathrm{supp}(b) \subseteq Q \quad \text{for some } \; Q, \tag{1.2}$$

where $Q = \mathbf{S}^{n-1} \cap \{y \in \mathbf{R}^n : |y - \varsigma| < \rho \text{ for some } \varsigma \in \mathbf{S}^{n-1} \text{ and } \rho \in (0, 1]\}$;

$$\|b\|_q \leq |Q|^{(\frac{1}{q}-1)}. \tag{1.3}$$

For $\mu \geq 0$ and $\nu \in \mathbf{R}$, define a non-negative function $\Phi_{\mu,\nu}$ by

$$\Phi_{\mu,\nu}(t) = \int_t^1 u^{-1-\mu} \log^\nu \frac{1}{u} du \qquad 0 < t < 1;$$
$$\Phi_{\mu,\nu}(t) = 0 \quad \text{if } \; t \geq 1.$$

The block spaces $B_q^{\mu,\nu}$ on $\mathbf{S}^{n-1}$ are defined by

$$B_q^{\mu,\nu}(\mathbf{S}^{n-1}) = \{\Omega \in L^1(\mathbf{S}^{n-1}) : \Omega(y') = \sum_m C_m b_m(y'),$$

each b_m is a q-block supported in Q_m, and $M_q^{\mu,\nu}(\{C_m\}) < \infty\}$ where

$$M_q^{\mu,\nu}(\{C_m\}) = \sum_m |C_m| \{1 + \Phi_{\mu,\nu}(|Q_m|)\}.$$

The "norm" $M_q^{\mu,\nu}(\Omega)$ of $\Omega \in B_q^{\mu,\nu}$ is defined by $M_q^{\mu,\nu}(\Omega) = \inf \{M_q^{\mu,\nu}(\{C_m\})\}$, where the infimum is taken over all q-block decompositions of Ω.

The method of block decomposition of functions was originated by M.H. Taibleson and G. Weiss in the study of the convergence of the Fourier series (see [16]). Later on, many applications of the block decomposition to harmonic analysis were discovered. For the details, readers may see the survey book [11], which is a good reference in this topic. Particularly, one can find in [11] (see also [13]) that

$$L^q(\mathbf{S}^{n-1}) \subseteq B_q^{\mu,\nu}(\mathbf{S}^{n-1}) \subseteq B_q^{\mu,\varepsilon}(\mathbf{S}^{n-1}) \quad \text{for } \nu > \varepsilon,\ \mu \in \mathbf{R};$$

and

$$B_q^{\mu,\alpha}(\mathbf{S}^{n-1}) \subseteq B_q^{\delta,\beta}(\mathbf{S}^{n-1}) \subseteq B_q^{0,\gamma}(\mathbf{S}^{n-1})$$

for $0 < \delta < \mu$ and any real numbers α, β, γ.

In [9], [10], Jiang and Lu proved the following theorem.

Theorem B *Suppose $n \geq 2$, Ω is a homogeneous function of degree zero satisfying* (1.1). *If h is a bounded radial function, then $T_{I,h}$ is bounded in $L^p(\mathbf{R}^n)$, $1 < p < \infty$, provided $\Omega \in B_q^{\mu,0}$ for some $q > 1$ and $\mu > 0$.*

It was noted by Keitoku and Sato in [12] that Theorem B is essentially the same of Theorem A, since for any $q > 1$ and $\mu > 0$, $B_q^{\mu,0}(\mathbf{S}^{n-1}) \subseteq \bigcup_{r>1} L^r(\mathbf{S}^{n-1})$. But Keitoku and Sato pointed out that $\bigcup_{r>1} L^r(\mathbf{S}^{n-1})$ is properly contained in $B_q^{0,0}(\mathbf{S}^{n-1})$ for any $q > 1$. The relationship between $B_q^{0,0}(\mathbf{S}^{n-1})$ and $L \log^+ L(\mathbf{S}^{n-1})$ remains open. Now we state the main result in this paper.

Theorem 1 *Suppose that Ω is a homogeneous function of degree zero satisfying* (1.1), *and that h is an $L^\infty(0,\infty)$ function. Then, the operator $T_{P_N,h}$ is bounded in $L^p(\mathbf{R}^n)$ if $\Omega \in B_q^{0,0}(\mathbf{S}^{n-1})$ for some $q > 1$, and the bound is independent of the coefficients of the polynomial P_N.*

Remarks.

1. For the reason of simplicity, in Theorem 1, we assume $h \in L^\infty(0,\infty)$. Actually, this condition can be weakened to the condition $\sup_{R>0} \frac{1}{R} \int_0^R |h(t)|^2 dt < \infty$. Readers can find this easy treatment in [8]. Thus, even in the case $P(t) = t$, our theorem is an improvement of Theorem A and B.

2. By the definition, $\Omega \in B_q^{0,0}(\mathbf{S}^{n-1})$ has the block decomposition

$$\Omega(y') = \sum_m C_m b_m(y') \tag{1.4}$$

where each b_m is a q-block, supported in Q_m and

$$\sum_m |C_m| \left\{1 + \log^+\left(\frac{1}{|Q_m|}\right)\right\} < \infty.$$

Throughout this paper, we always use the letter C to denote a positive constant that may vary at each occurrence but is independent of the essential variables.

2. Some Lemmas

Lemma A (Van der Corput [15]) *Suppose ϕ and f are smooth functions on $[a,b]$ and ϕ is real-valued. If $|\phi^{(k)}(x)| \geq 1$ for $x \in [a,b]$ then*

$$\left|\int_a^b e^{i\lambda\phi(t)} f(t)dt\right| \leq C_k |\lambda|^{\frac{-1}{k}} \left[\|f\|_\infty + \|f'\|_1\right]$$

holds when

(i) $k \geq 2$

(ii) *or $k = 1$, if in addition it is assumed that $\phi'(t)$ is monotonic.*

Let h and $\Omega(y') = \sum C_m b_m(y')$ be as in Theorem 1. For the polynomial $P_N(t) = \sum_{\lambda=1}^N \beta_\lambda t^\lambda$, we denote $P_r(t) = \sum_{\lambda=1}^r \beta_\lambda t^\lambda$ for $r = 1, 2, \ldots, N$. Define the following functions and operators:

$$B_{m,r}(f)(x) = \int_{\mathbf{R}^n} h(|y|)|y|^{-n} b_m(y') f(x - P_r(|y|)y')dy;$$

$$\hat{\sigma}_{\Omega,k,r}(\xi) = \int_{2^k \leq |y| \leq 2^{k+1}} h(|y|)|y|^{-n}\Omega(y') e^{-iP_r(|y|)\langle y',\xi\rangle}dy;$$

$$\hat{\mu}_{b_m,k,r}(\xi) = \int_{2^k \leq |y| \leq 2^{k+1}} |y|^{-n}|h(|y|)b_m(y')| e^{-iP_r(|y|)\langle y',\xi\rangle}dy;$$

$$\hat{\mu}_{\Omega,k,r}(\xi) = \int_{2^k \leq |y| \leq 2^{k+1}} |y|^{-n}|h(|y|)\Omega(y')| e^{-iP_r(|y|)\langle y',\xi\rangle}dy;$$

$$\hat{\Lambda}_{\Omega,k,r}(\xi) = \int\limits_{2^k \le |y| \le 2^{k+1}} |y|^{-n}|\Omega(y')|e^{-iP_r(|y|)\langle y',\xi\rangle}dy;$$

$$\sigma^*_{b_m,r}f(x) = \sup_k |\mu_{b_m,k,r} * f(x)|; \quad \sigma^*_{\Omega,r}f(x) = \sup_k |\mu_{\Omega,k,r} * f(x)|;$$

$$\Lambda^*_{\Omega,r}f(x) = \sup_k |\Lambda_{\Omega,k,r} * f(x)|;$$

Clearly, we have

$$T_{P_N,h}(f) = \sum_k \sigma_{\Omega,k,N} * f$$

Also, for each $r = 1, 2, \ldots, N$, we can write

$$T_{P_r,h}(f) = \sum_m \sum_k C_m \ \sigma_{b_m,k,r} * f$$

where

$$\hat{\sigma}_{b_m,k,r}(\xi) = \int\limits_{2^k \le |y| \le 2^{k+1}} h(|y|)|y|^{-n}b_m(y')e^{-iP_r(|y|)\langle \xi,y'\rangle}dy.$$

It is easy to see that $\|\hat{\sigma}_{\Omega,k,N}\|_\infty \le C$, $\|\hat{\sigma}_{b_m,k,N}\|_\infty < C$ uniformly for k and m.

Lemma 2.1 *For $1 < p < \infty$ and $h \in L^\infty(0,\infty)$, there is a constant C such that $\|\sigma^*_{b_m,r}(f)\|_p \le C\|f\|_p$ and $\|\sigma^*_{\Omega,r}f\|_p \le C\|\Lambda^*_{\Omega,r}(f)\|_p \le C\|f\|_p$, where the constant C is independent of the block $b_m(.)$ and the coefficients of the polynomial P_r (it may depend on r).*

Proof. Clearly, we only need to prove the first inequality.

Now

$$\mu_{b_m,k,r} * f(x) = \int\limits_{2^k \le |y| \le 2^{k+1}} |h(|y|)|y|^{-n}b_m(y')|f(x - P_r(|y|)y')dy.$$

Write $P_r(|y|)y' = (\Pi_1(|y|), \Pi_2(|y|), \ldots, \Pi_n(|y|))$, where each Π_j is a polynomial of $|y|$ whose coefficients depend on y'.

We denote $\tilde{P}(|y|) = (\Pi_1(|y|), \Pi_2(|y|), \ldots, \Pi_n(|y|))$, then

$$\mu_{b_m,k,r} * f(x) = \int\limits_{2^k \le |y| \le 2^{k+1}} |h(|y|)|y|^{-n}b_m(y')|f(x - \tilde{P}(|y|))dy.$$

Thus

$$\sup_{k\in Z} |\mu_{b_m,k,r} * f(x)|$$
$$\leq \sup_{k\in Z} \|h\|_\infty 2^{-nk} \int_{2^k\leq|y|\leq 2^{k+1}} |b_m(y')f(x-\tilde{P}(|y|))|dy$$
$$\leq C \sup_{k\in Z} 2^{-k} \int_{|t|\leq 2^{k+1}} \int_{\mathbf{S}^{n-1}} |b_m(y')|\,|f(x-\tilde{P}(t))|d\sigma(y')dt.$$

Noting $\|b_m\|_{L^1(\mathbf{S}^{n-1})} \leq C$, where C is independent of b_m, by Hölder's inequality we have

$$\left(\sup_{k\in Z} |\mu_{b_m,k,r} * f(x)|\right)^p$$
$$\leq C \int_{\mathbf{S}^{n-1}} \left(\sup_{s>0} \frac{1}{s} \int_0^s |f(x-\tilde{P}(t))|dt\right)^p |b_m(y')|d\sigma(y').$$

Therefore, we have

$$\left\|\sup_{k\in Z} |\mu_{b_m,k,r} * f(x)|\right\|^p_{L^p(R^n)}$$
$$\leq C \int_{\mathbf{S}^{n-1}} |b_m(y')| \left\|\sup_{s>0} \frac{1}{s} \int_0^s |f(.-\tilde{P}(t))|dt\right\|^p_{L^p(R^n)} d\sigma(y').$$

It was shown that

$$\left\|\sup_{s>0} \frac{1}{s} \int_0^s |f(.-\tilde{P}(t))|dt\right\|^p_{L^p(R^n)} \leq C\|f\|^p_{L^p(R^n)} \quad \text{for } 1 < p < \infty$$

with C independent of the coefficients of $\tilde{P}$ (thus independent of y') (see [15] pages 476–478). So we have

$$\left\|\sup_{k\in Z} |\mu_{b_m,k,r} * f(x)|\right\|_{L^p(R^n)} \leq C\Big(\int_{\mathbf{S}^{n-1}} |b_m(y')|d\sigma(y')\Big)\|f\|^p_{L^p(R^n)}$$
$$\leq C\|f\|_{L^p(R^n)} \quad \text{for } 1 < p < \infty.$$

□

Now we prove a key estimate in this paper:

Lemma 2.2 *Let $h \in L^\infty(0,\infty)$. For $q > 1$, suppose that b_m is a q-block with support in Q_m. Denote $\hat\sigma_{\Omega,k,0}(\xi) \equiv 0$, then for $r = 1, 2, \dots, N$*

(i) $|\hat\sigma_{\Omega,k,r}(\xi) - \hat\sigma_{\Omega,k,r-1}(\xi)| \le C|2^{rk}\beta_r\xi|$;

(ii) $|\hat\sigma_{b_m,k,r}(\xi)| \le C|2^{rk}\xi\beta_r|^{\frac{1}{2r\log|Q_m|}}$ *if* $|Q_m| < e^{\frac{q}{1-q}}$ *and* $\beta_r \ne 0$;

(iii) $|\hat\sigma_{b_m,k,r}| \le C|2^{rk}\beta_r\xi|^{-\omega}$ *if* $|Q_m| \ge e^{\frac{q}{1-q}}$ *and* $\beta_r \ne 0$

where C and ω are positive constants independent of $k \in \mathbf{Z}$, $\xi \in \mathbf{R}^n$, the block b_m, and the coefficients of the polynomials P_r.

Proof.

$$\begin{aligned}
&|\hat\sigma_{\Omega,k,r}(\xi) - \hat\sigma_{\Omega,k,r-1}(\xi)| \\
&= \left| \int_{2^k\le|y|\le 2^{k+1}} h(|y|)|y|^{-n}\Omega(y')e^{-iP_r(t)\langle y',\xi\rangle}dy \right. \\
&\qquad \left. - \int_{2^k\le|y|\le 2^{k+1}} h(|y|)|y|^{-n}\Omega(y')e^{-iP_{r-1}(t)\langle y',\xi\rangle}dy \right| \\
&\le C \int_{2^k}^{2^{k+1}} t^{-1}\left| \int_{\mathbf{S}^{n-1}} \Omega(y')e^{-iP_{r-1}(t)\langle y,\xi\rangle}\{e^{-i\beta_r t^r\langle y',\xi\rangle} - 1\}d\sigma(y')\right|dt \\
&\le C \int_{2^k}^{2^{k+1}} t^{-1} \int_{\mathbf{S}^{n-1}} |\Omega(y')|\,|\beta_r t^r\langle y',\xi\rangle|d\sigma(y')dt \\
&= C \int_{2^k}^{2^{k+1}} |\beta_r|t^{r-1}\left| \int_{\mathbf{S}^{n-1}} |\Omega(y')|\,|\langle y',\xi\rangle|d\sigma(y')\right|dt \\
&\le C\|\Omega\|_{L^1(\mathbf{S}^{n-1})}|\beta_r\xi| \int_{2^k}^{2^{k+1}} t^{r-1}dt \\
&\le C|\beta_r\xi|2^{rk} \quad \text{and (i) is proved.}
\end{aligned}$$

If $|Q_m| < e^{\frac{q}{1-q}}$, let $p_m = \log|Q_m|/\{1+\log|Q_m|\}$, then $1 < p_m$. By Hölder's inequality we have

$$|\hat\sigma_{b_m,k,r}(\xi)|^2 \le C \int_{2^k}^{2^{k+1}} \left| \int_{\mathbf{S}^{n-1}} b_m(y')e^{-iP_r(t)\langle y',\xi\rangle}d\sigma(y')\right|^2 t^{-1}dt.$$

Now

$$\left| \int\limits_{\mathbf{S}^{n-1}} b_m(y')e^{-iP_r(t)\langle y',\xi\rangle} d\sigma(y') \right|^2$$
$$= \iint\limits_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}} b_m(y')\bar{b}_m(x')e^{-iP_r(t)\langle y'-x',\xi\rangle} d\sigma(y')d\sigma(x'),$$

and by the Van der Corput lemma (Lemma A), if $r > 1$,

$$\int\limits_{2^k}^{2^{k+1}} e^{-iP_r(t)\langle y'-x',\xi\rangle} t^{-1}dt = \int\limits_1^2 e^{iP_r(2^kt)\langle y'-x',\xi\rangle} t^{-1}dt$$
$$\leq C(|2^k\beta_r^{\frac{1}{r}}||\langle y'-x',\xi\rangle|^{\frac{1}{r}})^{-1}.$$

By the easy fact $|\int_{2^k}^{2^{k+1}} e^{-iP_r(t)\langle y'-x',\xi\rangle}t^{-1}dt| \leq \log 2$, we have

$$\left|\int\limits_{2^k}^{2^{k+1}} e^{-iP_r(t)\langle y'-x',\xi\rangle}t^{-1}dt\right| \leq C|2^{kr}\beta_r\langle y'-x',\xi\rangle|^{\frac{-\delta}{r}}$$

for any $0 < \delta \leq 1$.

Let $\delta = \frac{1}{p'_m}$, we have

$$|\hat{\sigma}_{b_m,k,r}(\xi)| \leq C|2^{rk}\beta_r\xi|^{\frac{-1}{2rp'_m}}$$
$$\times \left\{ \iint\limits_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}} b_m(y')\bar{b}_m(x')(|\xi'.(y'-x')|)^{\frac{-1}{rp'_m}} d\sigma(y')d\sigma(x')\right\}^{\frac{1}{2}}$$

where $\xi' = \xi/|\xi|$. Thus by Hölder's inequality

$$|\hat{\sigma}_{b_m,k,r}(\xi)| \leq C|2^{rk}\beta_r\xi|^{\frac{-1}{2rp'_m}} \|b_m\|_{L^{p_m}(\mathbf{S}^{n-1})}$$
$$\times \left\{ \iint\limits_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}} (|\xi'.(y'-x')|)^{\frac{-1}{r}} d\sigma(y')d\sigma(x')\right\}^{\frac{1}{2p'_m}}.$$

By the choice of p_m, and Hölder's inequality, we have

$$\|b_m\|_{L^{p_m}(\mathbf{S}^{n-1})} \leq C\|b_m\|_{L^q(\mathbf{S}^{n-1})}|Q_m|^{\frac{1}{p_m}-\frac{1}{q}}$$
$$\leq C|Q_m|^{\frac{1}{p_m}-1} \leq C|Q_m|^{\frac{1}{\log|Q_m|}} \leq C.$$

So, by recalling $p'_m = -\log|Q_m|$, we obtain

$$|\hat{\sigma}_{b_m,k,r}(\xi)| \le C|2^{rk}\beta_r\xi|^{\frac{1}{2r\log|Q_m|}}.$$

For $r = 1$, note the easy fact

$$\left|\int_1^2 e^{iP_1(2^k t)\langle y'-x',\xi\rangle}t^{-1}dt\right| \le \min\left\{\log 2, \frac{1}{|2^k\beta_1\langle\xi,(y'-x')\rangle|}\right\}$$

where the first inequality can be obtained by bringing the absolute value inside the integral and the second inequality can be obtained by integration by parts.

So we have

$$\left|\int_1^2 e^{iP_1(2^k t)\langle y'-x',\xi\rangle}t^{-1}dt\right| \le \frac{C}{|2^k\beta_1\langle\xi,(y'-x')\rangle|^{\frac{1}{2}}}.$$

Thus, using the same argument as in $r > 1$, we have

$$|\hat{\sigma}_{b_m,k,1}(\xi)| \le C|2^k\beta_1\xi|^{\frac{1}{2\log|Q_m|}}.$$

This proves (ii).

To prove (iii), we take a $\lambda > 1$ such that $\lambda \le \min\{q, 2\}$. Following the proof in (ii), we have

$$\begin{aligned} |\hat{\sigma}_{b_m,k,r}(\xi)| \;&\le\; C|2^{rk}\beta_r\xi|^{\frac{-\delta}{2r}}\|b_m\|_{L^\lambda(\mathbf{S}^{n-1})} \\ &\times\left\{\iint_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}} (|\xi'.(y'-x')|^{\frac{1}{r}})^{-\delta\lambda'}d\sigma(y')d\sigma(x')\right\}^{\frac{1}{2\lambda'}} \end{aligned}$$

for any $\delta \in (0,1]$.

Since $\|b_m\|_{L^\lambda(\mathbf{S}^{n-1})} \le C$, letting $\delta < \frac{1}{\lambda'}$ we obtain

$$|\hat{\sigma}_{b_m,k,r}(\xi)| \le C|2^{rk}\beta_r\xi|^{\frac{-\delta}{2r}}.$$

□

We now choose and fix a function $\phi \in C_0^\infty(\mathbf{R})$ such that $\phi(t) \equiv 1$ for $|t| \le 1$ and $\phi(t) \equiv 0$ for $|t| > 2$.

Let $\varphi(t) = \phi(t^2)$. In order to use an induction argument, we define the measures $\{\tau_{\Omega,k,N-\lambda}\}$ and $\{\tau_{b_m,k,N-\lambda}\}$ by

$$\begin{aligned}\hat{\tau}_{\Omega,k,N-\lambda}(\xi) &= \hat{\sigma}_{\Omega,k,N-\lambda}(\xi) \prod_{N-\lambda<l\le N} \varphi(|2^{lk}\beta_{N-\lambda}\xi|) \\ &\quad - \hat{\sigma}_{\Omega,k,N-\lambda-1}(\xi) \prod_{N-\lambda-1<l\le N} \varphi(|2^{lk}\beta_{N-\lambda}\xi|), \\ \hat{\tau}_{b_m,k,N-\lambda}(\xi) &= \hat{\sigma}_{b_m,k,N-\lambda}(\xi) \prod_{N-\lambda<l\le N} \varphi(|2^{lk}\beta_{N-\lambda}\xi|) \\ &\quad - \hat{\sigma}_{b_m,k,N-\lambda-1}(\xi) \prod_{N-\lambda-1<l\le N} \varphi(|2^{lk}\beta_{N-\lambda}\xi|)\end{aligned}$$

for $k \in \mathbf{Z}$ and $\lambda = 0, 1, \ldots, N-1$, where we use the convention $\prod_{j\in\emptyset} a_j = 1$.

Since $\hat{\sigma}_{\Omega,k,0} = 0$, we find that

$$\sigma_{\Omega,k,N} = \sum_{\lambda=0}^{N-1} \tau_{\Omega,k,N-\lambda}. \tag{2.4}$$

Note that

$$T_{P_N,h}(f) = \sum_{k=-\infty}^{\infty} \sigma_{\Omega,k,N} * f = \sum_{\lambda=0}^{N-1} \sum_{k=-\infty}^{\infty} \tau_{\Omega,k,N-\lambda} * f,$$

so we have

$$\|T_{P_N,h}(f)\|_{L^p(\mathbf{R}^n)} \le \sum_{\lambda=0}^{N-1} \left\| \sum_{k=-\infty}^{\infty} \tau_{\Omega,k,N-\lambda} * f \right\|_{L^p(\mathbf{R}^n)}.$$

Thus, to prove the theorem, it suffices to show

$$\left\| \sum_{k=-\infty}^{\infty} \tau_{\Omega,k,N-\lambda} * f \right\|_{L^p(\mathbf{R}^n)} \le C\|f\|_{L^p(\mathbf{R}^n)} \quad \text{for } \lambda = 0, 1, 2, \ldots, N-1. \tag{2.5}$$

It is easy to see that $\tau_{\Omega,k,N-\lambda} = 0$ if $\beta_{N-\lambda} = 0$. Thus, without loss of generality we assume $\beta_{N-\lambda} \neq 0$ for $\lambda = 0, 1, \ldots, N-1$.

By Lemma (2.2) (i) we find

$$\begin{aligned}|\hat{\tau}_{\Omega,k,N-\lambda}(\xi)| \le &[|\hat{\sigma}_{\Omega,k,N-\lambda}(\xi) - \hat{\sigma}_{\Omega,k,N-\lambda-1}(\xi)| \\ &+ \|\hat{\sigma}_{\Omega,k,N-\lambda-1}(\xi)\|\,|1 - \varphi(|2^{(N-\lambda)k}\beta_{N-\lambda}\xi|)|] \\ &\times \prod_{N-\lambda<l\le N} |\varphi(|2^{lk}\beta_{N-\lambda}\xi|)|\end{aligned}$$

$$\leq C2^{(N-\lambda)k}|\beta_{N-\lambda}\xi|. \tag{2.6}$$

By Lemma (2.2) (ii) and (iii) we find that if $\operatorname{supp}(b_m) \subseteq Q_m$ with $|Q_m| < e^{\frac{q}{1-q}}$ then

$$|\hat{\tau}_{b_m,k,N-\lambda}(\xi)| \leq C|2^{(N-\lambda)k}\beta_{N-\lambda}\xi|^{\frac{1}{2(N-\lambda)\log|Q_m|}}, \tag{2.7}$$

and if $\operatorname{supp}(b_m) \subseteq Q_m$ with $|Q_m| \geq e^{\frac{q}{1-q}}$ then

$$|\hat{\tau}_{b_m,k,N-\lambda}(\xi)| \leq C|2^{(N-\lambda)k}\beta_{N-\lambda}\xi|^{-\omega} \quad \text{for some } \omega > 0. \tag{2.7$'$}$$

Also, by Lemma (2.1), it is easy to see that

$$\left\|\sup_{k\in\mathbf{Z}} |\tau_{\Omega,k,N-\lambda}| * f\right\|_p \leq C\|f\|_p \tag{2.8}$$

and

$$\left\|\sup_{k\in\mathbf{Z}} |\tau_{b_m,k,N-\lambda}| * f\right\|_p \leq C\|f\|_p \tag{2.9}$$

and the bounds are independent of b_m, and the coefficients of the polynomials.

By applying (2.9), we can obtain the following modified lemma in [4].

Lemma 2.3 *For arbitrary functions g_k,*

$$\left\|\left(\sum_k |\tau_{b_m,k,N-\lambda} * g_k|^2\right)^{\frac{1}{2}}\right\|_{p_0} \leq C\left\|\left(\sum_k |g_k|^2\right)^{\frac{1}{2}}\right\|_{p_0}$$

for $1 < p_0 < \infty$, where C is independent of b_m, and the coefficients of the polynomials.

Proof. The proof is a minor modification of those in Lemma of [4]. In fact, it suffices to consider the case $p_0 \geq 2$ so that $q = (\frac{p_0}{2})$.

There exist $u \in L^q_+$ of unit norm such that

$$\begin{aligned}
&\left\|\left(\sum_k |\tau_{b_m,k,N-\lambda} * g_k|^2\right)^{\frac{1}{2}}\right\|_{p_0}^2 \\
&\quad = \int_{\mathbf{R}^n}\left(\sum_k |\tau_{b_m,k,N-\lambda} * g_k(x)|^2\right)u(x)dx \\
&\quad \leq \sum_k \int_{\mathbf{R}^n}\left(|\tau_{b_m,k,N-\lambda}| * |g_k(x)|^2\right)u(x)dx
\end{aligned}$$

$$\leq \sum_k \int_{\mathbf{R}^n} |g_k(x)|^2 \sup_{k\in\mathbf{Z}} ||\tau_{b_m,k,N-\lambda}| * u(x)|dx$$
$$\leq \left\|\left(\sum_k |g_k(x)|^2\right)^{\frac{1}{2}}\right\|_{p_0}^2 \left\|\sup_{k\in\mathbf{Z}} |\tau_{b_m,k,N-\lambda}| * u\right\|_q.$$

By (2.9) we obtain

$$\left\|\left(\sum_{k\in\mathbf{Z}} |\tau_{b_m,k,r} * g_k|^2\right)^{\frac{1}{2}}\right\|_{p_0}^2 \leq C_{p_0}\left\|\left(\sum_{k\in\mathbf{Z}} |g_k|^2\right)^{\frac{1}{2}}\right\|_{p_0}.$$

□

3. Proof of main theorem

As we discussed in Section 2, to prove the theorem, it suffices to prove (2.5). By 2 in Remarks, we write $\Omega(y') = \sum_m C_m b_m(y')$. By (2.7) and (2.7′), without loss of generality, we may assume that the support Q_m of b_m are uniformly small such that $|Q_m| < e^{\frac{q}{1-q}}$.

Let $\{\Phi_j\}_{-\infty}^{\infty}$ be a smooth partition of unity in $(0, \infty)$ adapted to the interval $(2^{-(N-\lambda)j}\beta_{N-\lambda}^{-1}, 2^{-(N-\lambda)j+1}\beta_{N-\lambda}^{-1})$. To be precise, we require the following:

$$\Phi_j \in C^\infty,\ 0 \leq \Phi_j \leq 1,\ \sum_j \Phi_j(t)^2 = 1,$$
$$\mathrm{supp}(\Phi_j) \subseteq (2^{-(N-\lambda)j-1}\beta_{N-\lambda}^{-1}, 2^{-(N-\lambda)j+1}\beta_{N-\lambda}^{-1}).$$

Define the multiplier operators S_j in $\mathbf{R}^n$ by

$$(S_j f)^\wedge(\xi) = f^\wedge(\xi)\Phi_j(|\xi|).$$

We have

$$\begin{aligned}\sum_{k=-\infty}^{\infty} \tau_{\Omega,k,N-\lambda} * f &= \sum_k \tau_{\Omega,k,N-\lambda} * \left(\sum_j S_{j+k}S_{j+k}f\right)\\ &= \sum_j\left(\sum_k S_{j+k}(\tau_{\Omega,k,N-\lambda} * S_{j+k}f)\right)\\ &= \sum_j I_j f.\end{aligned}$$

Thus

$$\left\| \sum_{k=-\infty}^{\infty} \tau_{\Omega,k,N-\lambda} * f \right\|_p \le \left\| \sum_{j\ge 0} I_j f \right\|_p + \left\| \sum_{j<0} I_j f \right\|_p .$$

From classical Littlewood-Paley theory and Lemma (2.3), we know that

$$\|I_j f\|_p \le C\|f\|_p \text{ with } C \text{ independent of } j. \tag{3.1}$$

By the Plancherel theorem

$$\|I_j f\|_2^2 \le C \sum_k \int_{E_{j+k,N-\lambda}} |\hat{f}(\xi)|^2 |\hat{\tau}_{\Omega,k,N-\lambda}(\xi)|^2 d\xi$$

where

$$E_{j+k,N-\lambda} = \left\{ \xi : 2^{-(N-\lambda)(j+k)-1} \beta_{N-\lambda}^{-1} \le |\xi| < 2^{-(N-\lambda)(j+k)+1} \beta_{N-\lambda}^{-1} \right\}.$$

Thus by (2.6) we have

$$\begin{aligned} \|I_j f\|_2^2 &\le C \sum_k \int_{E_{j+k,N-\lambda}} |\hat{\tau}_{\Omega,k,N-\lambda}(\xi)|^2 |\hat{f}(\xi)|^2 d\xi \\ &\le C \sum_k \int_{E_{j+k,N-\lambda}} |2^{(N-\lambda)k} \beta_{N-\lambda} \xi|^2 |\hat{f}(\xi)|^2 d\xi \\ &\le C 2^{-2(N-\lambda)j} \|\hat{f}(\xi)\|_2^2. \end{aligned} \tag{3.2}$$

Using interpolation between (3.1) and (3.2), we obtain

$$\left\| \sum_{j\ge 0} I_j f \right\|_p \le C\|f\|_p. \tag{3.3}$$

On the other hand

$$\left\| \sum_{j<0} I_j f \right\|_p \le \sum_{j<0} \|I_j f\|_p \le \sum_{j<0} \sum_m |C_m| \, \|I_{j,m} f\|_p$$

where

$$I_{j,m} f = \sum_k S_{j+k} (\tau_{b_m,k,N-\lambda} * S_{j+k} f).$$

By Lemma (2.3) and the Littlewood-Paley theorem, one has

$$\|I_{j,m} f\|_{p_0} \le C\|f\|_{p_0} \quad \text{for } 1 < p_0 < \infty \tag{3.4}$$

where C is independent of b_m and j. By the Plancherel theorem and (ii) in Lemma (2.2), we have

$$\begin{aligned}\|I_{j,m}f\|_2^2 &\leq C\sum_k \int_{E_{k+j,N-\lambda},} |\hat{f}(\xi)|^2 |\hat{\tau}_{b_m,k,N-\lambda}(\xi)|^2 d\xi \\ &\leq C\sum_k \int_{E_{k+j\ ,N-\lambda}} |\hat{f}(\xi)|^2 |2^{(N-\lambda)k}\beta_{N-\lambda}\xi|^{\frac{1}{(N-\lambda)\log|Q_m|}} d\xi \\ &\leq C|2^{-(N-\lambda)j}|^{\frac{1}{(N-\lambda)\log|Q_m|}} \sum_k \int_{E_{k+j\ ,N-\lambda}} |\hat{f}(\xi)|^2 d\xi \\ &\leq C|2^{-(N-\lambda)j}|^{\frac{1}{(N-\lambda)\log|Q_m|}} \|\hat{f}\|_2^2.\end{aligned}$$

Therefore we obtain

$$\|I_{j,m}\|_{L^2\longrightarrow L^2} \leq C2^{\frac{-j}{\log|Q_m|}}. \tag{3.5}$$

Using interpolation again we obtain

$$\|I_{j,m}f\|_p \leq 2^{\frac{-j\theta}{\log|Q_m|}} \|f\|_p \quad \text{for some } \ \theta > 0. \tag{3.6}$$

This shows that

$$\begin{aligned}\sum_{j<0} \|I_j f\|_p &\leq C\sum_{j<0}\sum_m |C_m| 2^{\frac{-j\theta}{\log|Q_m|}} \|f\|_p \\ &\leq C\|f\|_p \sum_m |C_m| \left(\log\frac{1}{|Q_m|}\right). \end{aligned}\tag{3.7}$$

Clearly, the constant C above is independent of the essential variables. (3.3) and (3.7) now imply

$$\left\|\sum_{k=-\infty}^{\infty} \tau_{\Omega,k,N-\lambda} * f\right\|_p \leq CM_q^{0,0}\|f\|_p, \quad \text{for all } \ 1<p<\infty, \quad \text{and } \ \lambda = 0,1,\ldots,N-1. \tag{3.8}$$

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