Hokkaido Mathematical Journal Vol. 28 (1999) p. 231–252

A remark on parabolic projective foliations

Bruno C. Azevedo SCÁRDUA

(Received November 28, 1995; Revised July 30, 1997)

Abstract. In this paper we consider parabolic foliations on the complex projective plane $\mathbb{C}P(2)$. It is known that if such a foliation has only with hyperbolic singularities then it must be linear after some rational change of coordinates [1]. Our results enforce the idea that projective parabolic foliations with nondegenerate singularities must be linear in the above sense. We prove that if we relax the hypothesis of hyperbolic singularities, allowing also Martinet-Ramis type singularities (definition in §1), then the foliation is also linear hyperbolic. This same conclusion holds for a parabolic foliation with simple singularities and having an algebraic leaf. If the algebraic leaf contains singularities which are either simple nonresonant, Martinet-Ramis and Poincaré-Dulac resonant singularities, or saddle-nodes in good-position (see §1) then the foliation is given by a closed rational 1-form. Several examples and an application to complete polynomial vector fields on $\mathbb{C}^2$ are given.

Key words: holomorphic foliation, parabolic Riemann surface, holonomy group.

1. Introduction

Let X be a polynomial vector field on the affine space $\mathbb{C}^2$. Since X is algebraic its local flow induces a singular (holomorphic) foliation by curves $\mathcal{F}$ on the projective space $\mathbb{C}P(2)$, and any foliation by curves on $\mathbb{C}P(2)$ is obtained this way. The leaves of $\mathcal{F}$ are open Riemann surfaces and their generic conformal type may (in some cases) be related to $\mathcal{F}$, and therefore to X.

A remarkable class of Riemann surfaces is the one of parabolic surfaces. A Riemann surface R is *parabolic* if it does *not* admit nonconstant negative subharmonic functions or equivalently does *not* admit a (finite) Green function, ([26], [30])). Examples are punctured compact Riemann surfaces and closed Riemann surfaces minus zero logarithmic capacity subsets.

A foliation by curves $\mathcal{F}$ is a *parabolic foliation* if its leaves are parabolic Riemann surfaces with the induced conformal structure. An outstanding theorem of M. Suzuki states that a parabolic foliation with proper leaves on a Stein surface has a meromorphic first integral:

1991 Mathematics Subject Classification : 32L30, 58F18.

Theorem 1.1 [26] *Let $\mathcal{F}$ be a holomorphic foliation by curves on a Stein space X^2 of complex dimension 2. Assume that the leaves of $\mathcal{F}$ are properly embedded and that the set $\mathcal{P}(\mathcal{F}) = \{p \in X^2 \setminus \operatorname{sing} \mathcal{F} \mid L_p$ is parabolic$\}$ has positive transverse logarithmic capacity. Then $\mathcal{F}$ is parabolic and admits a nonconstant meromorphic first integral on X.*

In this paper we regard the problem of classifying parabolic foliations on $\mathbb{C}P(2)$. This problem has also been considered in [1] where we find the following result:

Theorem 1.2 [1] *Let $\mathcal{F}$ be a foliation with hyperbolic singularities on $\mathbb{C}P(2)$. Assume that the set $\mathcal{P}(\mathcal{F}) = \{p \in \mathbb{C}P(2) \setminus \operatorname{sing} \mathcal{F} \mid L_p$ is parabolic$\}$ has positive transverse logarithmic capacity. Then $\mathcal{F}$ is parabolic and therefore it is a linear hyperbolic foliation $xdy - \lambda ydx = 0$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$ for some affine chart $(x, y) \in \mathbb{C}^2 \subset \mathbb{C}P(2)$.*

Let X be a holomorphic vector field on a neighborhood of the origin $0 \in \mathbb{C}^2$, with an isolated singularity at 0. We consider the foliation $\mathcal{F}$ defined by X in a neighborhood of the origin. The singularity is called *nondenegerate* if the linear part $DX(0)$ is nonsingular. In this case we may write $\mathcal{F}$: $xdy - \lambda ydx + \text{h.o.t.} = 0$, for some $\lambda \in \mathbb{C}^*$. When $\lambda \in \mathbb{Q}$ the singularity will be called *resonant.* A nondegenerate singularity is called *simple* if $\lambda \notin \mathbb{Q}_+$. A simple singularity exhibits exactly two (smooth and transverse) separatrices [16]. If $\lambda \in \mathbb{C} \setminus \mathbb{R}_-$ the singularity is in the *Poincaré domain* and *the leaves of $\mathcal{F}$ are transverse to the small 3-spheres $\mathbb{S}^3_\epsilon(0)$ centered at the singularity.* If $\lambda \in \mathbb{R}_-$ then it is in the *Siegel domain* and exhibits a saddle-like behaviour: if a local leaf (which is not a separatrix) accumulates the singularity then it accumulates both separatrices. The singularity is *hyperbolic* if $\lambda \notin \mathbb{R}$. Hyperbolicity implies linearization of the foliation around the singular point [16]. The Reduction Theorem of Seidenberg [24] gives two final types of singularities for holomorphic foliations in dimension 2: (i) simple singularities (ii) *saddle-node singularities* of the form $y^{p+1}dx - [x(1 + \lambda y^p) + \text{h.o.t.}]dy = 0$, $\mathbb{N} \ni p \geq 1$. We call $(y = 0)$ the *strong separatrix* of the saddle-node.

If λ or $\lambda^{-1} \in \mathbb{N}$ the singularity is either linearizable or can be put in the (analytic) *Poincaré-Dulac* form: $xdy - (ny + x^n)dx = 0$ [17]. In this last case there exists only one separatrix, its holonomy map is tangent to the identity and nonperiodic.

Definition 1.3 (Martinet-Ramis singularity) A germ of nondegenerate singularity will be called *of Martinet-Ramis type* if $\lambda \in \mathbb{Q}_-$ and the singularity is not linearizable[1].

We recall that an *algebraic leaf* for a foliation on an algebraic surface is an algebraic invariant curve (assumed to be irreducible).

Definition 1.4 [4] Let Λ be an algebraic leaf of $\mathcal{F}$. A germ of saddle-node singularity $q \in \Lambda \cap \operatorname{sing} \mathcal{F}$ is *in good position* (with respect to Λ) if its strong separatrix is contained in Λ.

Finally, we recall the definition of the Robin constant [29]: Given a Riemann surface R and a point $x \in R$, the *Robin constant* $\lambda(x)$, of R with respect to the point x is defined by the equation:

$$G_x(Z) = -\log|\varphi_x(Z)| + \lambda(x) + h_x(\varphi_x(Z))$$

where $\varphi_x(Z)$ is a local parametrization of a neighborhood of $x \in R$ onto a disk $\mathbb{D} \subset \mathbb{C}$, $\varphi_x(x) = 0$, $h_x(w)$ is a harmonic function on $\mathbb{D}$, with $h_x(0) = 0$, and $G_x(Z)$ is the *Green function* of R with pole on x. Clearly $\lambda(x)$ depends on the local chart, but the fact that $\lambda(x) = +\infty$ or not, does not depend. By definition R is parabolic if and only if $\lambda(x) = +\infty$.

Examples of parabolic foliations are given by linear foliations, *rational foliations* (with rational first integrals) and by *Bernoulli foliations* as in the example below.

Example 1.5 Let $\mathcal{F}$ be a *Bernoulli foliation* on $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$, that is, $\mathcal{F} : \omega = p(x)dy - (y^{k+1}a(x) + yb(x))dx = 0$ in some affine chart $(x, y) \in \mathbb{C}^2 \subset \overline{\mathbb{C}} \times \overline{\mathbb{C}}$. We can assume that $k = 1$, since the finite ramified covering $(x, y) \mapsto (x, y^k)$ does not affect the parabolicity of the leaves (Lemma 1.8 below). Now, since $\mathcal{F}$ is a particular case of a Riccati foliation ([3]) it follows that the leaves of $\mathcal{F}$ are either invariant vertical lines (given on $\mathbb{C}^2$ by $p(x) = 0$) or are transverse to the vertical fibration $\pi(x, y) = x$. In fact, given any leaf L of $\mathcal{F}$, the restriction $\pi|_L : L \to \pi(L) \subset \overline{\mathbb{C}} \setminus \operatorname{sing} \mathcal{F}$ is a covering map [15]. This shows that $\mathcal{F}$ has parabolic leaves in the case $\#\{\operatorname{sing} \mathcal{F} \cap \overline{(y = 0)}\} \leq 2$. The horizontal line $\overline{(y = 0)}$ is invariant. If $\mathcal{F}$ has nondegenerate singularities on $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$, then $a(x)$, $b(x)$ must be constant, so that $\mathcal{F}$ is given by a closed rational 1-form, namely $\omega = \frac{dy}{ay^{k+1}+yb} + \frac{dx}{p(x)}$. On the other hand *in general, a*

[1]This is equivalent to the fact that the local holonomy of any separatrix is not periodic [16].

Riccati foliation $\mathcal{F} : p(x)dy-(a(x)y^2+b(x)y+c(x))dx = 0$, *is not a parabolic foliation*. In fact, mostly if $c(x) \not\equiv 0$, then there exists no algebraic invariant curve transverse to the invariant vertical lines. Thus the holonomy group of the line $\overline{(y=0)}$, (see [15] for a definition), is a subgroup of $\mathbb{P}SL(2;\mathbb{C})$ without fixed points, and *free*. However, except for a countable set of leaves, the leaves have trivial holonomy. Therefore, using the projection $(x, y) \mapsto x$, we conclude that the generic leaf is simply-connected and thus (by Picard's Theorem) it is diffeomorphic to the disk $\mathbb{D}$.

Example 1.6 Consider a polynomial Poincaré-Dulac form $\mathcal{F} : \omega = xdy - (\lambda y + x^\lambda)dx = 0$, $\lambda \in \mathbb{N}$, on $\mathbb{C}^2$. To this 1-form we associate the vector field $X = x\frac{\partial}{\partial x} + (\lambda y + x^\lambda)\frac{\partial}{\partial y}$. The affine leaves of $\mathcal{F}$ are the orbits of X. Using the flow of X we can parametrize the orbits and conclude that they are all diffeomorphic to $\mathbb{C}$ except for the one contained on $(x = 0)$, which is diffeomorphic to $\mathbb{C}^*$. $\mathcal{F}$ is parabolic on $\mathbb{C}P(2)$ and is given by the closed rational 1-form $\frac{\omega}{x^{\lambda+1}} = d(\frac{y}{x^\lambda}) - \frac{dx}{x}$.

Example 1.7 Let $\mathcal{F}$ be a *logarithmic foliation* on $\mathbb{C}P(2)$ [21], say $\mathcal{F}|_{\mathbb{C}^2}$ is given by

$$\omega = \left(\prod_{i=1}^{r} f_i\right) \cdot \sum_{j=1}^{r} \lambda_j . \frac{df_j}{f_j} = 0$$

where f_j is a polynomial, $\lambda_j \in \mathbb{C}^*$, and $\mathbb{C}P(2)\backslash\mathbb{C}^2$ is generically transverse to $\mathcal{F}$. According to the Residue Theorem we know that $\sum_{j=1}^{r} \lambda_j . \deg(f_j) = 0$, so that $\Omega := \frac{\omega}{(\prod_{i=1}^{r} f_i)} = d\log\left(\left(\frac{f_2^{\deg(f_1)}}{f_1^{\deg(f_2)}}\right)^{\lambda_2} \cdots \left(\frac{f_r^{\deg(f_1)}}{f_1^{\deg(f_r)}}\right)^{\lambda_r}\right)$ is closed. In particular, if $r = 3$ then ω is the rational pull-back by $\pi : \mathbb{C}P(2) \to \mathbb{C}P(2)$, $\pi(x, y) = \left(\frac{f_2^{\deg(f_1)}}{f_1^{\deg(f_2)}}(x, y), \frac{f_3^{\deg(f_1)}}{f_1^{\deg(f_3)}}(x, y)\right)$ of the linear (parabolic) foliation $\mathcal{L} : \lambda_2\frac{dx}{x} + \lambda_3\frac{dy}{y} = 0$. It follows from the lemma below that $\mathcal{F}$ is (in this case $r = 3$) a parabolic foliation. On the other hand, by Poincaré's Lemma $\mathcal{F}$ may have nonhyperbolic singularities (outside the polar set of Ω), with holomorphic first integral.

Lemma 1.8 *Let* $\pi : \tilde{S} \to S$ *be a proper holomorphic mapping where* S *and* $\tilde{S}$ *are Riemann surfaces. Then,* S *is parabolic if, and only if, so it is* $\tilde{S}$.

Proof. If S is not parabolic then we have two possibilities: If S is compact then since π is proper it follows that $\tilde{S}$ is also compact and therefore it is not parabolic. If S is hyperbolic, i.e, if there exists a nonconstant holomorphic mapping $f : S \to \mathbb{D}$, then clearly there exists a nonconstant holomorphic mapping $f \circ \pi : \tilde{S} \to \mathbb{D}$, so that $\tilde{S}$ is hyperbolic. Conversely, let us assume that $\tilde{S}$ is not parabolic. If $\tilde{S}$ is compact then so it is S. Thus we may assume that $\tilde{S}$ is hyperbolic. In this case, there exists a nonconstant holomorphic mapping $\tilde{f} : \tilde{S} \to \mathbb{D}$. On the other hand, the proper mapping π is a finite ramified covering, i.e., there exists a discrete subset $\mathcal{B} \subset S$, such that $\pi|_{\tilde{S}\backslash\pi^{-1}(\mathcal{B})} : \tilde{S}\backslash\pi^{-1}(\mathcal{B}) \to S\backslash\mathcal{B}$ is a finite covering of order say $r \in \mathbb{N}$. We fix a symmetric function $\sigma \in \mathbb{C}[x_1, \ldots, x_r]$, of the r variables $x_1, \ldots, x_r$, and an arbitrary point $p \in S\backslash\mathcal{B}$. Using σ and $\tilde{f}$ we define a germ of holomorphic mapping $f_p^\sigma : U_p \subset S \to \mathbb{C}$, as $f_p(z) = \sigma(\tilde{f}(z_1), \ldots, \tilde{f}(z_r))$, where $\{z_1, \ldots, z_r\} = \pi^{-1}(z)$. Standard arguments on analytic functions (as Riemann's Extension Theorem) show that these mappings f_p^σ glue and extend into a global holomorphic mapping $f^\sigma : S \to \mathbb{C}$. Since $\tilde{f}(\tilde{S}) \subset \mathbb{D}$, it follows that f^σ is bounded. Now, it is easy to see that some f^σ must be nonconstant (otherwise $\tilde{f}$ would be constant) and this shows that S is hyperbolic. □

Our main results are as follows:

1.9 Theorem A *Let $\mathcal{F}$ be a parabolic foliation on $\mathbb{C}P(2)$ having hyperbolic and Martinet-Ramis singularities. Then $\mathcal{F}$ is a linear hyperbolic foliation $xdy - \lambda ydx = 0$, $\lambda \in \mathbb{C}\backslash\mathbb{R}$ in some affine chart $(x, y) \in \mathbb{C}^2 \subset \mathbb{C}P(2)$.*

This result enforces the idea that the only parabolic foliations on $\mathbb{C}P(2)$ with nondegenerate singularities are the linear. In addition we have the following result (see Example 1.7):

1.10 Theorem B *Let $\mathcal{F}$ be a parabolic foliation with simple singularities on $\mathbb{C}P(2)$ having an algebraic leaf. Then $\mathcal{F}$ is logarithmic.*

Examples of foliations considered in Theorem C below may be produced by taking linear hyperbolic foliations on $\mathbb{C}P(2)$ and performing pull-back by rational maps $\pi : \mathbb{C}P(2) \to \mathbb{C}P(2)$ (Lemma 1.8).

1.11 Theorem C *Let $\mathcal{F}$ be a parabolic foliation with nondegenerate singularities on $\mathbb{C}P(2)$ such that any leaf containing a separatrix, contains the separatrix of a hyperbolic, Martinet-Ramis or Poincaré-Dulac singularity. Then $\mathcal{F}$ has an algebraic leaf Λ with solvable holonomy. If moreover the* *resonant singularities in Λ are Martinet-Ramis or Poincaré-Dulac singularities then $\mathcal{F}$ is given by a closed rational 1-form.*

Example 1.12 Let $\mathcal{F}|_{\mathbb{C}^2} : y^2dx - x(1+y)dy = 0$. We call $\Lambda : \overline{(y=0)} \subset \mathbb{C}P(2)$. Then $\operatorname{sing}\mathcal{F} \cap \Lambda = \{0, p_\infty\}$ where p_∞ appears in the line at the infinity and is a Poincaré-Dulac form $(u+v)dv - vdu = 0$ for $(u,v) = (1/x, y/x)$. It follows that $\mathcal{F}$ is parabolic and given by a closed rational 1-form *which is not logarithmic* (Example 1.6). The singular set $\operatorname{sing}\mathcal{F} \cap \Lambda$ satisfies the conditions of the following theorem:

1.13 Theorem D *Let $\mathcal{F}$ be a parabolic foliation on $\mathbb{C}P(2)$ having an algebraic leaf Λ. Assume that the resonant singularities in Λ are Martinet-Ramis or Poincaré-Dulac singularities, and that the degenerate singularities are saddle-nodes in good-position. Then $\mathcal{F}$ is given by a closed rational 1-form.*

A closed integrating factor for a holomorphic integrable 1-form ω is a closed meromorphic 1-form η, with simple poles, such that $d\omega = \eta \wedge \omega$, $(\eta)_\infty$ is invariant. Such a 1-form defines an affine transverse structure for $\mathcal{F} : \omega = 0$ outside the (invariant) polar set $(\eta)_\infty$ [21], [3].

1.14 Theorem E *Let $\mathcal{F}$ be a parabolic foliation on $\mathbb{C}P(2)$ having an algebraic leaf containing simple singularities, Poincaré-Dulac singularities and saddle-nodes in good-position. Then $\mathcal{F}|_{\mathbb{C}^2}$ is given by a polynomial 1-form ω which admits a rational closed integrating factor η.*

Example 1.15 Not all parabolic foliations of $\mathbb{C}P(2)$ are given by closed rational 1-forms. Let us consider $\mathcal{F} : \omega = xdy - (a(x)y + b(x))dx = 0$ in affine coordinates. Then as it is easy to see from the integration of the associated vector field, the leaves of $\mathcal{F}$ are covered by the plane $\mathbb{C}$ and are therefore parabolic. On the other hand, the 1-form ω admits the closed integrating factor $\eta = \frac{1+a(x)}{x}dx$. $\mathcal{F}$ is therefore transversely affine in $\mathbb{C}^2 \setminus \{x = 0\}$ [21]. In particular if we write $\eta = d\log H$, then we obtain $d(\frac{\omega}{H}) = 0$. In general, the function H is not rational, that is, in general the foliation $\mathcal{F}$ is not given by a closed rational 1-form. On the other hand, as it is easy to see, $\mathcal{F}$ exhibits degenerate singularities along the invariant line $L_\infty = \mathbb{C}P(2) \backslash \mathbb{C}^2$.

Example 1.16 This example illustrates a parabolic foliation given by a closed 1-form, whose algebraic leaves always contain some degenerate singularity. For relatively prime $k, \ell \in \mathbb{N}$ we consider the polynomial 1-

form $\omega = kxdy + \ell y(1 + \frac{\sqrt{-1}}{2\pi}x^\ell y^k)dx$, which admits the integrating factor $h = x^{\ell+1}y^{k+1}$, and gives us the following closed rational 1-form $\frac{1}{x^{\ell+1}y^{k+1}}.\omega = d\left(-\frac{1}{x^\ell y^k} + \log(\frac{\sqrt{-1}\ell}{2\pi}x)\right)$. Thus the leaves of $\omega = 0$ are parametrized by $x(t) = x_o e^{kt}$, $y(t) = \frac{y_o}{(1+\frac{\sqrt{-1}\ell}{2\pi}kty_o^k x_o^\ell)^{\frac{1}{k}}}$; so that the generic leaf is diffeomorphic to $\mathbb{C}^*$. It follows that $\omega = 0$ defines a parabolic foliation $\mathcal{F}$ on $\mathbb{C}P(2)$. We have $\operatorname{sing} \mathcal{F} \cap \mathbb{C}^2 = \{0\}$ and since this singularity is a *Martinet-Ramis normal form* [17] it is not linearizable and therefore of Martinet-Ramis type. On the other hand there are two other singularities (both degenerate) contained in L_∞ which is an algebraic leaf of $\mathcal{F}$. One given by $-ku^{\ell+k+1}dv + [(\ell+k)vu^{\ell+k} + \frac{\sqrt{-1}\ell}{2\pi}v^{k+1}]du = 0$ for $(u,v) = (1/x, y/x)$. The other singularity is of the form $-(k+\ell)\frac{ds}{s} + \ell\frac{dr}{r} + \frac{\sqrt{-1}}{2\pi}\frac{r^\ell}{s^{k+\ell}}(\frac{ds}{s} + \frac{dr}{r}) = 0$, for $(r,s) = (x/y, 1/y)$.

Remark 1.17 From $\mathbb{C}P(2)$ to $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$.

Now we make a simple remark, but which is useful in the search of examples of parabolic foliations (see for instance Example 1.5 above). First we recall that $\mathbb{C}P(2)$ is obtained from $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$ by a sequence of one blow-up and two blow-downs: Given an affine chart $(x,y) \in \mathbb{C}^2 \subset \overline{\mathbb{C}} \times \overline{\mathbb{C}}$, first we blow-up the point $(x = \infty, y = \infty)$. The transforms of the lines $\overline{(x = \infty)}$ and $\overline{(y = \infty)}$ have Chern class -1 and can be blowed-down. First we blow-down the transform of $\overline{(x = \infty)}$. Then we blow-down the transform of the line $(x = 0)$ to obtain $\mathbb{C}P(2)$. Notice that the afine system $(x,y) \in \mathbb{C}^2$ is "preserved" and we introduce $L_\infty = \mathbb{C}P(2)\backslash\mathbb{C}^2$ as an "exceptional curve" on $\mathbb{C}P(2)$. Let us call $\sigma : \overline{\mathbb{C}} \times \overline{\mathbb{C}} \to \mathbb{C}P(2)$ such standard morphism. Given a foliation $\tilde{\mathcal{F}}$ on $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$, the morphism σ induces a foliation $\mathcal{F}$ on $\mathbb{C}P(2)$ which satisfies $\tilde{\mathcal{F}} = \sigma^*\mathcal{F}$. As it is plain to see, we can assume that σ preserves the singularities of $\tilde{\mathcal{F}}$, and introduces three new singularities. Two of these are dicritical singularities p_1, p_2, of the *radial type*, that is of the local form $xdy - ydx = 0$. The remaining singularity p_3 is holomorphic first integral type, in particular it is in the Siegel domain. However, by [16], p_3 is linearizable and has finite local holonomy. The singularities $p_1, p_2 \in L_\infty$ come from the lines $\{0\} \times \overline{\mathbb{C}}$ and $\overline{\mathbb{C}} \times \{0\}$, not completely transverse to $\mathcal{F}$, and p_3 appears in the middle of the "exceptional curve" on $\mathbb{C}P(2)$.

Lemma 1.18 *The following conditions are equivalent:*
(i) *$\tilde{\mathcal{F}}$ has parabolic leaves.*

(ii) *$\mathcal{F}$ has parabolic leaves.*

(iii) *The restriction $\mathcal{F}^* = \mathcal{F}|_{\mathbb{C}^2}$ has parabolic leaves, for any affine space $\mathbb{C}^2 \subset \mathbb{C}P(2)$, such that $L_\infty = \mathbb{C}P(2)\backslash\mathbb{C}^2$ is (generically) transverse to $\mathcal{F}$.*

(iv) *We have*

$$\log \operatorname{capac}\{p \in \mathbb{C}^2\backslash \operatorname{sing} \mathcal{F}^*,\ L_p^* \textit{ is parabolic } \} > 0$$

where $\mathcal{F}^*$ is as in (iii).

Proof. We consider an affine chart $\mathbb{C}^2 \subset \mathbb{C}P(2)$, such that $\mathbb{C}P(2) \setminus \mathbb{C}^2$ is not invariant by the foliation, and for any leaf L of $\mathcal{F}$ we denote by L^* the corresponding leaf of $\mathcal{F}^*$ as above. We denote by $\widehat{L^*}$ the *holonomy covering* of the leaf L^* on $\mathbb{C}^2$, ([26]). Assume (iv). Since except for a countable set of leaves the leaf L^* has trivial holonomy it follows from the hypothesis that $\log \operatorname{capac}\{p \in \mathbb{C}^2\backslash \operatorname{sing} \tilde{\mathcal{F}}^*,\ \widehat{L_p^*} \text{ is parabolic}\} > 0$. Since $\mathbb{C}^2$ is a Stein manifold [25] it follows that $\{p \in \mathbb{C}^2\backslash \operatorname{sing} \mathcal{F}^*,\ \widehat{L_p^*} \text{ is parabolic}\} = \mathbb{C}^2\backslash \operatorname{sing} \mathcal{F}^*$ ([26], [30]). This implies that *the leaves of $\mathcal{F}^*$ are parabolic.* Now, given any leaf L^* of $\mathcal{F}^*$, we denote by $\tilde{L}$ and L the corresponding leaves of $\tilde{\mathcal{F}}$ (on $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$) and $\mathcal{F}$ (on $\mathbb{C}P(2)$) respectively. Then $\tilde{L}\backslash L^*$ is discrete on $\tilde{L}$. Since a discrete subset has null logarithmic capacity, it follows that $\tilde{L}$ is parabolic if, and only if L^* is parabolic. The same argument shows that L is parabolic if, and only if, L^* is parabolic. The remaining equivalences can be found in [1]. This ends the proof of the proposition. □

Remark 1.19 A well-known theorem of Huber [13], [28] asserts that given a Riemann surface M, if there exists a complete minimal immersion $\psi : M \to \mathbb{R}^n$, $n \geq 3$, having finite total curvature then M is parabolic. Using this and our above results one may study polynomial vector fields over $\mathbb{C}^2$ whose orbits have bounded geometry with respect to the standard hermitian geometry of $\mathbb{C}^2$ [22].

2. Construction of harmonic measures

In this section we construct harmonic measures (with respect to a suitable hermitian metric) for a given foliation with nondegenerate singularities on $\mathbb{C}P(2)$. Such measures will be supported also outside the singular set of $\mathcal{F}$.[2]

[2] I am grateful to P. Sad for showing me Proposition 2.1

Proposition 2.1 *Let $\mathcal{F}$ be a foliation with nondegenerate singularities over $\mathbb{C}P(2)$. There exists an harmonic measure μ whose support is not contained in the singular set of $\mathcal{F}$.*

2.2. The case of Poincaré type singularities [1]

Let us first assume that $\mathcal{F}$ is a foliation on $\mathbb{C}P(2)$ and that the singularities are all in the Poincaré domain. We know that in this case the local leaves around a singularity are transverse to the small spheres centered at this singularity. This allows the following construction [1]:

Let $\{p_1, \ldots, p_r\} = \operatorname{sing} \mathcal{F}$. Choose small balls B_j centered at p_j and construct the *double* $\mathcal{F}_d$ of the restriction $\mathcal{F}|_{\mathbb{C}P(2)-\cup B_j}$ (see also paragraph 2.4 below). This is a regular C^∞ foliation on a compact real manifold denoted $M_d = (\mathbb{C}P(2) - \cup B_j)_d$ and its leaves are naturally Riemann surfaces endowed with the complex structure given by the Schwarz Reflection Principle. Moreover (as it is noticed in [1]) these leaves are still parabolic as a consequence of [14]. Fix a Riemannian metric g on $(\mathbb{C}P(2) - \cup B_j)_d$, hermitian along the leaves of $\mathcal{F}_d$. Denote by $\Delta = \Delta^{\mathcal{F}_d}$ the *foliated laplacian* associated to the pair $(\mathcal{F}_d, g)$. According to [9], [10] we have: *Given any compact $\mathcal{F}_d$-saturated $K \subset (\mathbb{C}P(2) - \cup B_j)_d$, there exists a harmonic measure μ whose support is contained in K.*

2.3. The general case

Now we consider the general case, i.e., $\mathcal{F}$ is a foliation with nondegenerate singularities on $\mathbb{C}P(2)$. We write $\operatorname{sing} \mathcal{F} = \{p_1, \ldots, p_r\} \cup \{q_1, \ldots, q_s\}$, where p_j is in the Poincaré domain, and q_i is in the Siegel domain. We consider a sequence of foliations $\mathcal{F}_n$ on $\mathbb{C}P(2)$, with $\mathcal{F}_n \underset{n\to\infty}{\longrightarrow} \mathcal{F}$, in the usual topology of the space of foliations [15], and such that $\mathcal{F}_n$ has all its singularities in the Poincaré domain. Moreover, we also have for each q_i a singularity $q_i^n \in \operatorname{sing} \mathcal{F}_n$, which converges to $q_i, i \in \{1, \ldots, s\}$. Fixed i we consider compact neighborhoods $V_n \ni q_i^n$ and holomorphic diffeomorphims $\varphi_n : V_n \to \overline{\mathbb{B}}_2 = \{(x, y) \in \mathbb{C}^2, |x|^2 + |y|^2 \leq 2\}$, $\varphi_n(q_i^n) = 0$, in such a way that φ_n^{-1} converges uniformly to $\varphi^{-1} : \overline{\mathbb{B}}_2 \to V_o$, $V_o \ni q_i$, $\varphi(q_i) = 0$, and we can write

$$(\varphi_n)_*(\mathcal{F}_n) : [x + xA_n(x, y)]dy - [(\mu + \sqrt{-1}.\delta_n)y + yB_n(x, y)]dx = 0$$

$$(\varphi)_*(\mathcal{F}) : [x + xA(x, y)]dy - [\mu y + yB(x, y)]dx = 0$$

where $\mu \in \mathbb{R}_-$, is the eigenvalue quotient for $\mathcal{F}$ at q_i, $\delta_n \neq 0$, $\delta_n \to 0$.

Then if $r_n > 0$ is sufficiently small, we have that $(\varphi_n)_*(\mathcal{F}_n)$ is transverse to the spheres $\mathbb{S}_r = \{|x|^2 + |y|^2 = r^2\}$, for $r \le 2r_n$.

2.4. The double

We take $X^{(n)}$ the holomorphic vector field $[x + xA_n(x,y)]\frac{\partial}{\partial x} + [\mu + \sqrt{-1}\delta_n)y + yB_n(x,y)]\frac{\partial}{\partial y}$, which is tangent to $(\varphi_n)_*(\mathcal{F}_n)$, and transverse to the spheres $\mathbb{S}_r = \partial\mathbb{B}_r$ for all $r \le 2r_n$. We restrict the flow of $X^{(n)}$ to a real flow $X_t^{(n)}$, with the same transversality property (in fact, $X_t^{(n)}$ "enters" the spheres). We take $-t_n \le t \le t_n$, for $t_n > 0$ small enough. We define $\psi_n : \Delta_n \to \Delta_n$ (Δ_n is the region between $X_{-t_o}^{(n)}(\mathbb{S}_r)$ and $X_{t_o}^{(n)}(\mathbb{S}_r)$) as $\psi_n(X_t^{(n)}(z)) = X_{-t}^{(n)}(z)$, for $z \in \mathbb{S}_{r_n}$. Then ψ_n is of class $\mathcal{C}^\infty$ and holomorphic along the leaves of $(\varphi_n)_*(\mathcal{F}_n)$. We define the *double* of $\mathcal{F}_n$ using the identification given by ψ_n. Given any point $(x,y) \in \Delta_n$, we denote by $[(x,y)]$ the corresponding point in the double, that is, the equivalence class of (x,y). In $\overline{\mathbb{B}}_1$ we consider the function $U(x,y) = |x|^2 + |y|^2$, extended in a $\mathcal{C}^\infty$ way to $\overline{\mathbb{B}}_2$ so that it is $\equiv 0$ in a small neighborhood of $\partial\mathbb{B}_2$. This function induces in the double a function $U_n([x,y]) = \max\{U(x,y), U(\psi_n(x,y))\}$, in $[\Delta_n]$; and $U_n(x,y) = U(x,y)$ outside $[\Delta_n] = \{[(x,y)] | (x,y) \in \Delta_n\}$. We observe that U_n is $\mathcal{C}^\infty$ except along $\mathbb{S}_{r_n}$, where it is continuous and has r_n^2 as its minimum value.

2.5. Estimatives on the double

In the *double* of $\mathcal{F}_n$, (say $\mathcal{F}_n^d$ on $(\mathbb{C}P(2) - \cup B_j)_d$), we consider a hermitian metric (fixed independently of n) defined outside $\varphi_n^{-1}(\overline{\mathbb{B}}_2)$, which extends also as an hermitian metric, being $|dx|^2 + |dy|^2$ in $\varphi_n^{-1}(\mathbb{B}_1 \backslash \Delta_n)$. The Laplacian of U in the region between $\mathbb{B}_1$ and Δ_n is $\operatorname{Lap} U \equiv 2$. If $t_n > 0$ is small enough then we can extend this metric to $[\Delta_n]$ in such a way that the smoothing of U has Laplacian $\ge 3/2$ in $[\Delta_n]$.

Let μ_n be an harmonic probability associated to this double, let $D_1^{(n)}$ be the region inside $\mathbb{B}_1$ (including $[\Delta_n]$) and $D_2^{(n)}$ be the complement of $\mathbb{B}_1$. Let us denote by M_n^d the double associated to $\mathcal{F}_n$, i.e., $M_n^d = D_1^{(n)} \cup D_2^{(n)}$. Then:

$$0 = \int_{M_n^d} \operatorname{Lap} U d\mu_n = \int_{D_1^{(n)}} \operatorname{Lap} U d\mu_n + \int_{D_2^{(n)}} \operatorname{Lap} U d\mu_n$$

Therefore

$$\int_{D_2^{(n)}} \operatorname{Lap} U d\mu_n = -\int_{D_1^{(n)}} \operatorname{Lap} U d\mu_n \le -\frac{3}{2}\mu_n(D_1^{(n)})$$

Let $c > 0$ be such that $\operatorname{Lap} U|_{D_2^{(n)}} \geq -c$ (recall that $\mathcal{F}_n \to \mathcal{F}$). Then $-c\mu_n(D_2^{(n)}) \leq -\frac{3}{2}\mu_n(D_1^{(n)})$ so that $c\mu_n(D_2^{(n)}) \geq \frac{3}{2}(1-\mu_n(D_2^{(n)})$ and finally $\mu_n(D_2^{(n)}) \geq \frac{\frac{3}{2}}{\frac{3}{2}+c}$. Therefore μ has nontrivial support outside B_1.

On $\mathbb{C}P(2)$ we define a probability μ_n from the above one, by taking it as zero on the Borelians of the interior of $X_{-t_o}^{(n)}(\mathbb{S}_{r_n})$. Let $\mu_n \to \mu$, up to passing to a subsequence. Then μ is a harmonic probability such that $\mu(D_2^{(n)}) \geq \alpha > 0$. This shows Proposition 2.1 $\square$

3. Existence of an algebraic leaf

Here we prove the existence of an algebraic leaf for $\mathcal{F}$ as in Theorem C. Our main inspiration comes from [1], nevertheless the techniques may differ. Consider the harmonic probability measure μ given in Proposition 2.1. Such a measure can be decomposed as a product of a holonomy invariant transverse measure with the area form along the leaves; provided that (see [10], [9]) $\mu(\{p,\ L_p$ admits a nonconstant negative harmonic function$\}) = 0$. We take $\phi \neq K \subset \mathbb{C}P(2)\backslash \operatorname{sing}\mathcal{F}$ as the support of $\mu|_{\mathbb{C}P(2)\backslash \operatorname{sing}\mathcal{F}}$. Notice that by Proposition 2.1 $K \neq \emptyset$. Since the leaves of $\mathcal{F}$ are parabolic and therefore do not support negative (sub)harmonic functions, the remark above applies to give us a holonomy invariant transverse measure μ' on $\mathbb{C}P(2)\backslash \operatorname{sing}\mathcal{F}$, satisfying $K = \operatorname{Supp}(\mu')$. We denote by $\mathcal{M}$ the closure of K on $\mathbb{C}P(2)$. Let $L \subset \mathcal{M}$ be any leaf of $\mathcal{F}$. We want to prove that either $\overline{L}$ is an algebraic invariant curve on $\mathbb{C}P(2)$, or it accumulates some singularity with a separatrix contained in some algebraic leaf. First we notice that $\overline{L} \cap \operatorname{sing}\mathcal{F} \neq \emptyset$: In fact, otherwise $\overline{L} \subset \mathcal{M}$ contains a nontrivial minimal set of $\mathcal{F}$ on $\mathbb{C}P(2)$. But this is not possible because the measure μ induces a holonomy invariant transverse measure supported on $\mathcal{M}$, which is not possible by [2]. Thus we may choose a singular point $q \in \operatorname{sing}\mathcal{F} \cap \overline{L}$, that can be written as $xdy - \lambda ydx + \text{h.o.t} = 0$ for some local coordinates $(x,y) \in U$ centered at q. We may also assume that $(x=0) \cup (y=0)$ contains the local separatrices of $\mathcal{F}$ at q and $(y=0)$ is actually a separatrix. Denote by L_q the leaf of $\mathcal{F}$ that contains the separatrix $(y=0)$. Let us fix local transverse disk $\Sigma : (x=1)$ to $\mathcal{F}$, $\Sigma \cong \mathbb{D}$, $\Sigma \cap (y=0) = q_1$, and let $h : (\Sigma, q_1) \to (\Sigma, q_1)$, be either the local holonomy associated to q (in case q is a hyperbolic, Martinet-Ramis or Poincaré-Dulac singularity) or the holonomy map associated to a hyperbolic, Martinet-Ramis or Poincaré-Dulac singularity having a local separatrix contained in L_q. We will prove:

Lemma 3.1 *Let $q \in \operatorname{sing}\mathcal{F}$ be accumulated by L. Then locally around q, $\overline{L}$ is contained in the union of the two separatrices of $\mathcal{F}$ at q. In particular $\overline{L}$ is an analytic subset of $\mathbb{C}P(2)$ of dimension one.*

Proof. We fix local coordinates $(x, y) \in U$ as above. We may reduce our argumentation to the following cases:

(i) $\lambda \in \mathbb{C}\backslash\mathbb{R}$: In this case the singularity is linearizable and the same holds for h so that we may assume that $h(y) = \nu.y$, for $\nu = \exp(2\pi i\lambda)$, with $|\nu| < 1$. It follows that h maps a disk $\mathbb{D}_1 \subset \mathbb{D}$ into a smaller disk $\mathbb{D}_2 \subset \mathbb{D}_1$ and therefore the measure μ restricted to Σ, vanishes outside the origin, so that $\operatorname{Supp}(\mu) \cap \Sigma = \{0\}$. Thus $\overline{L} \cap U$ must be contained in the union of separatrices of $\mathcal{F}$ at q.

(ii) q is a Martinet-Ramis singularity: A well-known consequence of the saddle-like behaviour of these type singularities, (see [16] for instance) is the following:

Lemma 3.2 *If a leaf L_1 of $\mathcal{F}$ accumulates the singularity q then, either it is locally contained in the union of separatrices of $\mathcal{F}$ at q, or it accumulates some of these separatrices.*

Let us assume that L is not locally contained in the union of separatrices of $\mathcal{F}$ at q. We regard the intersection $L \cap \Sigma$. We already know (Lemma 3.2) that $L \cap \Sigma$ accumulates the origin $0 \in \Sigma$. Choose a flow-box neighborhood $V \times \mathbb{D}_1$ where $\mathbb{D}_1 \subset \Sigma$ is a subdisk, and $V \subset (y = 0)$ is a small disk centered at $0 \in \Sigma$. We may assume that in some coordinates (u, v) in $V \times \mathbb{D}_1$, the foliation $\mathcal{F}|_{V \times \mathbb{D}_1}$ is given by $v = cte$. In particular, if L_1 is any leaf of $\mathcal{F}$ which intersects $V \times \mathbb{D}_1$, then L_1 contains a plaque $v = cte$ and may prolonged to outside $V \times \mathbb{D}_1$. We know from [17] that the holonomy local diffeomorphism $h \in \operatorname{Diff}(\mathbb{D}_1, 0)$ associated to a separatrix of $\mathcal{F}$ through q has the following property: there are invariant sectors $U_\theta \subset \mathbb{D}_1$, with $0 \in \overline{U_\theta}\backslash U_\theta$, $\theta \in (0, 2\pi)$, where h behaves like an attractor. In particular, given a small open disk $D \subset U_\theta$ we have that $h^n(D) \subset U_\theta$ is a sequence of disks converging uniformly to the origin, and also $h^n(D) \cap h^m(D) = \emptyset$ if $n \neq m$. If we take such a disk D with $D \cap L \neq \emptyset$ then $\mu(V \times D) = \epsilon > 0$ because $L \subset \operatorname{Supp}(\mu)$. Also we have $\mu(V \times h^n(D)) = \epsilon$ because the measure $\mu|_L$ is a product measure and is invariant by holonomy. Finally, $[V \times h^n(D)] \cap [V \times h^m(D)] = \emptyset$ if $n \neq m$.

(iii) q_o is a Poincaré-Dulac singularity. This case is similar to the preceeding case, it uses the fact that the local holonomy of q_o must be tangent to the identity but nontrivial.

This all implies that (since L is not contained in the union of local separatrices of $\mathcal{F}$ at q) the leaf L has infinite area for the fixed Riemannian metric. This is an absurd because μ is a probability measure. This ends the proof of Lemma 3.1. □

Lemma 3.1 shows that there is a leaf L whose closure $\overline{L}$ is an analytic dimension one subset ot $\mathbb{C}P(2)$. Using Chow's Theorem [11] we conclude that $\Lambda = \overline{L}$ is an algebraic leaf of $\mathcal{F}$.

4. Solvable holonomy

In this section we prove that the algebraic leaf in Theorem C has solvable holonomy group. Let $\Lambda \subset \mathbb{C}P(2)$ be an algebraic leaf of a parabolic foliation $\mathcal{F}$. Denote by $\Lambda_o = \Lambda \backslash \operatorname{sing} \mathcal{F}$. Then Λ_o is parabolic and we have two possibilities:

(i) Λ_o is uniformized by the plane $\mathbb{C}$. In this case we have clearly $\Lambda_o \equiv \mathbb{C}^*$ (because since Λ is algebraic invariant it must contain some singularity of $\mathcal{F}$), and therefore $\pi_1(\Lambda_o) \cong \mathbb{Z}$ is abelian. It follows that the holonomy of Λ_o (denoted by $\operatorname{Hol}(\Lambda_o)$) is abelian.

(ii) Λ_o is uniformized by the disc $\mathbb{D}$. In this case we have $\Lambda_o \equiv \mathbb{D}/H$ for some subgroup $H \subset \operatorname{Diff}(\mathbb{D})$, such that $H \cong \pi_1(\Lambda_o)$. The *holonomy covering* $\widehat{\Lambda}_o$ of Λ_o is given by $H/\widehat{H}$ where $H/\widehat{H}$ is a finite extension of $\mathbb{Z}$ and $\widehat{H}$ is the kernel of the holonomy homomorphism $\pi_1(\Lambda_o) \to \operatorname{Diff}(\mathbb{C}, 0)$, [1].

Proposition 4.1 *The holonomy group* $\operatorname{Hol}(\Lambda_o)$ *of the leaf* Λ_o *is solvable. Moreover if* $\operatorname{Hol}(\Lambda_o)$ *is nonabelian then the simple singularities of* $\mathcal{F}$ *in* Λ *must consist of resonant singularities, i.e., with rational quotient of eigenvalues.*

We use the following lemma.

Lemma 4.2 *Let* $G \subset \operatorname{Diff}(\mathbb{C}, 0)$ *be a nonsolvable subgroup. There exist* $f, h \in G$ *such that* $1 \neq f^n \neq h^m \neq 1$, $\forall n, m \in \mathbb{N}$.

Proof. Since G is nonsolvable the derived subgroup $[G, G]$ contains elements f, h of distinct orders of flatness [7], [19], say $f = z + az^k + \text{h.o.t.}$, $h =$

$z + bk^\ell + \text{h.o.t.}$ Then we have $f^n = z + naz^k + \text{h.o.t.}$, $h^m = z + mbz^\ell + \text{h.o.t.}$, so that clearly we have $1 \neq f^n \neq h^m \neq 1$. $\square$

According to the lemma above (since a finite extension of $\mathbb{Z}$ cannot contain two infinity order *disjunct* cyclic subgroups), it follows that $\mathrm{Hol}(\Lambda_o) \cong H/\widehat{H}$ must be solvable. This same remark shows that if $\mathrm{Hol}(\Lambda_o)$ is non-abelian, then all flat elements must have the same flatness order (which in fact shows that the subgroup of commutators $[G, G]$ must be ciclic), and the nonflat ones must be of finite order and therefore they are linearizable as rational rotations. By [16], [17] we conclude that the simple singularities in $\operatorname{sing} \mathcal{F} \cap \Lambda$ must be resonant. This proves Proposition 4.1 $\square$

From the discussion above we obtain:

Corollary 4.3 *Let $\mathcal{F}$ be a parabolic foliation on $\mathbb{C}P(2)$, with an algebraic leaf Λ. Then the holonomy of Λ is solvable.*

5. Construction of closed 1-forms and of closed integrating factors

We refer to [8], [12] for the notion of transversely formal object over a divisor on a projective surface.

Proposition 5.1 *Let $\mathcal{F}$ be a foliation on a projective surface M and $\Lambda \subset M$ an algebraic leaf. Assume that* $\operatorname{sing} \mathcal{F} \cap \Lambda$ *consists of simple singularities, Poincaré-Dulac singularities and saddle-nodes in good-position. Let $\mathcal{F}$ be given by a rational 1-form ω on M.*

(i) *If the holonomy of Λ is abelian then ω admits a transversely formal integrating factor $\hat{h}$ over Λ, i.e., $\hat{h}$ is a transversely formal function over the divisor Λ, such that $\frac{\omega}{\hat{h}}$ is closed.*

(ii) *If the holonomy of Λ is solvable then ω admits a transversely formal closed integrating factor $\hat{\eta}$ defined over Λ, i.e., $\hat{\eta}$ is a transversely formal 1-form over Λ, closed, with simple poles and such that $d\omega = \hat{\eta} \wedge \omega$.*

Proof. We give here just the main ideas. Further details are found in [8], [4], they also come from a careful reading of [20]. We may assume for simplicity that the polar set of ω is transverse to Λ, and cuts Λ outside $\operatorname{sing} \mathcal{F}$. First we recall that according to [16], [17], [18] a nondegenerate singularity as well as a saddle-node always admits a formal integrating factor. Moreover, if q_o is such a singularity, and $\hat{h}_o$ is such an integrating factor (defined as a formal expression at q_o), with respect to ω (that is, $\frac{\omega}{\hat{h}_o}$ is closed as a formal 1-form), then we can extend $\hat{h}_o$ as a transversely formal integrating factor for ω, over a small disk $\mathbb{D}_{q_o} \subset \Lambda$, centered at q_o, using the resommation properties of the integrating factors along the separatrices for simple and Martinet-Ramis saddle-node singularities. This is done by means of choosing a local system of coordinates (x, y), centered at q_o, and such that $\Lambda : (y = 0)$. Then, in these coordinates, we consider formal expressions $\hat{h}_o(x, y) = \sum_{j=0}^{+\infty} a_j(x) y^j$, where the $a_j(x)$ are also formal positive series in the variable x. Now, imposing that $\hat{h}_o$ is an integrating factor for ω, we obtain a differential equation which has a formal solution as remarked above, and the coefficients $a_j(x)$ are in fact analytic functions of x, in a fixed small disk centered at the origin, this is a consequence of Briot-Bouquet's Theorem type argument [8].

Now we proceed: first we assume that $\mathrm{Hol}(\Lambda)$ is abelian. According to [8], [20] there exists a transversely formal integrating factor $\hat{h}$ for ω, defined over the open curve $\Lambda_o = \Lambda \backslash \operatorname{sing} \mathcal{F}$. We will show that $\hat{h}$ extends in a transversely formal way to $\operatorname{sing} \mathcal{F} \cap \Lambda$ as a consequence of the fact that $\frac{\hat{h}}{\hat{h}_o}$ is a transversely formal first integral for the foliation near q_o, that is, $\omega \wedge d(\frac{\hat{h}}{\hat{h}_o}) = 0$ as a formal expression. Given any singularity $q_o \in \Lambda$, we have the following possibilities:

(1) q_o is formally linearizable with a holomorphic local first integral. According to [16] q_o admits a holomorphic first integral, and therefore we may assume that $\hat{h}_o$ is in fact holomorphic in a neighborhood of q_o. Thus $\frac{\hat{h}}{\hat{h}_o}$ extends to q_o as a consequence of the fact that it is already defined over the separatrix through q_o tangent to Λ_o. In fact, we can find analytic coordinates (x, y) centered at q_o, such that $(y = 0)$ corresponds to Λ, and $\mathcal{F}$ is given in these coordinates by $pxdy + qydx = 0$, with $p, q \in \mathbb{N}$, $\langle p, q\rangle = 1$. We take $\hat{h}_o = xyg$ where g is the meromorphic function defined by $\omega(x, y) = g.(pxdy + qydx)$. Then we have $d(\frac{\omega}{\hat{h}_o}) = 0$. Now, the fact that $d(\frac{\omega}{\hat{h}}) = 0$, outside $(x = 0)$, implies that $d(\frac{\hat{h}_o}{\hat{h}}) \wedge \omega = 0$. Thus $\hat{f} = \frac{\hat{h}_o}{\hat{h}}$ is a meromorphic first integral for $\mathcal{F}$ along $(x \neq 0)$, $(y = 0)$. Then $\hat{f} = \hat{\varphi}(\hat{f}_o)$, for some holomorphic one variable function $\varphi \in \mathbb{C}\{z\}$, where $\hat{f}_o = x^q y^p$ is a *primitive* holomorphic formal first integral for $\mathcal{F}$ at q_o and so $\hat{f} = \varphi(x^q y^p)$. The fact that $\hat{f}$ is holomorphic formal along $(y = 0) \subset \Lambda$ minus q_o, and the fact that $y = 0 \Longrightarrow x^q y^p = 0$, implies that $\hat{\varphi}$ is holomorphic and therefore $\hat{f}$ extends

holomorphically as $\varphi(\hat{f}_o)$ to q_o. This shows that $\hat{h}$ extends in a transversely formal way to q_o in the case $\hat{f}$ is nonconstant. If $\hat{f}$ is constant then the extension of $\hat{h}$ to q_o is immediate.

(2) q_o is formally linearizable but admits no formal holomorphic first integral (called *nonresonant*). Here $\hat{h}$ extends to q_o as a consequence of the fact that the quotient $\frac{\hat{h}}{\hat{h}_o}$ over a punctured neighborhood of q_o in Λ, must be a transversely formal first integral, and q_o admits no such nonconstant first integrals.

(3) q_o is a resonant singularity but not formally linearizable (simple Martinet-Ramis singularity). In this case we have $(\mathcal{F}, q_o) : kxdy + \ell ydx +$ h.o.t. $= 0$ and $k, \ell \in \mathbb{N}$, $(k, \ell) = 1$. Once again $\frac{\hat{h}_o}{\hat{h}}$ is a formal meromorphic first integral for ω over a punctured disk in Λ centered at q_o. But the singularity is supposed to be nonlinearizable, so that its local holonomy associated to Λ is not periodic and so $\frac{\hat{h}}{\hat{h}_o}$ must be constant, which implies that $\hat{h}$ extends formally to q_o.

(4) q_o is of the form $xdy - \lambda ydx +$ h.o.t. $= 0$ with $\lambda = n \in \mathbb{N}$ and nonlinearizable. In this case by Poincaré-Dulac Theorem [17] there exists a *holomorphic* system of coordinates (still denoted (x, y)) that puts q_o in the form $ydx - (nx + y^n)dy = 0$ with $\Lambda : (y = 0)$. Thus we have $(\mathcal{F}, q_o)$: $\frac{dy}{y} - d(\frac{x}{y^n}$ which is a closed meromorphic 1-form admitting no holomorphic first integral. Again we find that $\hat{h}/\hat{h}_o$ extends must be constant and this implies the extension of $\hat{h}$ to q_o.

(5) q_o is a saddle-node in good-position. Here we have use the fact that the strong manifold is contained in Λ and its local holonomy is tangent to the identity, but nontrivial [18], and therefore leaves invariant no formal meromorphic function, except the constants. This implies again that $\hat{h}$ extends to q_o as a constant multiple of $\hat{h}_o$.

Now we assume that $\mathrm{Hol}(\Lambda)$ is solvable nonabelian. Using the techniques of [8], [20] we obtain a transversely formal closed meromorphic 1-form $\hat{\eta}$, defined over Λ_o, and satisfying $d\omega = \hat{\eta} \wedge \omega$. Moreover, according to [7] we have a formal embedding $\mathrm{Hol}(\Lambda) \subset \mathbb{H}_k$, where by definition $\mathbb{H}_k = \{\varphi \in \mathrm{Diff}(\mathbb{C}, 0); \varphi(z)^k = \frac{\mu_\varphi z^k}{1 + a_\varphi z^k}, \mu_\varphi \in \mathbb{C}^*, a_\varphi \in \mathbb{C}\}$. The number k is a formal invariant called the *ramification order of the group* [7]. The construction of the 1-form $\hat{\eta}$ gives $\mathrm{Res}_{\Lambda_o} \hat{\eta} = k + 1$ [8], [20]. Fix a singularity $q_o \in \Lambda$. Using the formal normal forms for q_o we may obtain a formal 1-form

$\hat{\eta}_o$ at q_o, which is a (formal) closed integrating factor for ω. As above we may extend $\hat{\eta}_o$ as a transversely formal 1-form over a small disk $\mathbb{D}_{q_o} \subset \Lambda$ centered at q_o. The difference $\hat{\eta} - \hat{\eta}_o$ is a closed multiple of ω, so that we may write it as $\hat{h}.\omega$ for some transversely formal integrating factor $\hat{h}$ over $\mathbb{D}_{q_o}$. This already shows (according to the above considerations) that $\hat{\eta}$ extends to q_o in a transversely formal way. However we want to remark that it is possible to choose $\hat{\eta}_o$ so that it coincides with $\hat{\eta}$ and we need this information later. Again five are the cases to be considered. We detail the following:

(a) q_o admits a holomorphic first integral $x^q y^p$. Using the remark above we find $\hat{\eta} - \hat{\eta}_o = d\hat{\varphi}(x^q y^p).(q\frac{dx}{x} + p\frac{dy}{y})$ for some formal holomorphic function $\hat{\varphi}(z) \in \mathbb{C}\{\{z\}\}$, which shows the extension of $\hat{\eta}$ to q_o.

(b) q_o is a Martinet-Ramis singularity. According to the inclusion $\mathrm{Hol}(\Lambda) \subset \mathbb{H}_k$ the local holonomy associated to the separatrix S at q_o is (tangent to 1) and formally conjugate to a map of the form $\varphi(z) = \frac{z}{(1+az^k)^{\frac{1}{k}}}$. In fact, we know that any homography which is not tangent to 1 is linearizable, and on the other hand the linearization on the local holonomy implies the linearization of the singularity. On the other hand φ is the holonomy of the germ of singularity $\omega_{k,\ell} = \ell x dy + ky(1 + \frac{\sqrt{-1}}{2\pi}x^k y^\ell)dx = 0$. Thus (see [17], [8]) the singularity q_o is formally conjugated to the foliation $\omega_{k,\ell} = 0$. Therefore there are formal coordinates $(\hat{x}, \hat{y})$ centered at q_o such that for some formal meromorphic function $\hat{g}, \omega(\hat{x}, \hat{y}) = \hat{g}\omega_{k,\ell}(\hat{x}, \hat{y})$. Moreover if we define $\hat{\eta}_o = (k+1)\frac{d\hat{y}}{\hat{y}} + (\ell+1)\frac{d\hat{x}}{\hat{x}} + \frac{d\hat{g}}{\hat{g}}$, then we obtain $d\omega = \hat{\eta}_o \wedge \omega$. We have $\hat{\eta} - \hat{\eta}_o = \hat{h}.\omega$ for some formal expression $\hat{h}$ which satisfies $d(\hat{h}.\omega) = 0$. On the other hand, we know that by construction $\mathrm{Res}_{\Lambda_o} \eta = k+1$, so that $\hat{h}.\omega$ is closed, and holomorphic along $\Lambda_o \backslash \{q_o\}$. Since the singularity q_o is of the (nonlinearizable) formal normal form above, it follows that $\hat{h}.\omega = 0$. Therefore we extend $\hat{\eta}$ as $\hat{\eta}_o$ to q_o.

(c) q_o is a Poincaré-Dulac normal form. In this case we have $\omega(x, y) = gy^n(\frac{dy}{y} - d(\frac{x}{y^n}))$ for some meromorphic function g and some local holomorphic chart (x, y), with $\Lambda : (y = 0)$. We define $\hat{\eta}_o := \frac{dg}{g} + n\frac{dy}{y}$. Then the difference $\hat{\eta} - \hat{\eta}_o$ must be of the form $c.((\frac{dy}{y} - d(\frac{x}{y^n}))$ for some constant $c \in \mathbb{C}$. Since $\hat{\eta}$ and $\hat{\eta}_o$ have simple poles on Λ it follows that $c = 0$ and $\hat{\eta} = \hat{\eta}_o$.

(d) q_o is formally linearizable but admits no holomorphic first integral. In this case we choose formal coordinates (x, y) such that $\omega(x, y) = g(xdy - \lambda y dx)$ with $\lambda \in \mathbb{C} \backslash \mathbb{Q}$ and g formal meromorphic. We define $\hat{\eta}_o = \frac{dg}{g} + \alpha\frac{dx}{x} +$

$\beta\frac{dy}{y}$ for $\alpha, \beta \in \mathbb{C}$ satisfying $\alpha + \beta\lambda = 1 + \lambda$. Then $\hat{\eta}_o$ is a closed integrating factor for $\omega(x,y)$ [21], [3]. On the other hand $\hat{\eta} - \hat{\eta}_o = f(\frac{dy}{y} - \frac{dx}{x})$ for some formal meromorphic function that satisfies $df \wedge (\frac{dy}{y} - \frac{dx}{x}) = 0$. Since $\lambda \notin \mathbb{Q}$ a formal computation shows that $f = c \in \mathbb{C}$ is constant and we conclude that $\hat{\eta}$ extends to q_o.

(e) q_o is a saddle-node in good-position. We have in analytic coordinates $(\mathcal{F}, q_o) : X^{k+1}dY - Y(1 + \lambda X^k)dX + (\text{h.o.t.})dX = 0$ with $(Y = 0) \subset \Lambda$. We may choose formal coordinates (x,y) at q_o such that give $\omega = g(x^{p+1}dy - y(1 + \lambda x^p)dx)$ [18], and $X|x$ as formal expressions. Take $\eta_o = \frac{d(gx^{p+1}y)}{gx^{p+1}y}$. Since q_o is in good position, Λ contains the (strong) separatrix $(x = 0)$ then the local holonomy being embeddable in $\mathbb{H}_k$ implies that $\lambda = 0$ and $p = k$ [4]. This gives $\text{Res}_\Lambda\, \hat{\eta}_o = k + 1$ and therefore the difference $\hat{\eta} - \hat{\eta}_o$ is holomorphic (both 1-forms have simple poles) over $\mathbb{D}_{q_o} - \{q_o\}$. Thus this difference must be zero. In particular the residues of the form $\hat{\eta}$ are k and 1.

We have therefore constructed $\hat{\eta}$ over the curve Λ as in the statement. This proves the proposition. □

6. Proof of the main results

It follows from [4], [8] that, since $\mathbb{C}P(2)\backslash\Lambda$ is a Stein manifold [25], Hironaka-Matsumura theorem [12] asserts that both $\hat{h}$ as well as $\hat{\eta}$, constructed in Proposition 5.1, extend meromorphically to $\mathbb{C}P(2)$. This already proves Theorem E. Theorem D is proved as follows:

Lemma 6.1 *Let $\mathcal{F}$, Λ be as in Theorem* D. *Assume that* $\text{Hol}(\Lambda)$ *is nonabelian, then $\hat{\eta}$ has entire residues and poles of order one on $\mathbb{C}P(2)$.*

Proof. In fact, according to the last part of Proposition 4.1 we may assume that all the simple singularities $q_o \in \Lambda$ are resonant of Martinet-Ramis type. Thus we just have to observe that from the proof of Proposition 5.1 above, it follows that $\hat{\eta}$ has simple poles and entire residues, all of these along Λ and the separatrices through these points q_o which are transverse to Λ (notice that cases (a) and (d) are excluded). □

It follows from the lemma above that if the holonomy of Λ is nonabelian then we can obtain $\hat{\eta}$ with simple poles and entire residues on $\mathbb{C}P(2)$. The Integration Lemma [5] implies therefore that $\hat{\eta}$ is of the form $\hat{\eta} = d\log(R)$

for some rational function R on $\mathbb{C}P(2)$. Thus it is clear that $d(\frac{\omega}{R}) = 0$. The proof of Theorem D is finished. Notice that this argumentation together with the result of section 3 implies Theorem C.

Now we proceed the proof of Theorem B. According to Theorem E $\mathcal{F}|_{\mathbb{C}^2}$ is given by a polynomial 1-form ω which admits a closed integrating factor η. We have two possibilities of argumentation: (1) since the singularities of $\mathcal{F}$ are simple it follows that $(\eta)_\infty$ contains some algebraic leaf Γ such that $\operatorname{Res}_\Gamma \eta \notin \{2, 3, 4, \ldots\}$ and therefore the holonomy of Λ is abelian linearizable (see [21]) which implies that $\mathcal{F}$ is logarithmic [21], [3]; (2) using the fact that the singularities of $\mathcal{F}$ are simple we may conclude as in [21] that the invariant part of $(\eta)_\infty$ is a nodal curve S (recall that a simple singularity exhibits two transverse separatrices) that satisfies the equality in *Poincaré Problem* $\deg(S) = \deg(\mathcal{F}) + 2$ where $\deg(\mathcal{F})$ is the *degree* of $\mathcal{F}$ (see [6], [21]). Thus using the main result of [6] we conclude that $\mathcal{F}$ is logarithmic. This ends the proof of Theorem B.

Finally, we prove Theorem A. According to Theorem C $\mathcal{F}$ is given by a closed rational 1-form and therefore it has some algebraic leaf [3]. It follows from Theorem B that $\mathcal{F}$ is logarithmic. Using the fact that a Martinet-Ramis singularity cannot be defined by a simple poles closed 1-form [3], we conclude that there are no Martinet-Ramis singularities. Therefore $\mathcal{F}$ has only hyperbolic singularities and must be linear as in [1]. Theorem A is now proved.

Let us give an application of our results to the study of holomorphic flows on $\mathbb{C}^2$. Holomorphic flows on a Stein surface have been studied by Suzuki in [27]. Our contribution is the following (a singularity is *dicritical* if it has infinitely many separatrices):

Corollary 6.2 *Let X be a complete polynomial vector field on $\mathbb{C}^2$ and denote by $\mathcal{F}$ the corresponding foliation on $\mathbb{C}P(2)$.*

(i) *If L_∞ is noninvariant then $\mathcal{F}$ admits a rational first integral.*

(ii) *Assume that L_∞ is invariant. If $\operatorname{sing}\mathcal{F} \cap L_\infty$ contains some dicritical singularity then X admits a meromorphic first integral. If $\operatorname{sing}\mathcal{F} \cap L_\infty$ consists of simple singularities then $\mathcal{F}$ is given by a closed rational 1-form. If the singularities in L_∞ are simple, Poincaré-Dulac and saddle-nodes in good position then $\mathcal{F}$ admits a rational closed integrating factor.*

Proof. Since $\mathbb{C}^2$ contains no compact complex Torus the orbits of X must be diffeomorphic to $\mathbb{C}$ or $\mathbb{C}^*$ and $\mathcal{F}$ is parabolic. If the line at infinity

L_∞ is noninvariant then there exists a rational first integral ([23]). Thus we may assume that L_∞ is an algebraic leaf of $\mathcal{F}$. If L_∞ contains some dicritical singularity then the generic leaf is diffeomorphic to $\mathbb{C}^*$ [23] and $\mathcal{F}|_{\mathbb{C}^2}$ admits a meromorphic first integral [27]. It remains to consider the case L_∞ contains only singularities as in Theorem D. Using Theorems D and E we complete the proof. □

Before finishing this paper we would like to state a conjecture:

Conjecture 6.3 Let $\mathcal{F}$ be a parabolic foliation on $\mathbb{C}P(2)$. Then we have the following possibilities:
(i) $\mathcal{F}$ admits a rational first integral
(ii) $\mathcal{F}$ is given by a closed rational 1-form
(iii) $\mathcal{F}$ admits a closed rational integrating factor and is a rational pull-back of a Bernoulli foliation.

Acknowledgments This work was partially conceived during a postdoctorade stage at the University of Rennes I. I want to thank the IRMAR and D. Cerveau for the hospitality. I am in debt with A. Lins Neto, J. Rebelo and P. Sad for valuable discussions. I am also grateful to the referee for his kind interest and valuable remarks which helped to improve this paper, including corrections in the main statements. This work was partially supported by CNPq-Brasil and Min. Affaires Étrangerès-France.

References

[1] Brunella M., *Foliations on the complex projective plane with many parabolic leaves.* Ann. Inst. Fourier, Grenoble. **44** 4 (1994), 1237–1242.

[2] Camacho C., Neto A. Lins and Sad P., *Minimal Sets of Foliations on Complex Projective Spaces.* Publ. Math. I.H.E.S. **68** (1988), 187–203.

[3] Camacho C. and Scárdua B. Azevedo, *Liouvillian first integrals of holomorphic foliations.* Pre-print, IMPA, 1998.

[4] Camacho C. and Scárdua B. Azevedo, *Foliations on complex projective spaces with algebraic limit sets.* to appear in Astérisque $\geq$ 1998.

[5] Cerveau D. and Neto A. Lins, *Irreducible components of the space of holomorphic foliations of degree two in $\mathbb{C}P(n)$, $n \geq 3$.* Ann. of Math. (1996), 577–612.

[6] Cerveau D. and Neto A. Lins, *Holomorphic foliations in $\mathbb{C}P(2)$ having an invariant algebraic curve.* Ann. Inst. Fourier, t.**41** (1991), Fasc. 4, 883–903.

[7] Cerveau D. and Moussu R., *Groupes d'automorphismes de $(\mathbb{C}, 0)$ et équations différentielles $ydy + \cdots = 0$.* Bull. Soc. Math. France **116** (1988), 459–488.

[8] Cerveau D. and Scárdua B. Azevedo, *On the integration of polynomial vector fields*

on dimension two. In preparation.

[9] Garnett L., *Foliations, the ergodic theorem and brownian motion.* Jour. of Funct. Anal. **51** (1983), 285–311.

[10] Ghys E., *Topologie de feuilles génériques.* Annals of Math. **141** (1995), 387–422.

[11] Gunning R. and Rossi H., *Analytic functions of several complex variables.* Prentice Hall, Englewood Cliffs, NJ, 1965.

[12] Hironaka H. and Matsumura H., *Formal functions and formal embeddings.* J. Math. Soc. Japan, Vol. **20**, Nos.1–2, 1968.

[13] Huber A., *On subharmonic functions and differential geometry in the large.* Comment. Math. Helv. **32** (1957), 13–72.

[14] Kusunoki Y. and Mori S., *On the harmonic boundary of an open Riemann surface.* I, Jap. Jour. of Math. **29** (1959), 52–56.

[15] Neto A. Lins, *Construction of singular holomorphic vector fields and foliations in dimension two.* J. of Diff. Geometry **26** (1987), 1–31.

[16] Mattei J.F. and Moussu R., *Holonomie et intégrales premières.* Ann. Sc. Ecole Norm. Sup. **13** (1980), 469–523.

[17] Martinet J. and Ramis J-P., *Classification analytique des équations différentielles nonlineaires resonnants du premier ordre.* Ann. Sc. Ec. Norm. Sup. **16** (1983), 571–621.

[18] Martinet J. and Ramis J-P., *Problème de modules pour des équations différentielles non lineaires du premier ordre.* Publ. Math. Inst. Hautes Études Scientifiques **55** (1982), 63–124.

[19] Nakai I., *Separatrices for nonsolvable dynamics on* $\mathbb{C}, 0$. Ann. Inst. Fourier, Grenoble **44**, 2 (1994), 569–599.

[20] Paul E., *Formes singuliéres à holonomie résoluble.* Prépublication, Univ. Paul Sabatier, Toulouse, (1995).

[21] Scárdua B. Azevedo, *Transversely affine and transversely projective foliations.* Ann. Sc. École Norm. Sup., 4^e série, t.**30**, (1997), 169–204.

[22] Scárdua B. Azevedo, *Polynomial vector fields having orbits with bounded geometry.* pre-print Impa - Rio de Janeiro, 1998.

[23] Scárdua B. Azevedo, *Parabolic uniform limits of holomorphic flows in* $\mathbb{C}^2$. Bol. Soc. Mat. Mexicana (3), Vol. **4** (1998), 123–130.

[24] Seidenberg A., *Reduction of singularities of the differential equation* $Ady = Bdx$. Amer. J. of Math. **90** (1968), 248–269.

[25] Siu Y., *Techniques of extension of analytic objects.* Marcel Dekker, New York, 1974.

[26] Suzuki M., *Sur les intégrales premières de certains feuilletages analytiques complexes.* Séminaire Norguet, Springer Lect. Notes **670** (1977), 53–79.

[27] Suzuki M., *Sur les opérations holomorphes de* $\mathbb{C}$ *et de* $\mathbb{C}^*$ *sur un space de Stein.* Séminaire Norguet, Springer Lect. Notes **670** (1977), 80–88.

[28] White B., *Complete surfaces of finite total curvature.* Journal of Differential Geometry **26** (1987), 315–326.

[29] Yamaguchi H., *Sur le mouvement des constants de Robin.* J. Math. Kyoto Univ. **15** (1975), 53–71.

[30] Yamaguchi H., *Parabolicité d'une fonction entiére.* J. Math. Kyoto Univ. **16**-1 (1976), 71–92.

Instituto de Matemática Pura e Aplicada
Estrada D. Castorina, 110, Jardim Botânico
Rio de Janeiro - RJ, CEP: 22460-320, Brasil
E-mail: scardua@impa.br